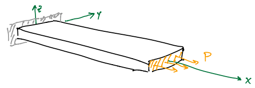
Strain Energy - Stretching



$U = \frac{1}{2}\int_V \sigma \varepsilon \, dV$

① load controlled / force controlled
- apply $P \longrightarrow$ measure ε

② displacement controlled
- apply $\varepsilon \longrightarrow$ measure P

① load controlled

$\sigma_{xx} = \frac{P}{A}$ $\quad \sigma_{ij} = \begin{bmatrix} P/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \varepsilon_{ij} = \begin{bmatrix} \delta/L & 0 & 0 \\ 0 & -\nu\,\delta/L & 0 \\ 0 & 0 & -\nu\,\delta/L \end{bmatrix}$

$\varepsilon_{ij} = ? \longrightarrow \varepsilon_{ij} = \frac{\sigma_{ij}}{E}$

$$U = \frac{1}{2}\int_V \sigma_{ij}\left(\frac{\sigma_{ij}}{E}\right) dV = \frac{1}{2}\int_V \frac{\sigma_{ij}^2}{E} = \frac{1}{2}\int_V \frac{P^2}{EA^2}\,dV$$

$$U_s = \frac{1}{2}\int_{-h/2}^{h/2}\int_{-b/2}^{b/2}\int_0^L \frac{P^2}{EA^2}\,dx\,dy\,dz = \frac{1}{2}\underbrace{hb}_{A}\int_0^L \frac{P^2}{EA^2}\,dx$$

$$\boxed{U_s = \frac{1}{2}\int_0^L \frac{P^2}{EA}\,dx}$$

② displacement controlled

$\sigma_{ij} = ? \longrightarrow \sigma_{ij} = E\varepsilon_{ij}$

$\varepsilon_{xx} = \frac{\delta}{L}$ $\quad \varepsilon_{ij} = \begin{bmatrix} \delta/L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$U = \frac{1}{2}\int_V \overbrace{(E\varepsilon_{ij})}^{\sigma_{ij}}\varepsilon_{ij}\,dV = \frac{1}{2}\int_V E\,\varepsilon_{ij}^2\,dV$$

$$\boxed{U_s = \frac{1}{2}EA\int_0^L \varepsilon^2\,dx}$$

Stretching Energy

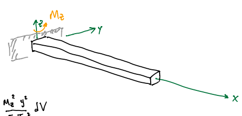
$$\boxed{\begin{aligned} U_s &= \frac{1}{2}EA\int_0^L \varepsilon^2\,dx \\ U_s &= \frac{1}{2EA}\int_0^L P^2\,dx \end{aligned}}$$

→ equivalent through Hooke's law

Scaling $\quad U_s \sim EA\,\varepsilon^2 L$
$\quad U_s \sim \frac{1}{EA}P^2 L$

$\int_0^L dx$

Strain Energy - Bending



① Beam: pure bending ⟶ applied moment

$\sigma_{xx} = -\frac{M_z y}{I_z}$

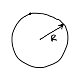
$$U_b = \frac{1}{2}\int_V \frac{\sigma_{xx}^2}{E}\,dV = \frac{1}{2}\int_V \frac{1}{E}\left(-\frac{M_z y}{I_z}\right)^2 dV = \frac{1}{2}\int_V \frac{M_z^2 y^2}{E I_z^2}\,dV$$

$$U_b = \frac{1}{2}\int_0^L \frac{M_z^2}{E I_z^2}\underbrace{\int_A y^2\,dA}_{I_z} \longrightarrow \boxed{U_b = \frac{1}{2}\int_0^L \frac{M_z^2}{E I_z}\,dx}$$

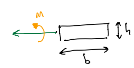
② Beam: pure bending ⟶ applied curvature

constitutive relationship: $M = \kappa E I$

$$U_b = \frac{1}{2}\int_0^L \frac{(\kappa E I)^2}{EI}\,dx \longrightarrow \boxed{U_b = \frac{EI}{2}\int_0^L \kappa^2\,dx}$$



$\kappa = \frac{1}{R}$



$I = \frac{h^3 b}{12}$

$U_b \sim E h^3 b\,\kappa^2 L$

Strain Energy - Torsion

$\sigma_{xx} \sim \frac{My}{I}$ $\qquad \sigma_{x\theta} \sim \frac{T\rho}{J}$

bending moments & torques ⟹ couple

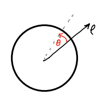
↳ $T = \tau G J$

amount of twist $\quad \neq$ shear stress

$\sigma = E\varepsilon$ ← young's modulus

$\sigma = G\gamma$ ← shear modulus

$$\boxed{U_t = \frac{GJ}{2}\int_0^L \tau^2\,dx}$$



Stretching & Bending

$$\left.\begin{aligned} U_s &\sim Ehb\int \varepsilon^2\,dx \sim EhbL\,\varepsilon^2 \\ U_b &\sim Eh^3 b\int \kappa^2\,dx \sim Eh^3 bL\,\kappa^2 \end{aligned}\right]$$

$$\frac{U_b}{U_s} \sim \frac{E h^{\cancel{3}\,2} \cancel{b} \cancel{L}\,\kappa^2}{E \cancel{h} \cancel{b} \cancel{L}\,\varepsilon^2} \sim h^2\left(\frac{\kappa}{\varepsilon}\right)^2$$

↳ thin objects are easy to bend and hard to stretch

① let's assume our beam is inextensible
$U_s = 0$

② thin objects are stiff, but prone to instability