AN INVERSE PROBLEM IN THE PRESENCE OF A MEAN FLOW

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INTRODUCTION

The need for nondestructive and nonintrusive sensing devices has motivated the research of several inverse acoustic problems. Many of these problems involve acoustically illuminating a specified region so as to use the scattered acoustics to determine the shape of an obstacle in the region (Colton & Kress 1992), to determine the surface roughness of an obstacle in the region (Schatzberg 1993; Spivack 1992) or to determine general characteristics of the region such as the number of different material layers (Kedzierawski 1993). Another inverse acoustic problem that has received much attention is acoustic holography, wherein the motion of a body is determined from the acoustic field it produces (Sarkissian, Gaumond, Williams, & Houston 1993). In acoustic holography there need not be an incident acoustic field; instead, an actuator on the body can induce the body vibrations. In all of these inverse acoustic applications, the medium which supports the acoustic wave is at rest.

Recently, within the fields of unsteady aerodynamics and aeroacoustics, interest in a new class of inverse problems has emerged. These inverse acoustic problems must be considered in the presence of a mean flow. Common systems analyzed within the fields of unsteady aerodynamics and aeroacoustics, include fixed and rotating single-blade and multi-blade configurations subject to an unsteady flow field and/or exhibiting aeroelastic vibrations. In both cases, the unsteady forces applied to the blades can cause fatigue and produce unwanted sound. In systems like turbomachinery and propellers, it is not only destructive, but it is very difficult to instrument the blades in order to experimentally measure the unsteady surface pressure. Hence, the use of the radiated sound to determine the unsteady surface pressure is ideal. It is also of interest in such applications, to determine the instigating unsteady phenomenon. To consider the inverse acoustic problems in the field unsteady aerodynamics and aeroacoustics, the assumption used in other inverse acoustic problems, that the body is immersed in a still medium, cannot be applied.

The first theoretical treatment of an inverse problem in unsteady aerodynamics and aeroacoustics analyzed a single flat-plate airfoil encountering a convected vortical disturbance. This work was presented at the 1994 ASME winter meeting (Patrick, Atassi, & Blake 1994). Since then further results for this problem have been ob-

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tained (Patrick 1995), and have been submitted to the AIAA Journal (Patrick, Atassi, & Blake 1995a; Patrick, Atassi, & Blake 1995b). For the flat-plate airfoil problem, it has been shown that the sound radiated from the interaction of the airfoil and a vortical disturbance can be used to determine the unsteady surface pressure and the magnitude of the vortical disturbance.

This *inverse aeroacoustic problem*, defined as determining the unsteady surface pressure from the radiated sound, is theoretically ill-posed, and as such, the first naive methods which were developed for performing the inversion were very sensitive to the accuracy of the input data. However, optimization and regularization techniques were soon incorporated, and now, even with noisy input data, very good reconstructions of the unsteady surface pressure can be obtained.

The second part of the inversion, termed the *inverse aerodynamic problem*, is defined as calculating the magnitude of the flow disturbance from the response of the airfoil. This part is well-posed. When the unsteady surface pressure is well known on the entire airfoil, the aerodynamic inversion is performed directly. If the surface pressure is only known at a few discrete locations, the inversion must be linked to a direct solver.

The methods developed thus far for the aeroacoustic and unsteady aerodynamic inversions have been partially verified through experiment (Minniti & Patrick 1995). Experimentally obtained acoustic data was used as input to the inversion schemes. The predicted unsteady surface pressure was then compared to experimental surface pressure data at a few locations and the agreement was quite good. Only a partial verification could be performed since the flat-plate airfoil was only instrumented with a small number of pressure transducers. The methods, using the experimentally obtained input data, predicted the unsteady surface pressure at those few locations well.

The present research extends the previous work on inverse aeroacoustic problems to include aeroelastic vibrations of the airfoil. The aeroelastic vibrations considered in this paper are modelled as a rotational oscillation, i.e. as a pitching airfoil. The inversion for an airfoil undergoing or translational oscillation, or a heaving motion, follows directly. For the oscillating airfoil application, the inverse aeroacoustic part remains the same and the unsteady surface pressure is determined from the radiated acoustics. The problem is ill-posed and the techniques developed for the inverse gust problem can be used here. The goal of the inverse aerodynamic part is now to determine the amplitude of the oscillation and, in the rotational oscillation case, to determine the axis of rotation. The aerodynamic inversion is well-posed and thus the amplitude of the oscillation and the rotation axis are determined uniquely and the inversion is stable even when the input data are noisy.

This paper outlines the direct problem for the oscillating airfoil in compressible flow, describes the inversion method, discusses uniqueness for the inverse problem and finally presents some results. The results are compared to the results obtained for a similar frequency vortical flow disturbance. The inversion, for a vortical disturbance and an oscillation both occurring at the same frequency, is also discussed.

MATHEMATICAL FORMULATION

For a flat-plate airfoil undergoing a small amplitude oscillation in subsonic, inviscid, isentropic flow, the continuity equation and Euler's equation can be linearized about the mean flow quantities and then combined to give a single, homogeneous, convective wave equation for the unsteady pressure (Goldstein 1976). Figure 1 depicts the basic problem. For either a rotational oscillation, represented by amplitude α and rotation axis x_0, or a translational oscillation, represented by amplitude h, with frequency ω, if we assume there is no spanwise variation in the oscillation, the convective wave equation can then be reduced to the Helmholtz equation as follows:

$$\left(\tilde{\nabla}^2 + K_1^2\right) P = 0 \tag{1}$$

where

$$\begin{aligned} P(\tilde{x}_1, \tilde{x}_2) &= p' e^{-i(k_1 t + M K_1 \tilde{x}_1)} \\ \tilde{x}_1 &= x_1 \\ \tilde{x}_2 &= \beta x_2 \\ M &= \frac{U_\infty}{c_0} \\ \beta &= \sqrt{1 - M^2} \end{aligned}$$

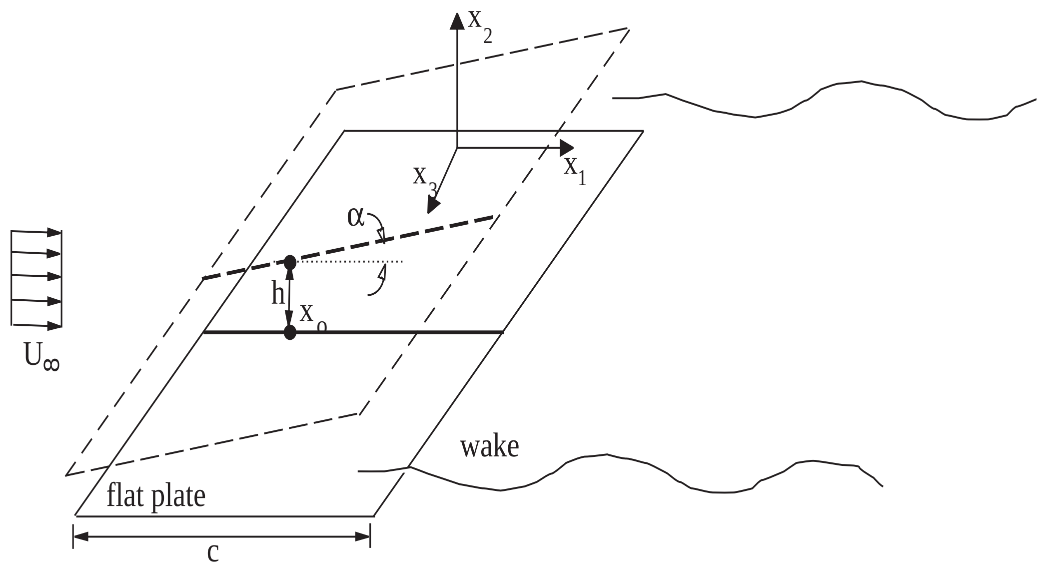

Figure 1: Single oscillating airfoil.

$$K_1 = \frac{k_1 M}{\beta^2}$$

The reduced frequency is $k_1 = \omega c/2U_\infty$, c_0 is the speed of sound, U_∞ is the velocity of the free stream, and $\tilde{\nabla} = (\partial/\partial\tilde{x}_1, \partial^2/\partial\tilde{x}_2)$. Note that in this formulation, lengths are normalized with respect to the half chord, $c/2$, velocities with respect to U_∞, time with respect to $c/2U_\infty$, and unsteady pressure with respect to $\rho_\infty u_2 U_\infty$. u_2 is the unsteady velocity component normal to the airfoil. For the gust problem, u_2 is the second component of the gust, which can be written as γU_∞ where γ usually takes on a value between 0 and 0.10. For a rotational oscillation, u_2 is represented by αU_∞ where α is the amplitude of the oscillation in radians. Finally, for a translational oscillation $u_2 = (h/c)U_\infty$ where h is the amplitude of the oscillation in units of length and c is the airfoil chord length.

The boundary Helmholtz equation (1) is subject to a radiation boundary condition. Also, the unsteady pressure must be continuous upstream and downstream of the airfoil but have a jump discontinuity across the airfoil.

DIRECT PROBLEM

In the 1930's, Theodorsen presented the analytical expression for the unsteady response of an oscillating airfoil in the limit of incompressible flow (Theodorsen 1935). For the case of compressible flow, however no such analytical expression exists; instead, the unsteady surface pressure and the velocity of the airfoil are related via an integral equation. (Credit for the first development of this integral equation is given to Possio (1938)). The integral equation is obtained by representing the unsteady surface pressure as a combination of plane waves and then transforming, to an integral equation, the differential relation in the second component of Euler's equation,

$$\rho_\infty \frac{D_\infty u_2}{Dt} = -\frac{\partial p'}{\partial x_2}, \tag{2}$$

where $D_\infty/Dt = \partial/\partial t + U_\infty \partial/\partial x_1$. The final form of the integral equation is

$$u_2(\tilde{x}_1, \tilde{x}_2 = 0) = \frac{-i\beta}{4\pi} \int_{-1}^{1} \Delta p'(\xi) \mathcal{K}(\tilde{x}_1 - \xi) d\xi \tag{3}$$

where

$$\mathcal{K}(\tilde{x}_1 - \xi) = \int_{-\infty}^{\infty} \frac{M\sqrt{\alpha^2 - K^2}}{(\alpha M - K_1)} e^{-i(\alpha - MK_1)(\tilde{x}_1 - \xi)} d\alpha \tag{4}$$

In the direct problem, once the unsteady pressure jump along the airfoil is known, the radiated sound can be determined. Green's theorem applied to Eq. (1) can be used to show that the transformed pressure in the field can be determined in terms of the transformed pressure jump along the airfoil. In two-dimensions the relationship is given by

$$P(\vec{\tilde{x}}) = \frac{-iK\tilde{x}_2}{4}\int_{-1}^{1}\Delta P(\tilde{y}_1)\frac{H_1^{(2)}(K|\vec{\tilde{x}}-\vec{\tilde{y}}|)}{|\vec{\tilde{x}}-\vec{\tilde{y}}|}d\tilde{y}_1 \quad (5)$$

(Atassi, Dusey, & Davis 1993).

The airfoil response and the radiated sound were calculated using Equations (3) - (5) for both a convected vortical gust interacting with the flat-plate airfoil and for an oscillating flat-plate airfoil. In both applications, the Mach number was set to 0.6 and a reduced frequency of 1.0 was assumed. The vortical disturbance considered is a transverse gust with amplitude equivalent to 10% of the mean flow velocity. The velocity of the gust at a specified frequency is represented by the single Fourier component as

$$u_2 = .1U_\infty e^{i(\omega t - k_1 x_1)} \quad (6)$$

The oscillating airfoil is chosen to have an oscillation amplitude of $\alpha = 1° = .01745$ radians and the axis of rotation is set at the quarter chord point. If the chord lies between $-c/2$ and $c/2$ the displacement of the airfoil is given by

$$x_2 = .01745(x_1 + .5c)e^{i\omega t} \quad (7)$$

Using $u_2 = D_\infty x_2/Dt$, the normal component of the velocity is

$$u_2 = (i\omega .01745(x_1 + .5c) + .01745U_\infty)e^{i\omega t} \quad (8)$$

Figure 2 shows the resulting airfoil responses for the two applications. The value plotted is the unsteady pressure jump, $\Delta p'$, normalized by $\rho_\infty U_\infty^2$. The solid line represents the response to the gust and the dotted line represents the response of the oscillating airfoil. Notice that if the normalization $\rho_\infty u_2 U_\infty$ were used, the unsteady pressure jump would appear to differ by a factor of 10. The far-field pressure was also calculated

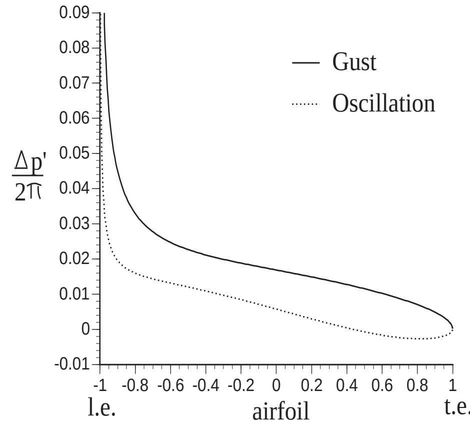

Figure 2: Magnitude of the airfoil unsteady pressure jump for an airfoil encountering a gust and an oscillating airfoil.

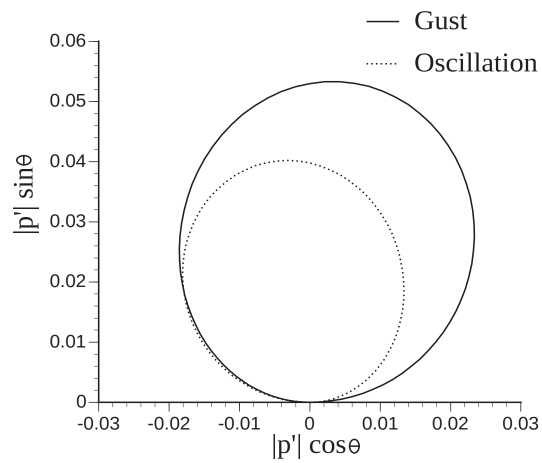

Figure 3: Acoustic directivity from an airfoil encountering a gust and an oscillating airfoil.

for these cases. Figure 3 shows the corresponding directivity pattern. Again the pressure plotted has been normalized by $\rho_\infty U_\infty^2$.

THE INVERSE PROBLEM

Just as the direct problem for the oscillating airfoil is performed in two parts, so to the inverse problem is done in two parts. First, the inverse aeroacoustic part is discussed.

The Inverse Aeroacoustic Problem The goal of the aeroacoustic inversion is to use the pressure in the far field to find the surface pressure. This entails solving the integral equation (5). Hence, the inversion for the oscillating airfoil is identical to the inversion for the gust problem.

The method developed for performing the inversion requires that the integral equation be cast as a matrix equation (Patrick, Atassi, & Blake 1994). Although a simple trapezoidal quadrature rule for discretizing the integral equation leads to adequate reconstructions of the surface pressure jump (Patrick 1995), the preferred method is to use a collocation series for ΔP in the integral equation (5). The characteristics of an aerodynamic response can be embedded in the collocation series leading to more accurate reconstructions and lower requirements on the amount of necessary input data. The collocation series is chosen as

$$\Delta P(\mu) = A_0 cot(\frac{\mu}{2}) + \sum_{n=1}^{\infty} A_n \sin n\mu \qquad (9)$$

where $y_1 = -\cos\mu$. Regardless of the method used to obtain the matrix equation, the resulting system of equations is ill-conditioned and must be solved using the singular value decomposition in combination with a regularization technique. When the collocation method is used to obtain the matrix equation, an *a priori* cut-off of the singular values serves as the regularizing technique. In essence the number of terms used in the series, Eq. (9), to represent the pressure jump is limited such that the singular values of the resulting matrix equation are all greater than the specified cut off. If the input data is very accurate the cut-off, or regularization parameter, should be set as 0.01.

Of most interest in application however, is the inversion's behavior when the input data contains noise. It was shown for the inverse gust problem, that to *ensure* that errors in the input data were not amplified in the solution, the cut-off parameter had to be larger than 1.0 (Patrick 1995; Patrick, Atassi, & Blake 1995b). However, there is a trade-off; while ensuring that there is no amplification of the input error, a perfect reconstruction will never be obtained even if very accurate input data are used.

To illustrate this point, as well as compare the aeroacoustic inversion for the oscillating airfoil to the results for the gust problem, the case $M = 0.6$, $k_1 = 1.0$ is used. First, the aeroacoustic inversion is performed for a gust of amplitude $.1U_\infty$, (as in Eq. (6)). Three different regularization parameters are tested using "perfect" input data obtained from a numerical direct solver. Figure 4 shows the real and imaginary parts of the reconstructed unsteady pressure jump as compared to the exact jump. The Figure shows the very accurate reconstruction obtained when the regularization parameter is 0.01 and the error in the reconstruction when the regularization parameter is increased.

Perturbing the input data by 20% in the real and imaginary parts gives the noisy far-field data shown in Figure 5. In this figure, both the magnitude and phase of the far-field unsteady pressure are shown. Using the noisy data as input, with the three choices of the regularization parameter we obtain the reconstructions shown in Figure 6. Here one can see the amplification of the input error when the regularization parameter is less than 1.0. Comparing Figures 4 and 6 shows almost no influence of the noise on the solution when the regularization parameter is 1.0.

For comparison, the same set of tests are shown for an airfoil oscillating with amplitude 1° at a reduced frequency of 1.0. Figure 7 has the reconstructions when "perfect" input data are used. Perturbing the input data, as shown in Figure 8, leads to the reconstructions in Figure 9. The same behavior is seen for the aeroacoustic inversion in the case of the oscillating airfoil as in the case of the vortical disturbance.

The Inverse Aerodynamic Problem The goal of the aerodynamic inversion is to use the airfoil surface pressure to find the nature and the amplitude of the unsteady disturbance. The inverse aerodynamic problem is quite different than the aeroacoustic problem. The relation between the unsteady pressure jump along the surface and the unsteady velocity is in the form of an integral; but, for the aerodynamic inversion, the integral simply needs to be evaluated. When the pressure jump is known everywhere along the airfoil, the integration in Eq. (3) can be performed. Notes on methods for performing the integration, (primarily concerning the associated kernel), are given in Appendix 5.B of (Goldstein 1976).

For the case of a vortical disturbance, the magnitude of the integral represents the magnitude of the com-

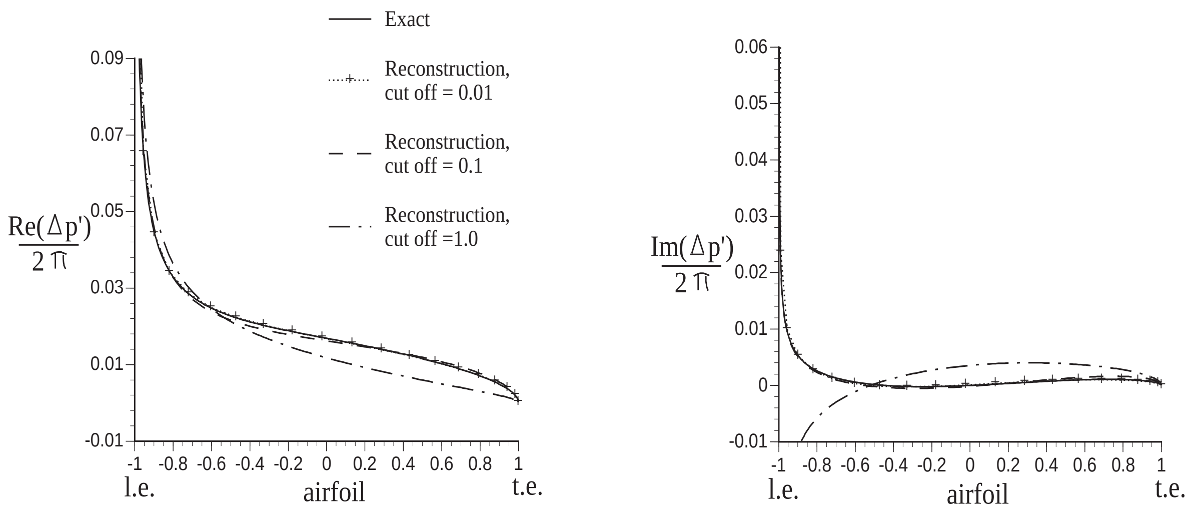

Figure 4: Reconstruction of the unsteady pressure jump for the case of a vortical disturbance using perfect input data with different regularization parameters. $M = 0.6, k_1 = 1.0$.

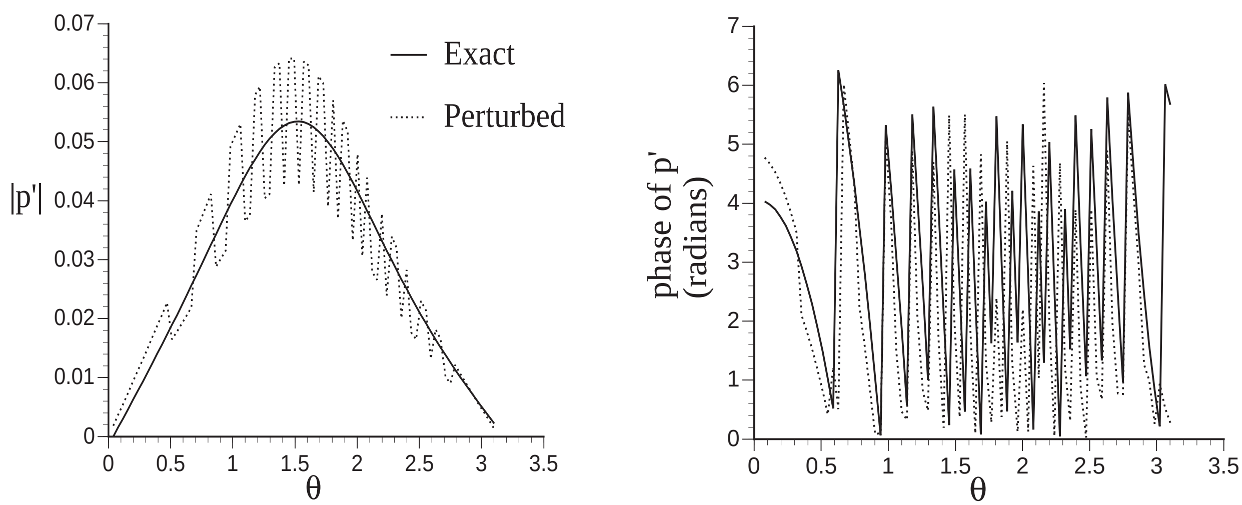

Figure 5: Magnitude (left) and phase (right) of perfect (solid line) and perturbed (dotted line) input data for the case of a vortical disturbance. $M = 0.6, k_1 = 1.0$.

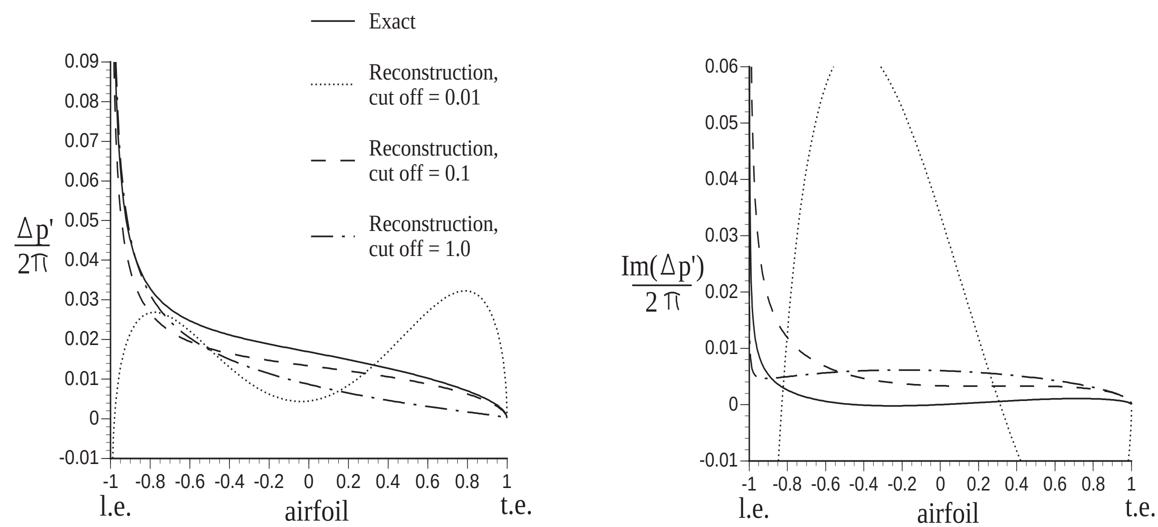

Figure 6: Reconstruction of the unsteady pressure jump for the case of a vortical disturbance using perturbed input data with different regularization parameters. $M = 0.6, k_1 1.0$.

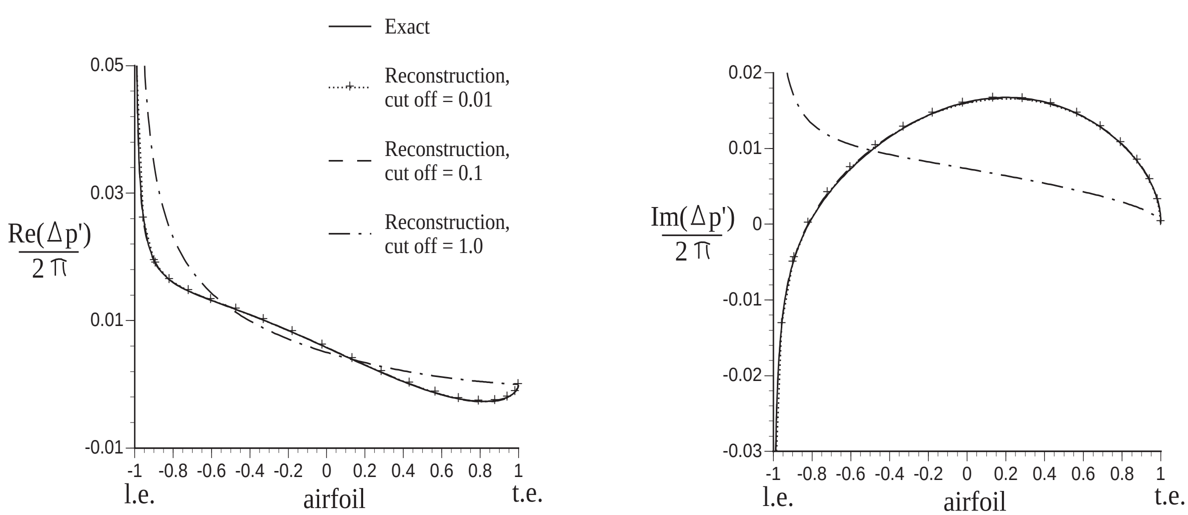

Figure 7: Reconstruction of the unsteady pressure jump for the case of an oscillating airfoil using perfect input data with different regularization parameters. $M = 0.6, k_1 = 1.0$.

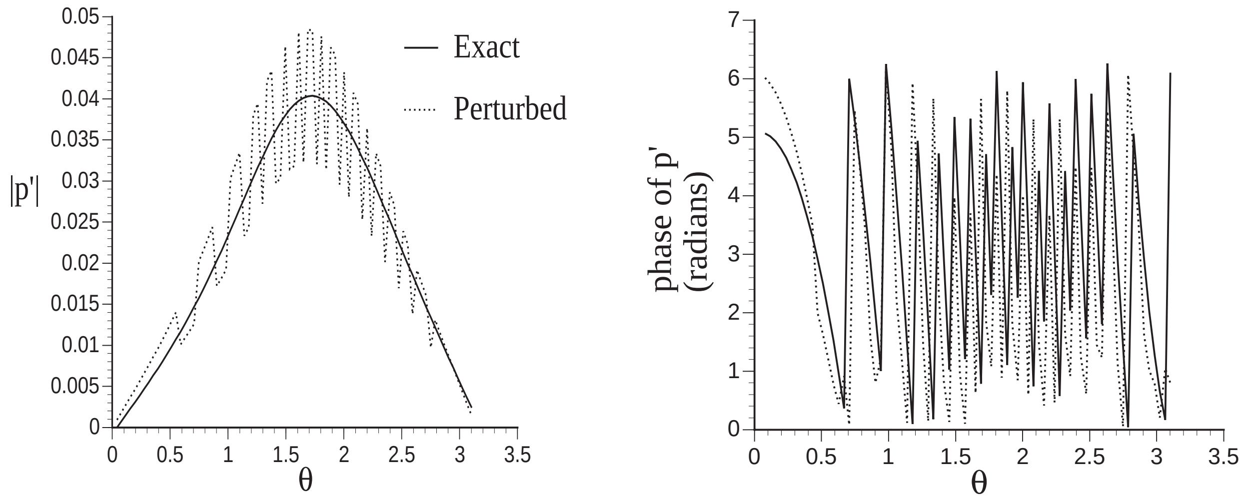

Figure 8: Magnitude (left) and phase (right) of perfect (solid line) and perturbed (dotted line) input data for the case of an oscillating airfoil. $M = 0.6, k_1 = 1.0$.

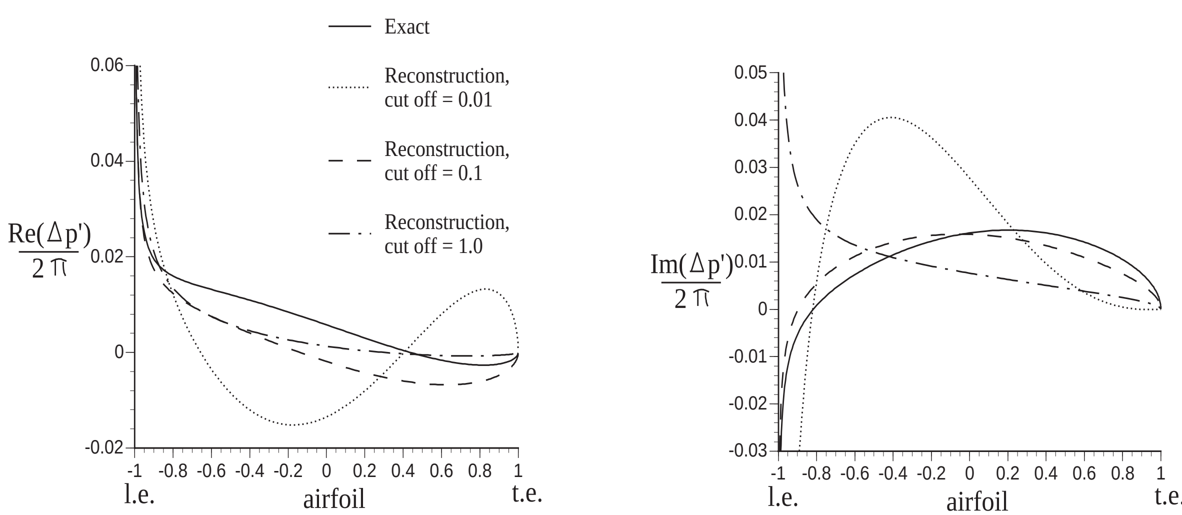

Figure 9: Reconstruction of the unsteady pressure jump for the case of an oscillating airfoil using perturbed input data with different regularization parameters. $M = 0.6, k_1 = 1.0$.

ponent of the unsteady vortical disturbance normal to the airfoil. Therefore, if the surface pressure is known quite accurately, the integration must be performed for only one $\tilde{x}_1$ location. For practical application however, the data have some associated uncertainty. Therefore, the amplitude of the vortical disturbance should be obtained by averaging the results of the integration at several airfoil locations.

For the rotationally oscillating airfoil, two parameters must be determined. We know that the left hand side of Eq. (3) for a general rotational oscillation is given by

$$u_2(\tilde{x}_1) = (i\omega\alpha U_\infty(\tilde{x}_1 + x_0) + \alpha U_\infty)e^{i\omega t} \tag{10}$$

Here, α is the amplitude of the angular oscillation in radians and x_0 represents the location about which the airfoil rotates. At first, it would seem that the velocity must be obtained at two locations along the airfoil in order to determine both α and x_0. However, since these parameters, along with the frequency ω and the velocity U_∞, are real numbers, while the pressure and velocity have associated phase and are represented by complex numbers, two equations are obtained at every $\tilde{x}_1$ location. i.e.

$$\alpha U_\infty = \Re\left(\frac{-i\beta}{4\pi}\int_{-1}^{1}\Delta p'(\xi)\mathcal{K}(\tilde{x}_1 - \xi)e^{-i\omega t}d\xi\right) \tag{11}$$

$$(\omega U_\infty \tilde{x}_1 + \omega U_\infty x_0)\alpha = \Im\left(\frac{-i\beta}{4\pi}\int_{-1}^{1}\Delta p'(\xi)\mathcal{K}(\tilde{x}_1 - \xi)e^{-i\omega t}d\xi\right) \tag{12}$$

where $\mathcal{K}$ is defined in Eq. (4). It is clear from these two equations that α and x_0 are obtained uniquely with one integration of the unsteady pressure, when the pressure is known accurately. If the pressure data contain errors, the integration only propagates these errors and does not amplify them. Therefore, performing the integration for several values of $\tilde{x}_1$ and averaging the corresponding values of α and x_0 provides satisfactory results.

When the unsteady pressure jump along the airfoil is obtained via the aeroacoustic inversion described earlier, the integration in Eq. (3) can be performed. However, if the collocation method presented above is not used for the aeroacoustic inversion or if the unsteady surface pressure is determined experimentally instead of through the aeroacoustic inversion, a sufficient amount of pressure data for the integration may not exist. In this case, the alternative is to assume that the velocity simply corresponds to a unit gust or a unit angular rotation and solve Eq. (3) as an integral equation. This provides the unsteady pressure jump associated with a unit disturbance. Calculating the ratio between the true unsteady pressure jump and the pressure jump corresponding to a unit amplitude disturbance, gives the value of the true velocity.

Notes The entire approach to the inverse unsteady aerodynamic and aeroacoustic problem is based on linear theory. This assumption is good for very slender streamlined bodies and small unsteady disturbances at moderate and low reduced frequencies. In these cases, any nonlinear effects are second order and will be treated as noise in the inversion. Since the inverse methods are able to handle noisy data, one can assume that nonlinear effects which exist in application will not invalidate the inversion. Some experiments have been run for the gust problem in an attempt to validate the techniques and the inversion methods have worked well. The same should hold for the oscillating airfoil.

Since the vortical gust and oscillating airfoil problems can be treated by linear analysis, if a gust and an oscillation should both occur at a single reduced frequency, the inversions are still feasible and performed in the exact manner as described above. The input data will be the sum of the radiated sound produced by both of the unsteady excitations. The aeroacoustic inversion will then give the unsteady pressure jump along the airfoil which is equivalent to the sum of the airfoil response to the gust and the airfoil response to the oscillation. Using this pressure jump then, the normal component of the vortical disturbance, the amplitude of the oscillation and the location of the rotational axis can all be determined. However, even if the surface pressure jump is known very accurately, two integrations of the surface pressure are required to obtain the desired parameters uniquely.

CONCLUSIONS

Previous work on inverse problems in the fields of unsteady aerodynamics and aeroacoustics is extended to include oscillating airfoils in subsonic flows. It is shown that the far-field radiated sound can be used to determine the unsteady pressure along the airfoil surface and subsequently the magnitude of the oscillation and the rotation axis. Very accurate reconstructions can be obtained even when the input data contain errors if the inverse aeroacoustic part is performed using optimization and regularization techniques and simple averaging is used in the inverse aerodynamic part.

Demonstrating the feasibility of using far-field acoustic measurements to reconstruct both the surface pressure and the parameters which characterize the unsteady phenomenon, for a single oscillating airfoil in a vortical flow, has been an essential first step for studying inverse problems in the fields of unsteady aerodynamics and aeroacoustics. Many open inverse problems of technological significance exist in these fields. One direct extension of the current work is the consideration of the effect of finite span on the inversion. Another interesting and challenging problem, which we hope to address in the near future is the generalization of the single-airfoil problem to a cascade of airfoils.

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