Applied Mathematics in Mechanics

Problem Set #2

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1. The equation for an elastic beam is:

$$EI\frac{\partial^4 u}{\partial x^4} + \rho\frac{\partial^2 u}{\partial t^2} = 0. \tag{1}$$

where the boundary conditions are $u = u_0 \sin(\omega t)$ and $\partial u/\partial x = 0$ at $x = 0$, while $u = \partial u/\partial x = 0$ at $x = \ell$. Assume the initial conditions are $u = 0$ and $\partial u/\partial t = 0$ at $t = 0$. Here, E is the elastic modulus, I is the second moment of area, and ρ is the mass per unit length of the beam. Nondimensionalize the problem in such a way that the resulting boundary conditions contain no nondimensional groups.

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2. The equations that account for the relativistic motion of a planet around the sun are

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -\frac{Gm}{r^2} + \frac{b}{r^3}, \tag{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = 0, \tag{2b}$$

where b is a constant. Assume the initial conditions are $r = r_0$, $r' = 0$, and $\theta = 0$ at $t = 0$.

- What are the dimensions of r_0, b?
- Nondimensionalize the problem. The scaling should be chosen so the only nondimensional group appearing in the problem involves b.

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3. The temperature $T(t)$ of a chemical sample in a furnace at time t is governed by the initial value problem

$$\frac{\mathrm{d}T}{\mathrm{d}t} = qe^{-\theta/T} - k(T - T_f) \tag{3a}$$

where the initial condition is $T(0) = T_0$, where T_0 is the initial temperature of the sample, T_f is the temperature of the furnace, and q, k, and θ are positive constants.

- What are the dimensions of q, k, and θ?
- Nondimensionalize the problem. Choose your time scale such that the heat loss term is large compared to the heat generated by the reaction.

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4. A rocket blasts off the surface of Earth. During the initial phase of flight, fuel is burned off at the maximum possible rate α, and the exhaust gas is expelled downward with a velocity β relative to the velocity of the rocket. The motion is governed by the following set of equations

$$\dot{m}(t) = -\alpha, \qquad m(0) = M, \tag{3a}$$

$$\ddot{x}(t) = \frac{\alpha\beta}{m(t)} - \frac{g}{\left(1 + \frac{x(t)}{R}\right)^2}, \qquad \dot{x}(0) = 0, \qquad x(0) = 0, \tag{3b}$$

where $m(t)$ is the mass of the rocket, $x(t)$ is the height above Earth's surface, M is the initial mass, g is the acceleration due to gravity, and R is the radius of Earth. Nondimensionalize the problem and determine the characteristic values. Assume that the acceleration is primarily due to fuel burning, and that the gravitational force is relatively small.