QUESTIONS FOR THE QUALIFYING 2011 EXAM — MA583

Exercise 1. Consider an urn that contains m balls, either black or white. At each step, a *fair* coin is flipped; then:

- if the coin lands heads, a *random* ball is picked from the urn and replaced by a *white* ball.
- if the coin lands tails, a *random* ball is picked from the urn and replaced by a *black* ball.

Let $(X_n)_{n=1,2,\ldots}$ be the stochastic process denoting the number of black balls in the urn at the nth step of the process.

(1) Argue why $(X_n)_{n=1,2,\ldots}$ is a (discrete-time) Markov chain with state space $S = \{0, 1, \ldots, m\}$.
(2) What is the period of each state? Are the states connected? Are they transient or recurrent?
(3) Specify the transition probabilities P_{ij}, for $i \in S$ and $j \in S$.
(4) In the case $m = 2$, show (by solving explicitly a 3×3 system of linear equations) that the stationary distribution of the chain is the Binomial distribution with $m = 2$ trials and success probability $1/2$.
(5) **[This is slightly harder]** Show that the previous result in part 4 (i.e., that the stationary distribution of the chain is the Binomial distribution with m trials and success probability $1/2$) holds for any m.

Exercise 2. A town has two outposts that measure seismic activity. The two leading scientists disagree on the annual average number of minor-or-less earthquakes (defined as less than 4.0 magnitude on the Richter scale). One claims that the average is 50 per year, while the other claims that it is 100 per year. Let $(N_t)_{t\geq 0}$ denote the process counting the number of minor-or-less earthquakes in the region up to time t, where time is measured in years. As you have no *a priori* reason the believe one scientist or the other, you assume that, *conditional* on the value of a random variable Λ, $(N_t)_{t\geq 0}$ is a Poisson process with rate Λ, where $P[\Lambda = 50] = 1/2$ and $P[\Lambda = 100] = 1/2$.

(1) Calculate $P[N(t) = n]$, where $n = 0, 1, \ldots$
(2) Is $(N_t)_{t\geq 0}$ a Poisson process?
(3) Does $(N_t)_{t\geq 0}$ have stationary increments? Explain your answer.
(4) Does $(N_t)_{t\geq 0}$ have independent increments? Explain your answer.
(5) After your team install a state-of-the-art seismograph, you record 120 minor-or-less earthquakes in the region during the first year. How does this affect your views on the distribution of Λ? (Basically, you are asked to calculate the *conditional* distribution of $\Lambda = 100$ given $N_1 = 120$.)