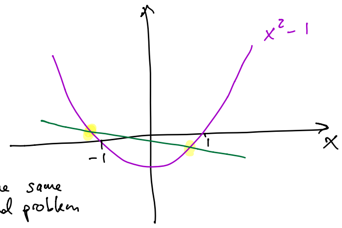
Perturbation Methods

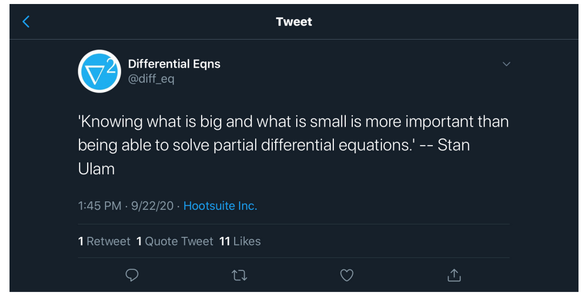
We have a procedure to find dimensionless groups and identify a small parameter ϵ

Algebraic Equation

$x^2 + 2\epsilon x - 1 = 0$ (ϵ small)

$x^2 - 1 = -2\epsilon x$





Regular Perturbation
- the reduced problem ($\epsilon = 0$) has the same number of solutions as the original problem

Analytical Solution: $x = -\epsilon \pm \sqrt{1+\epsilon^2}$

$\sqrt{1+\epsilon^2} = 1 + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots$

$x = -\epsilon \pm \left(1 + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots\right) = \pm 1 - \epsilon \pm \frac{1}{2}\epsilon^2 \mp \frac{1}{8}\epsilon^4 + \cdots$

$O(1)$: $x = \pm 1$

$O(\epsilon)$: $x = \pm 1 - \epsilon$

$O(\epsilon^2)$: $x = \pm 1 - \epsilon \pm \frac{1}{2}\epsilon^2$

Approximations

- Taylor's theorem : our guess

$f(x) = f(0) + x f'(0) + \frac{1}{2!}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \cdots$

$x(\epsilon) = x(0) + \epsilon x'(0) + \frac{1}{2}\epsilon^2 x''(0) + \cdots$

$x(\epsilon) \sim x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \cdots$ → "well-ordering" assumption $0 < \alpha < \beta < \gamma \cdots$

$x(\epsilon) \sim \pm 1 + O(\epsilon)$

$x(\epsilon) \sim \pm 1 - \epsilon + O(\epsilon^2)$

$x(\epsilon) \sim \pm 1 - \epsilon \pm \frac{1}{2}\epsilon^2 + O(\epsilon^3)$

$\frac{dx}{d\epsilon} = -1 + \epsilon$

$x'(0) = \left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = -1 \equiv x_1$

$\frac{1}{2}x''(0) = \frac{1}{2}\frac{d^2x}{d\epsilon^2} = \frac{1}{2} \equiv x_2$

Regular Perturbation

$x^2 + 2\epsilon x - 1 = 0$ → $\underbrace{x_0^2 + 2\epsilon^\alpha x_0 x_1 + \cdots}_{x^2} + \underbrace{2\epsilon(x_0 + \epsilon^\alpha x_1 + \cdots)}_{2\epsilon x} \underbrace{- 1}_{-1} = 0$

x^2: $(x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \cdots)(x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \cdots)$

$x_0^2 + 2\epsilon^\alpha x_0 x_1 + 2\epsilon^\beta x_0 x_2 + \epsilon^{2\alpha} x_1^2 + \epsilon^{2\beta} x_2^2 + \cdots$

$2\epsilon x$: $2\epsilon(x_0 + \epsilon^\alpha x_1 + \cdots)$

$O(1)$: let $\epsilon \to 0$

$x_0^2 - 1 = 0$ → $\boxed{x_0 = \pm 1}$

$\pm 2\epsilon^\alpha x_1 + \cdots + 2\epsilon(\pm 1 + \epsilon^\alpha x_1 + \cdots) = 0$

$\boxed{\pm 2\epsilon^\alpha x_1} + \cdots \boxed{\pm 2\epsilon} + 2\epsilon^{1+\alpha} x_1 + \cdots \boxed{= 0}$ → $\pm 2\epsilon x_1 + \cdots \pm 2\epsilon + \boxed{2\epsilon^2 x_1} + \cdots = 0$

$\boxed{\alpha = 1}$

$O(\epsilon)$: $\pm 2\epsilon x_1 \pm 2\epsilon = 0$

$\boxed{x_1 = -1}$

$\pm 2\epsilon^\beta x_2 + \epsilon^2 + \cdots + 2\epsilon(-\epsilon + \epsilon^\beta x_2 + \cdots) = 0$

$\boxed{\pm 2\epsilon^\beta x_2} \boxed{+ \epsilon^2} + \cdots \boxed{-2\epsilon^2} + 2\epsilon^{2+\beta} x_2 + \cdots = 0$

$\boxed{\beta = 2}$

$O(\epsilon^2)$: $\pm 2\epsilon^2 x_2 + \epsilon^2 - 2\epsilon^2 = 0$

$x_2 = \pm \frac{1}{2}$

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