Qualifying Exam: CAS MA 583

Boston University, Spring 2007

Problem 1. Let $S_1, S_2, ...$ be the occurrence times of an inhomogeneous Poisson process with rate function $\lambda(t)$. Let $N(t)$ be its counting process, and let $X_1, X_2, ...$ be the interarrival times such that $X_1 = S_1$ and $X_i = S_i - S_{i-1}$ for $i \geq 2$.

(a) Let $\Delta N_{(a,b]} = N(b) - N(a)$. What is the distribution of $\Delta N_{(a,b]}$?

(b) What is the joint distribution of (X_i, X_{i+1})?

(c) Let $(a,b]$ be a subinterval of $[0,T]$. Compute the distribution of $\Delta N_{(a,b]} \mid N(T) = n$.

(d) Assume again that $N(T) = n$. Let W be a random variable with $\Pr(W = i) = 1/n,$ for $i = 1, ..., n$. Compute the distribution of $S_W \mid N(T) = n$.

Problem 2. Suppose each animal can be one of three genotypes, AA, Aa, or aa. An animal with AA is normal, an animal with aa is diseased, and an animal with Aa looks normal but can have diseased offspring. A rancher decides to eliminate a disease from his animal population by inbreeding them. In this scheme, two non-diseased animals are mated and among their descendants, two non-diseased individuals of opposite sex are selected at random. These animals are mated and the process continues. The genotype of the offspring is determined by selecting one gene from each parent with equal probability.

(a) Construct a Markov chain describing the genotypes of the two individuals selected for mating, and calculate the transition probabilities. (Remember: diseased animals are never selected for mating)

(b) Classify each state of this Markov chain as transient, null-recurrent, or positive recurrent.

(c) How many stationary distributions will this Markov chain have? Write down a stationary distribution if at least one exists. If not, prove that no stationary distribution exists.

(d) Assume now that there is another disease that cannot be detected by the rancher with genotypes BB (healthy), Bb (healthy), bb (diseased). When the rancher inbreeds his animals to eliminate disease a, he ends up randomly selecting from the offspring no matter what genotype for disease b. Construct a Markov chain for the genotype of the mating animals for disease b. How many stationary distributions will this Markov chain have?

(e) Assume the initial pair of animals each has genotype Bb. After many generations, what is the probability that there is at least one diseased animal (bb)? What is the probability that all the animals in the population are diseased?