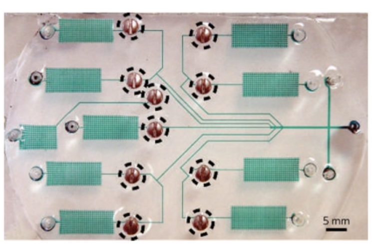
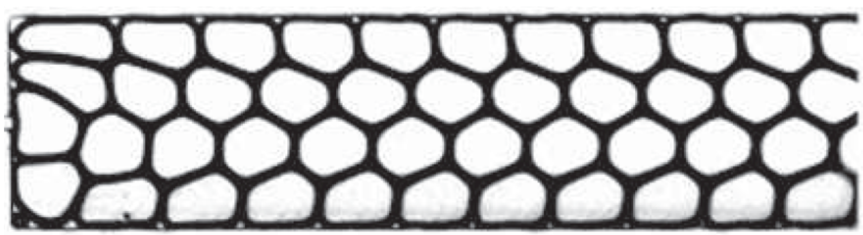
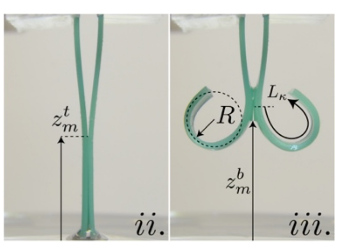
Confined Fluid Flow: Microfluidics and Capillarity

Douglas P. Holmes

Mechanical Engineering
Boston University





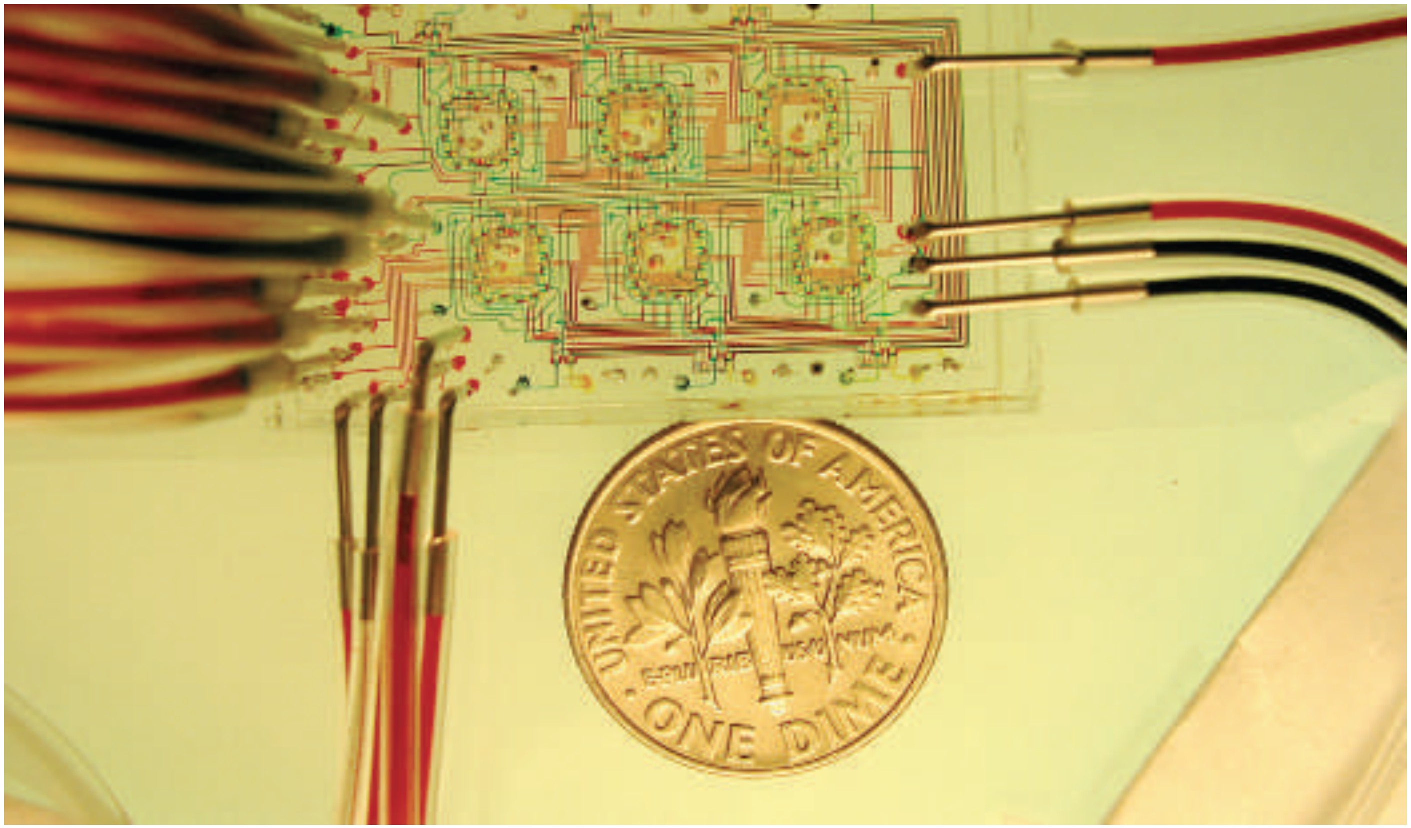


Sapienza, Università di Roma – Short Course (2015)

Why Small?

Microfluidic chemostat

…static chemical environment used as bioreactor…
fresh medium is continuously added, culture liquid is continuously removed



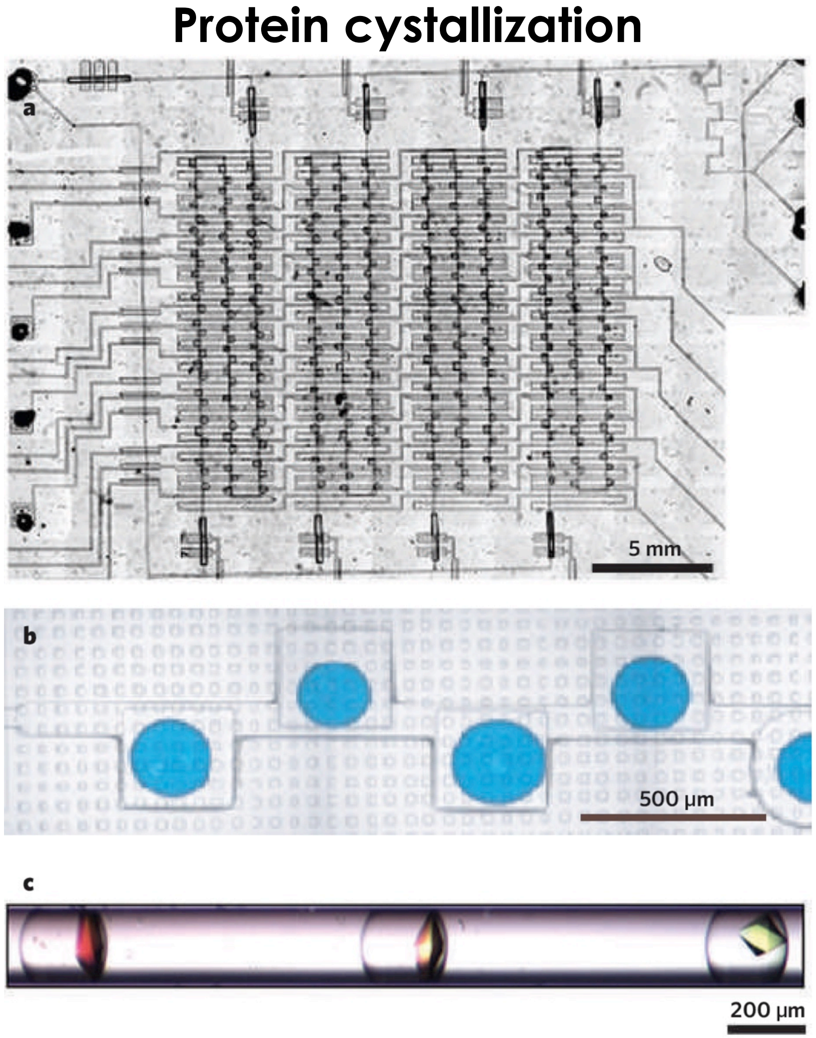
Device provided single-cell resolution to study microbial population growth.

Balagaddé, Frederick K., Lingchong You, Carl L. Hansen, Frances H. Arnold, and Stephen R. Quake. "Long-term monitoring of bacteria undergoing programmed population control in a microchemostat." Science 309, no. 5731 (2005): 137-140.

Whitesides, George M. "The origins and the future of microfluidics." Nature 442, no. 7101 (2006): 368-373.

Why Small?

Protein cystallization



Droplets containing proteins are trapped in microchannel wells.

Dyed droplets in the wells of the device.

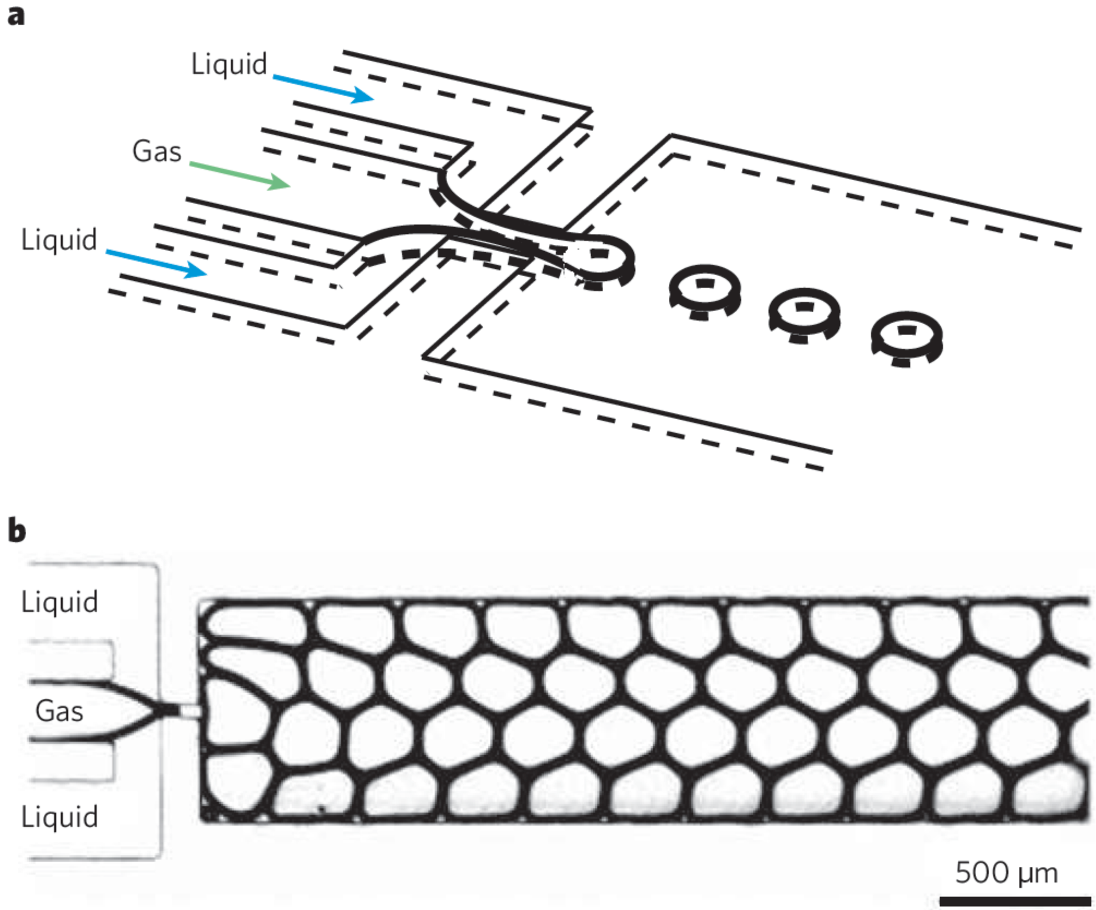
Proteins crystallized within droplets in a glass capillary.

Device used to efficiently screen for optimal protein crystallization conditions

Zheng, Bo, Joshua D. Tice, L. Spencer Roach, and Rustem F. Ismagilov. "A Droplet-Based, Composite PDMS/Glass Capillary Microfluidic System for Evaluating Protein Crystallization Conditions by Microbatch and Vapor-Diffusion Methods with On-Chip X-Ray Diffraction." Angewandte chemie international edition 43, no. 19 (2004): 2508-2511.
Whitesides, George M. "The origins and the future of microfluidics." Nature 442, no. 7101 (2006): 368-373.

Why Small?

Bubbles and foams

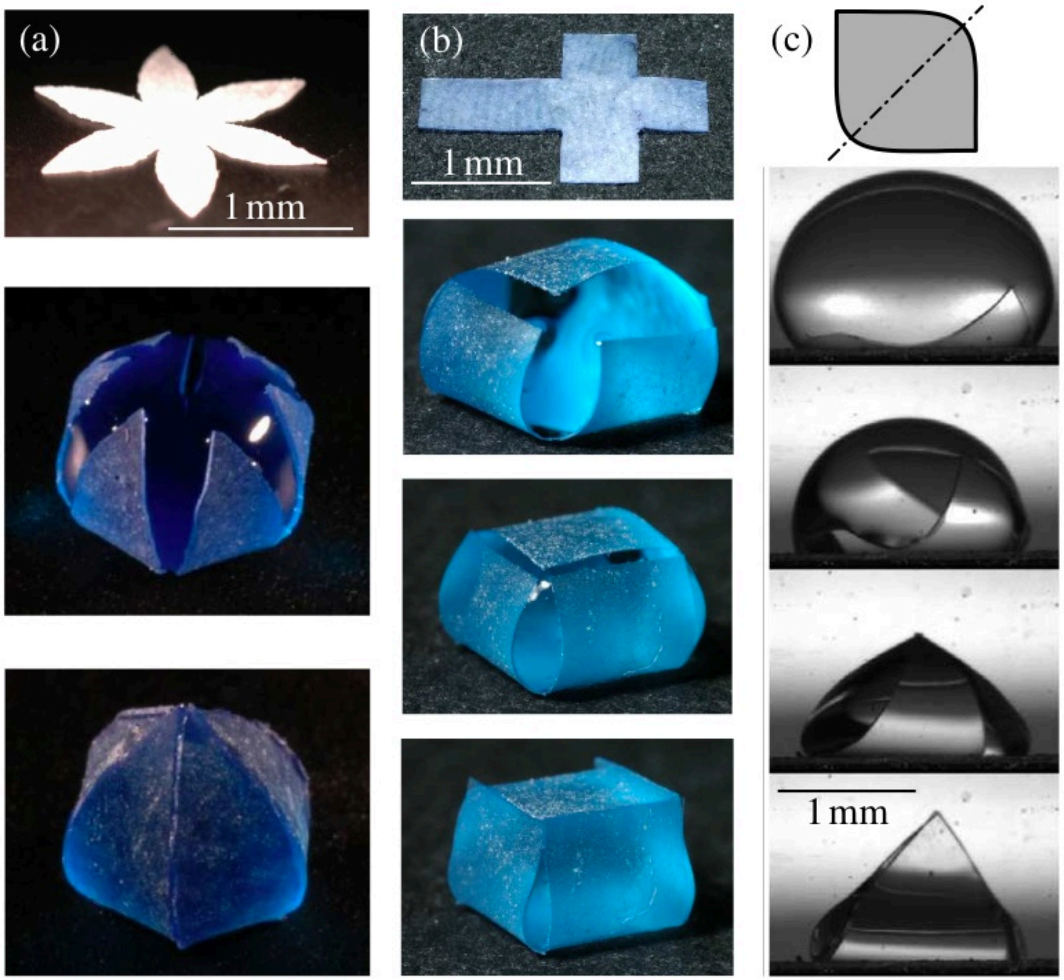


Bubble generation with tunable, monodisperse sizes

Garstecki, Piotr, Irina Gitlin, Willow DiLuzio, George M. Whitesides, Eugenia Kumacheva, and Howard A. Stone. "Formation of monodisperse bubbles in a microfluidic flow-focusing device." Applied Physics Letters 85, no. 13 (2004): 2649-2651.
Whitesides, George M. "The origins and the future of microfluidics." Nature 442, no. 7101 (2006): 368-373.

Why Small?

Capillary origami

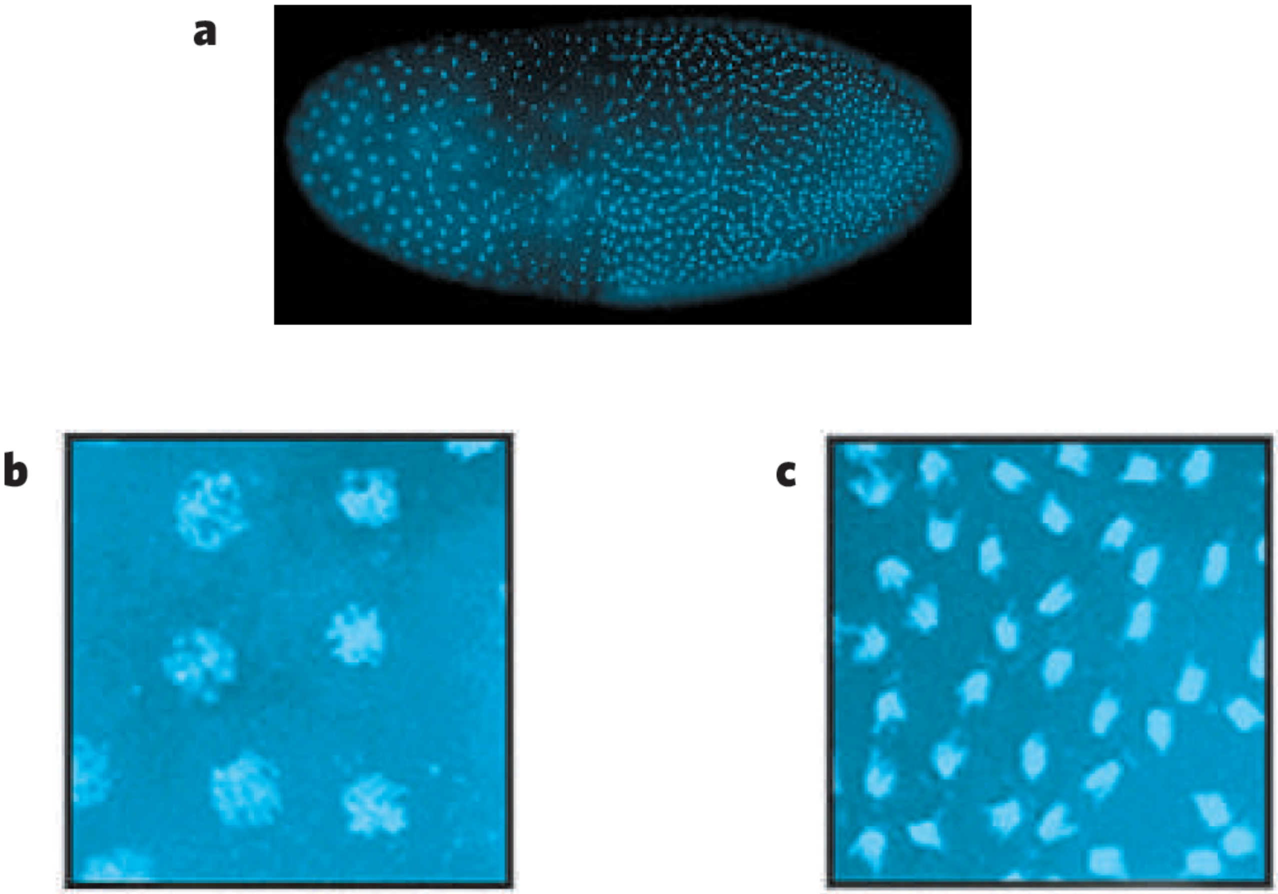


Using surface tension to deform thin structures

Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

Why Small?

Cellular and developmental biology



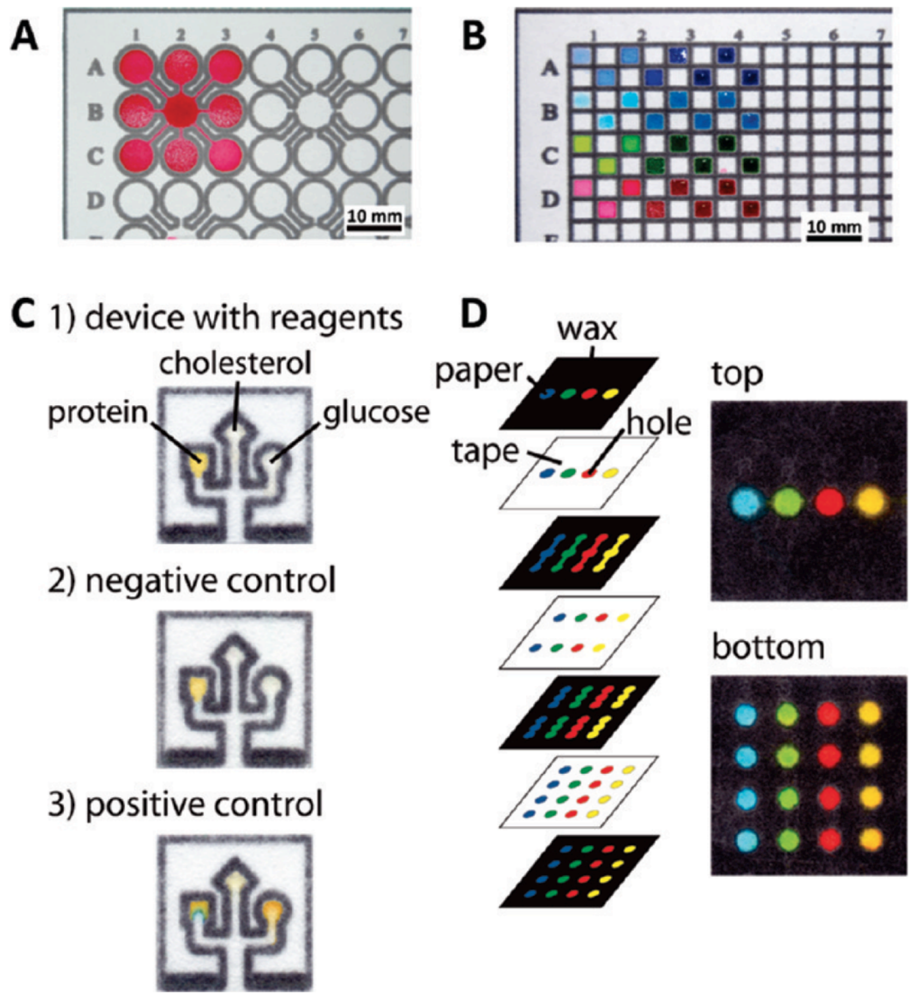
Study effect of temperature on the development of a fruitfly embryo.
Immobilized embryo has water with cold (left) and warm (right) flow over it.

Lucchetta, Elena M., Ji Hwan Lee, Lydia A. Fu, Nipam H. Patel, and Rustem F. Ismagilov. "Dynamics of Drosophila embryonic patterning network perturbed in space and time using microfluidics." Nature 434, no. 7037 (2005): 1134-1138.

Whitesides, George M. "The origins and the future of microfluidics." Nature 442, no. 7101 (2006): 368-373.

Why Small?

Inexpensive diagnostics

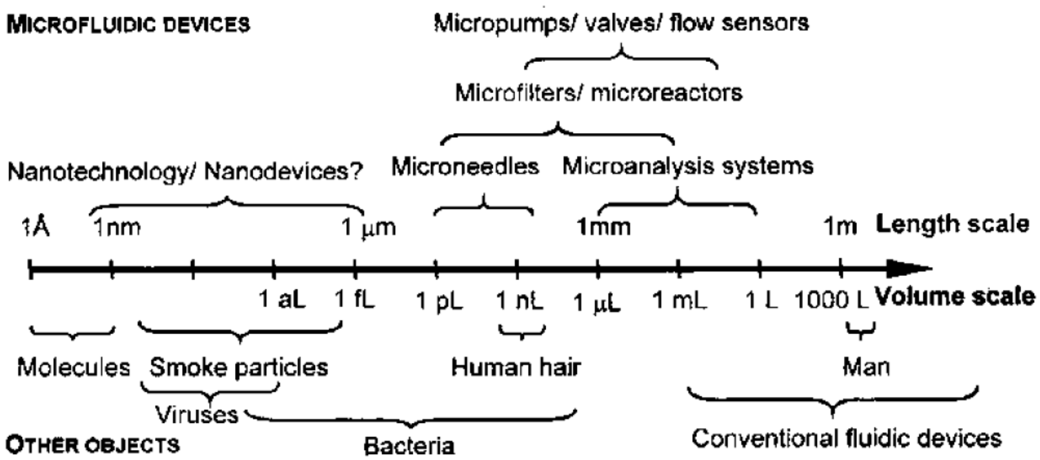


Costs as low as $0.001 per device (€0.00089)

Low-cost, simple paper-based microfluidics for diagnostics

Carrilho, Emanuel, Andres W. Martinez, and George M. Whitesides. "Understanding wax printing: a simple micropatterning process for paper-based microfluidics." Analytical chemistry 81, no. 16 (2009): 7091-7095.

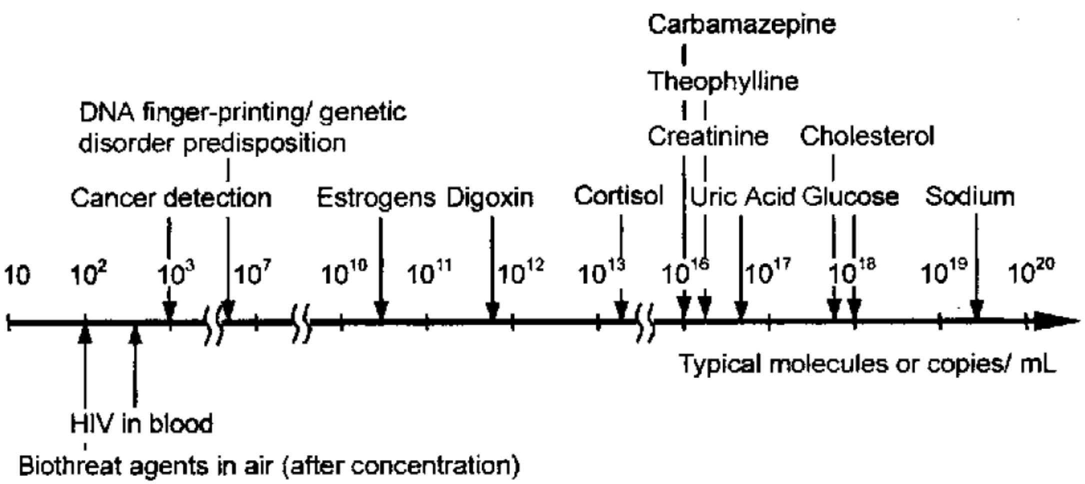
Why Small?



Volume goes as **L³**

- Small decrease in size = large reduction of sample volume.

How much sample volume do we need?



Detection of biomolecules:

Digoxin – heart stimulating drug

Cortisol – stress hormone from adrenal gland

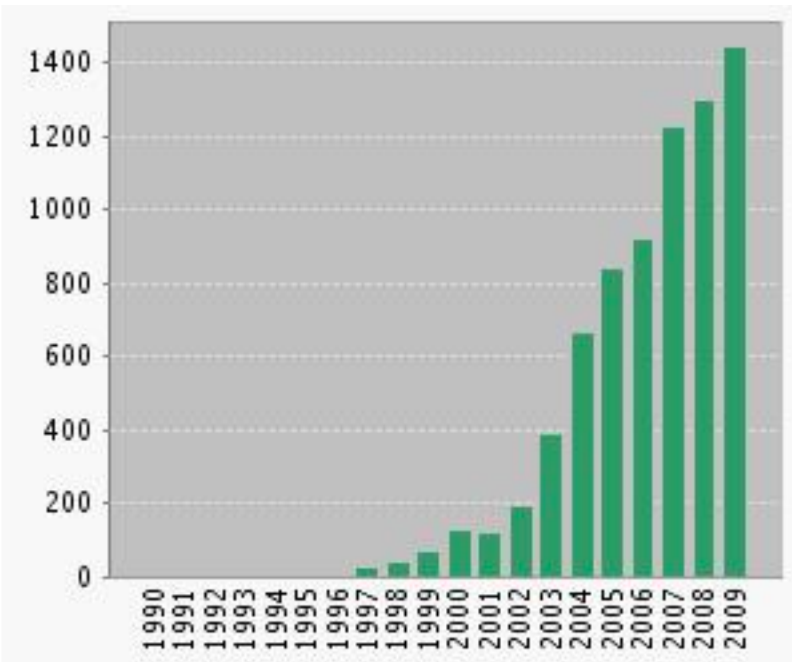
Creatinine – level in blood is measure of kidney function

Theophylline – drug used to treat respiratory diseases, e.g. asthma

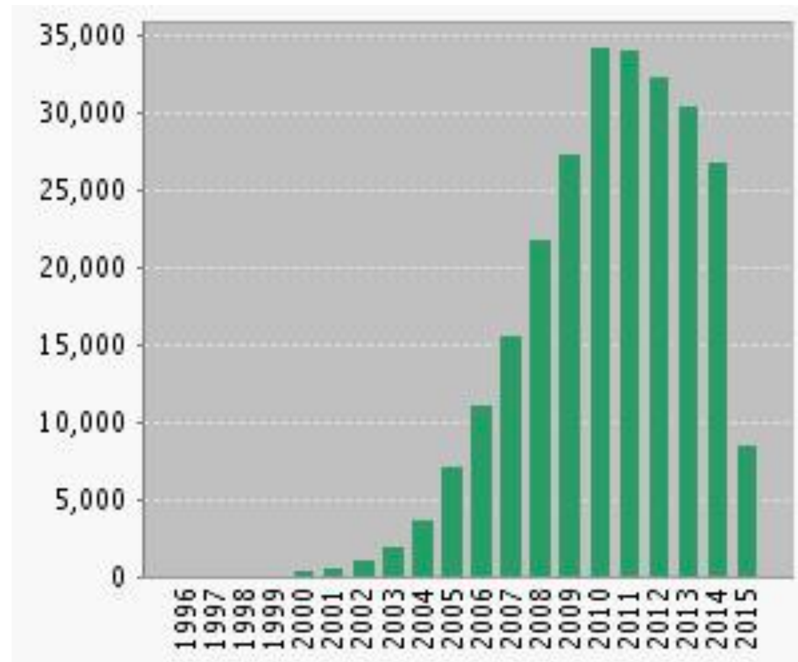
http://homes.nano.aau.dk/lg/Lab-on-Chip2008_files/Lab-on-chip2008_1.pdf

Microfluidics

Published items in each year…



Citations in each year…



ISI Web of Knowledge: Citation Report "Topic: Microfluidics", 1990-2009.

Confined Fluid Flow: Microfluidics and Capillarity

Reynolds Number: Inertia vs. Viscous effects

- Review of characteristic flows…

Péclet Number: Transport phenomena in a continuum

- Diffusion, separation, and mixing…

Geometric confinement: Controlling and manipulating fluid flow

- Microfluidic fabrication, valving, pumping…

Capillary Number: Viscosity vs. Surface tension

- Droplet formation, capillary rise, elasticity…

Fluid Dynamics

Navier-Stokes Equations (momentum conservation)

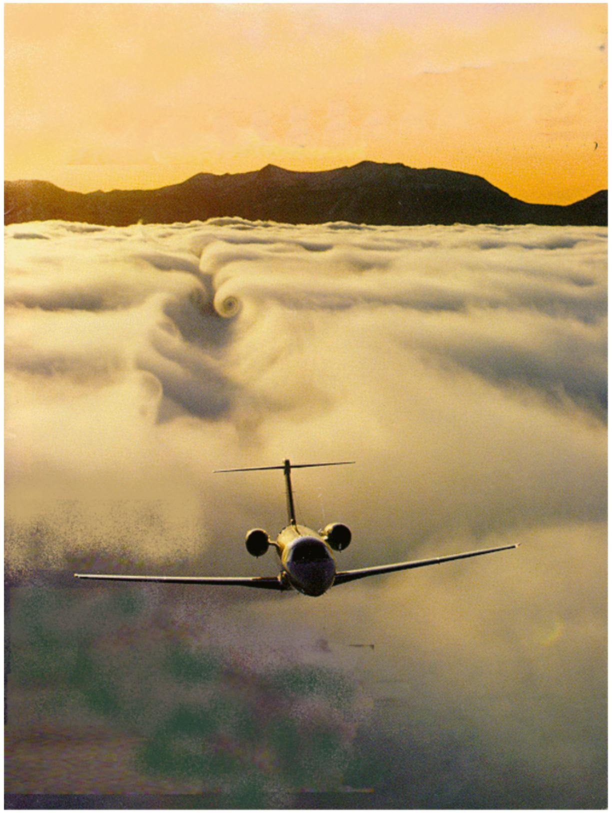
$$\overbrace{\rho\left(\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{variation}} + \underbrace{\mathbf{u}\cdot\nabla\mathbf{u}}_{\text{convection}}\right)}^{\text{Inertial acceleration}} = \overbrace{\underbrace{-\nabla p}_{\text{pressure}} + \underbrace{\mu\nabla^2\mathbf{u}}_{\text{diffusion}} + \underbrace{\mathbf{f}}_{\text{body forces}}}^{\text{Forces}}$$

Continuity Equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

Fluid Behavior

Inertial Forces



Trailing airplane vortices

Viscous Forces

Coiling honey

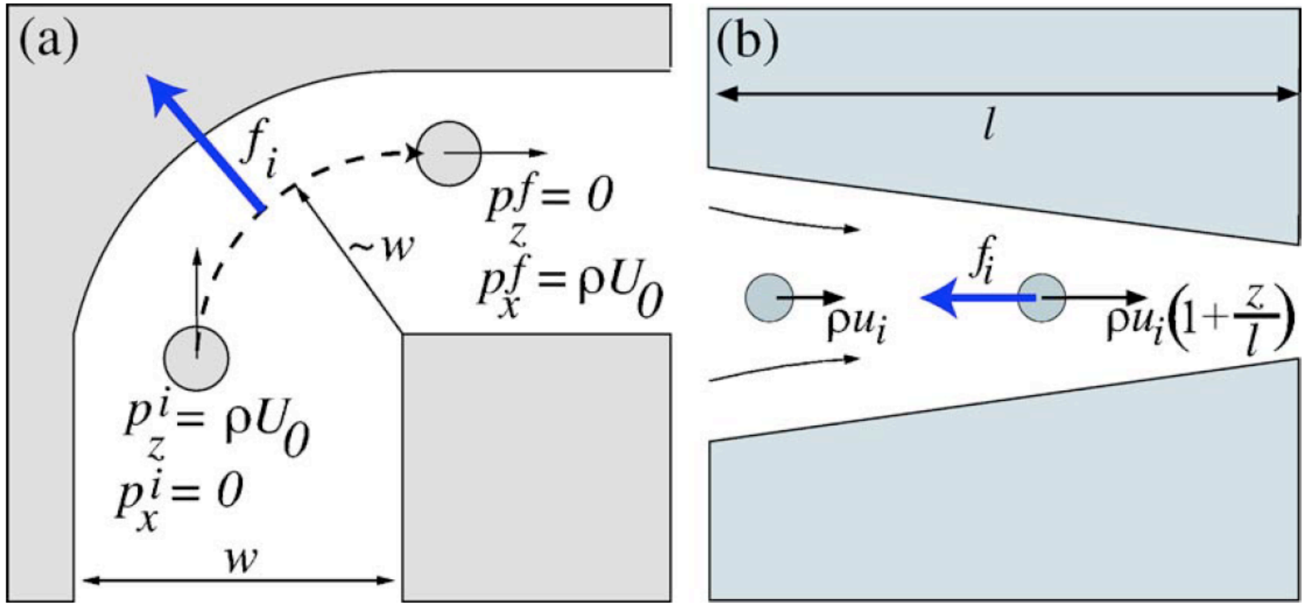
Reynolds Number: inertial/viscous

Fluid Behavior

Inertial Forces

Viscous Forces

Reynolds Number: inertial/viscous



Fluid element accelerating around curve.

- During a turn time:
  $\tau_0 \sim w/U_0$
- Loss of momentum density:
  ρU_0
- By exerting an inertial centrifugal force density:

$$f_i \sim \rho U_0/\tau_0 = \rho U_0^2/w$$

Fluid element in a channel of contracting length.

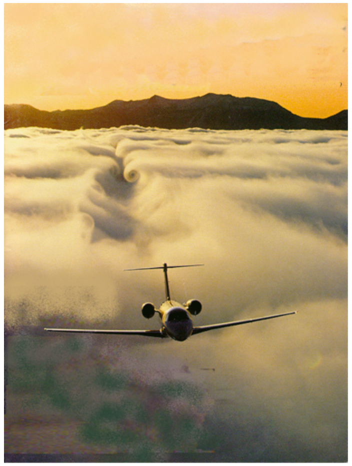
- By mass conservation, velocity increases as:
  $u \sim U_0(1 + z/l)$
- Gain momentum at a rate:

$$f_i \sim \rho \frac{\mathrm{d}u}{\mathrm{d}t} = \rho U_0 \frac{\mathrm{d}u}{\mathrm{d}z} \sim \frac{\rho U_0^2}{l}$$

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
Airplane: http://eis.bris.ac.uk/~glhmm/gfd/Airplane-ChrisWillcox.jpg
Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg

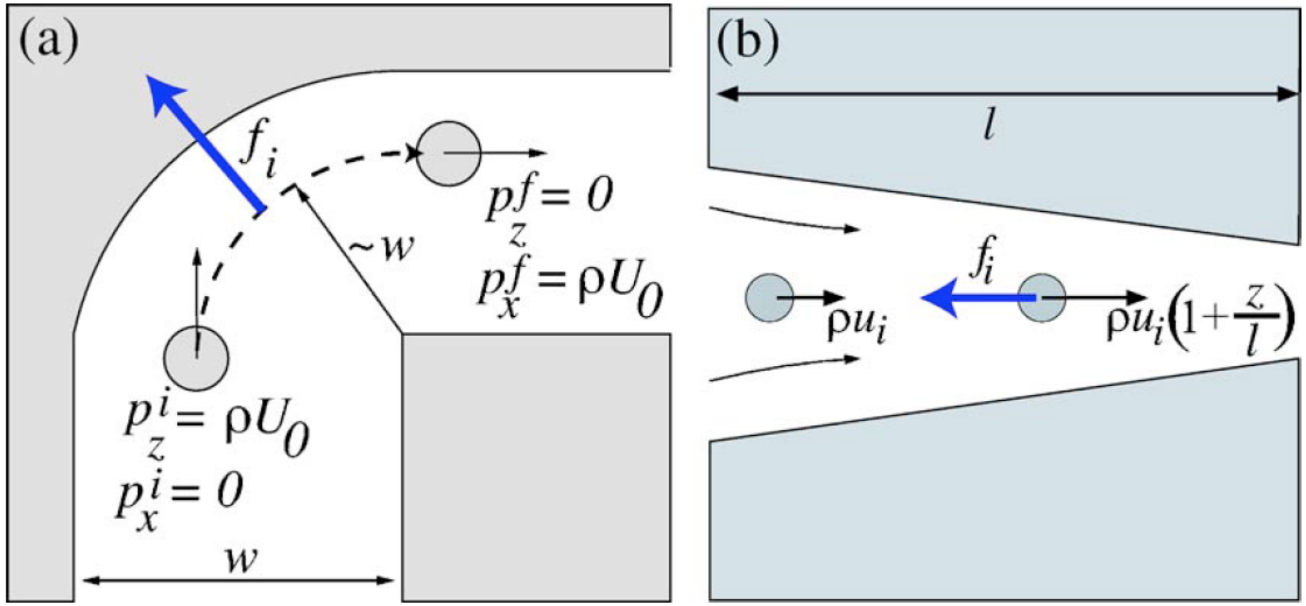
Fluid Behavior

Inertial Forces



Viscous Forces

Reynolds Number: inertial/viscous



Inertial Forces $f_i \sim \rho U_0/\tau_0 = \rho U_0^2/w$ $\qquad f_i \sim \rho \frac{\mathrm{d}u}{\mathrm{d}t} = \rho U_0 \frac{\mathrm{d}u}{\mathrm{d}z} \sim \frac{\rho U_0^2}{l}$

Viscous Forces

- Viscous force densities result from gradients in viscous stress: $f_v \sim \mu U_0 / L_0^2$

Ratio of these two force densities is the **Reynolds number**:

$$\frac{f_i}{f_v} = \frac{\rho U_0 L_0}{\mu} \equiv \mathscr{R}$$

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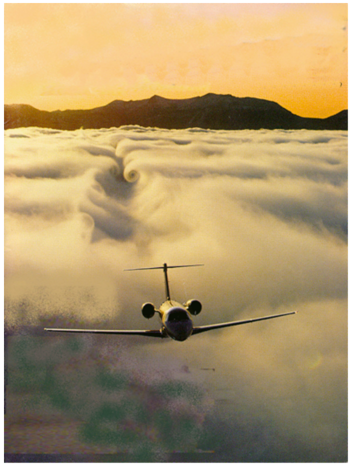
Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg

Fluid Behavior

Inertial Forces



Viscous Forces

Reynolds Number: inertial/viscous $\frac{f_i}{f_v} = \frac{\rho U_0 L_0}{\mu} \equiv \mathscr{R}$

Estimation of Reynolds numbers for common microfluidic devices.

- Typical fluid – water
  - Viscosity: 1.025 cP @ 25°C
  - Density: 1 g/mL
- Typical channel dimensions
  - Radius/height (smaller than width): 1 – 100 μm
- Typical velocities
  - Average velocity: 1 μm/s – 1 cm/s

Typical Reynolds number:

$$\mathscr{R} \sim \mathcal{O}(10^{-6}) - \mathcal{O}(10^{1})$$

Low Reynolds number: viscous forces > inertial forces

- Flows are **linear.**
- Nonlinear terms in Navier-Stokes disappear
  - Linear, predictable **Stokes flow**

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
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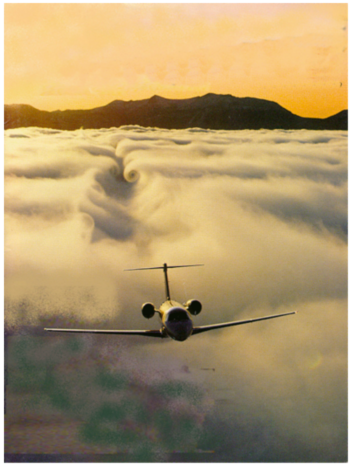
Fluid Behavior

Inertial Forces



Viscous Forces

Reynolds Number: inertial/viscous

$$\overbrace{\rho\left(\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{variation}} + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\text{Inertial acceleration}} = \overbrace{-\nabla p + \underbrace{\mu\nabla^2\mathbf{u}}_{\text{diffusion}} + \mathbf{f}}^{\text{Forces}}$$

Reynolds number is focused on **steady inertial forcing** due to the convective derivative:

- Almost always unimportant in microfluidic flows.
- Time dependence arises from U_0

$$\rho\mathbf{u}\cdot\nabla\mathbf{u}$$

Linear unsteady term sets the inertial time scale to **establish steady flows**:

$$\rho\partial\mathbf{u}/\partial t$$

Time scale estimated by balancing unsteady **inertial force density** with **viscous force density**

$$f_u \sim \rho U_0/\tau_i \qquad f_v \sim \mu U_0/L_0^2$$

$$\tau_i \sim \frac{\rho L_0^2}{\mu}$$

Time required for a vorticity to diffuse a distance L_0, with a diffusivity $\nu=\mu/\rho$, $\tau_i \sim 10$ms

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
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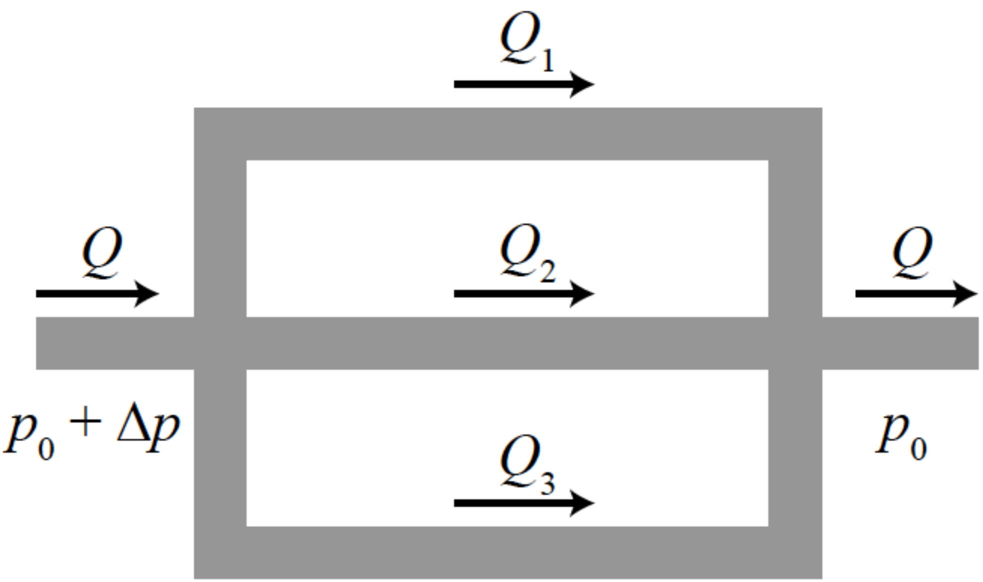
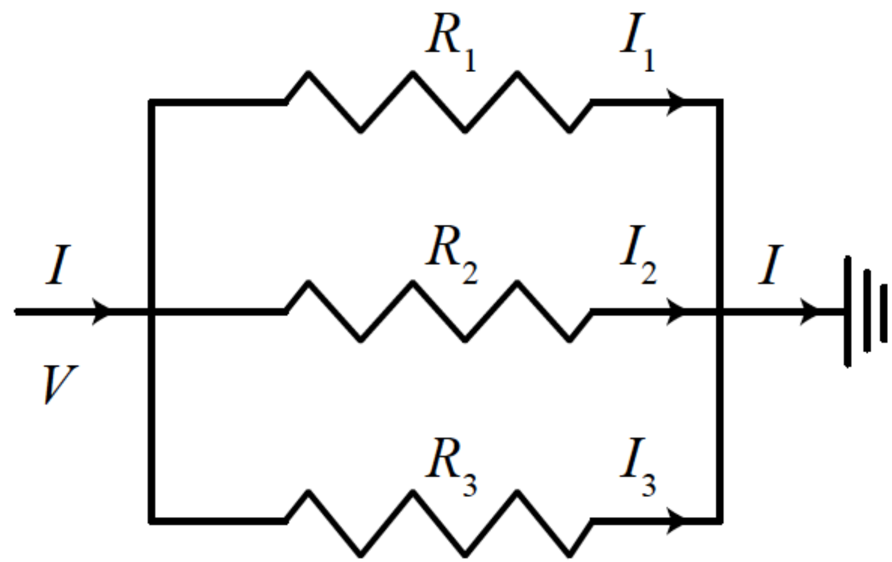
Fluids and Circuits





Pressure-driven flow in a network of parallel channels.

- Flow rates Q_i in three parallel channels.
- Total flow rate: $Q = Q_1 + Q_2 + Q_3$
- Pressure drop and flow rate related by:

$$\Delta p = Q R_H$$

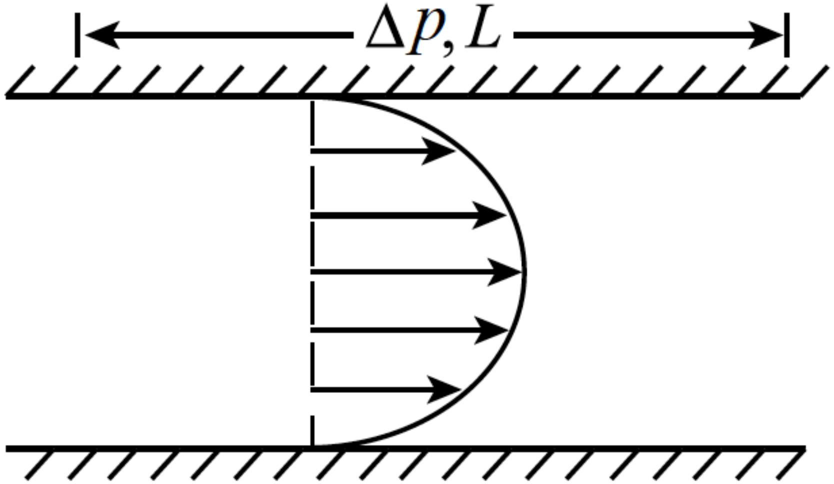
Electrical circuit with resistances in parallel.

- First elementary rule of circuit design is Ohm's law
- Relates electrical potential to current:

$$\Delta V = I R$$

Stone, Howard A. "Introduction to fluid dynamics for microfluidic flows." In CMOS Biotechnology, pp. 5-30. Springer US, 2007.

Flow: Cylindrical Tube



Poiseuille flow

Assumptions:

1. Neglect gravity: $g_z = 0$
2. Steady state: $\frac{\partial(\cdot)}{\partial t} = 0$
3. Axisymmetric: $\frac{\partial(\cdot)}{\partial \theta} = 0$

Continuity Equation (cylindrical coords):

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial(ru_\theta)}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Flow only in z direction: $\frac{\partial u_z}{\partial z} = 0$

Navier-Stokes equations:

$$\overbrace{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\text{Inertial acceleration}} = \overbrace{-\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}}^{\text{Forces}}$$

Reduces to the following:

$$0 = -\frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)$$

Pressure driven flow – linear change

$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L} = \frac{\Delta p}{L}$$

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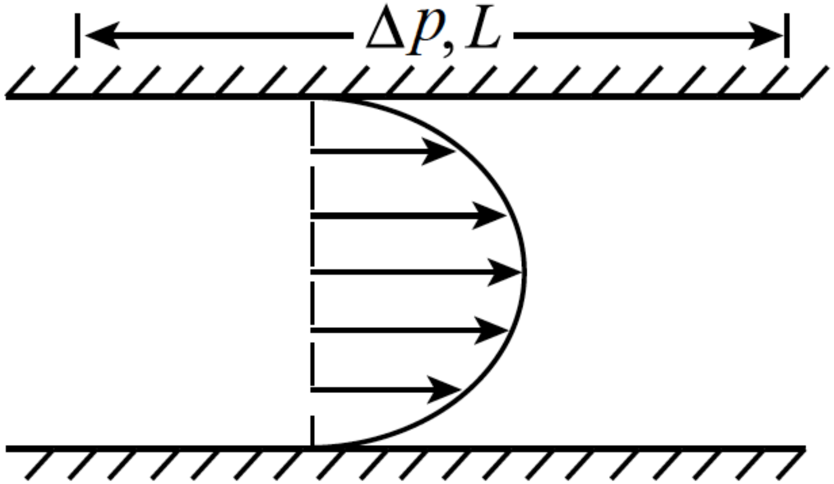
Flow: Cylindrical Tube



Poiseuille flow

Boundary conditions:

1. Velocity is at a maximum at the center of the piper:

$$\frac{\mathrm{d}u_z}{\mathrm{d}r} = 0 \quad @ \quad r = 0$$

2. No-slip along the walls.

$$u_z = 0 \quad @ \quad r = R$$

Navier-Stokes equations - cylindrical pipe:

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)$$

$$\int_0^r \mathrm{d}\left(r\frac{\mathrm{d}u_z}{\mathrm{d}r}\right) = \frac{1}{\mu}\frac{\Delta p}{L}\int_0^r r\ \mathrm{d}r$$

...integrate once...

$$r\frac{\mathrm{d}u_z}{\mathrm{d}r} = \frac{1}{\mu}\frac{\Delta p}{L}\frac{r^2}{2} + \mathrm{C}_1$$

$$\int_0^r \mathrm{d}u_z = \int_0^r \left(\frac{1}{\mu}\frac{\Delta p}{L}\frac{r}{2} + \frac{\mathrm{C}_1}{r}\right)\mathrm{d}r$$

...integrate twice...

$$u_z = \frac{1}{\mu}\frac{\Delta p}{L}\frac{r^2}{4} + \mathrm{C}_1 \ln(r) + \mathrm{C}_2$$

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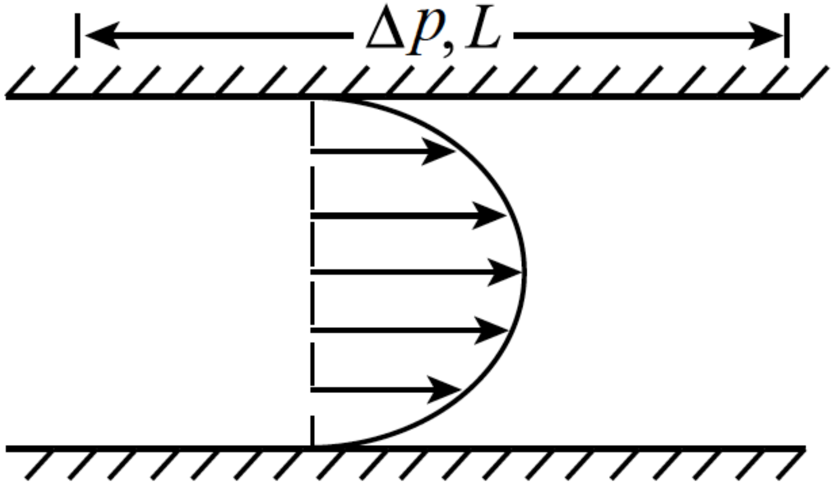
Flow: Cylindrical Tube



Poiseuille flow

Parabolic velocity distribution:

$$u(r) = \frac{1}{4\mu} \underbrace{\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|}_{\text{axial pressure gradient}} \left(a^2 - r^2\right)$$

Flow rate (volume/time)

$$Q = 2\pi \int_0^a u(r)\; r\; \mathrm{d}r = \frac{\pi a^4}{8\mu} \left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|$$

The flow rate depends on the ***fourth*** power of the radius: important in small systems

- *e.g.* For the same pressure gradient, reducing the radius by a factor of **two** causes a **16-fold** reduction in flow rate.
- *e.g.* Consider a blood vessel – a **10% decrease in radius** produces more than a **40% decrease in the flow rate** of blood (PSA: Eat more kale!)

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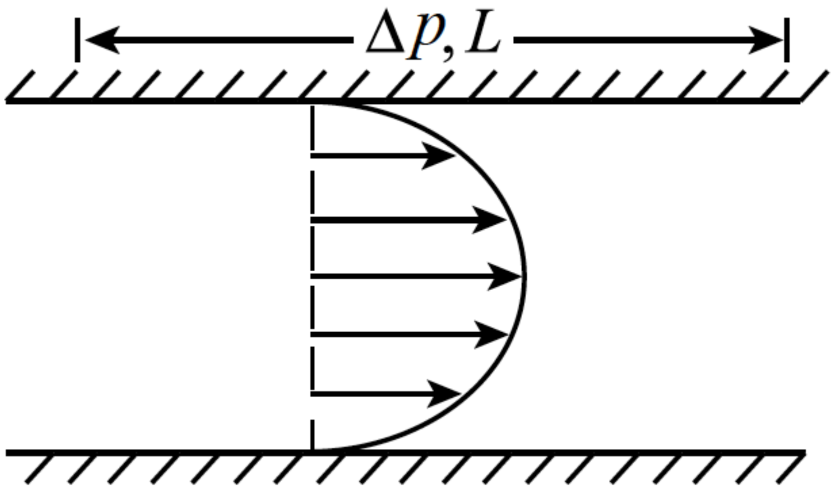
Flow: Cylindrical Tube



Poiseuille flow

Average velocity

$$U = \frac{Q}{\pi a^2} = \frac{a^2}{8\mu}\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|$$

Recall the Reynolds number:

$$\mathscr{R} = \frac{\rho U a}{\mu}$$

For a fixed pressure gradient, $\mathscr{R} \sim a^3$ and so a **factor** of **two** change in **radius** produces a **factor** of **eight** change in **Reynolds number**.

The average velocity is proportional to the pressure gradient, writing the flow field in vector form gives Darcy's law:

$$\frac{\mu U}{k} = -\nabla p$$

- *k* is the **permeability** (dimensions length^2)
- linear relation between pressure and velocity is due to **lack of inertia** effects
- order-of-magnitude of permeability is typically the **square of smallest dimension**.

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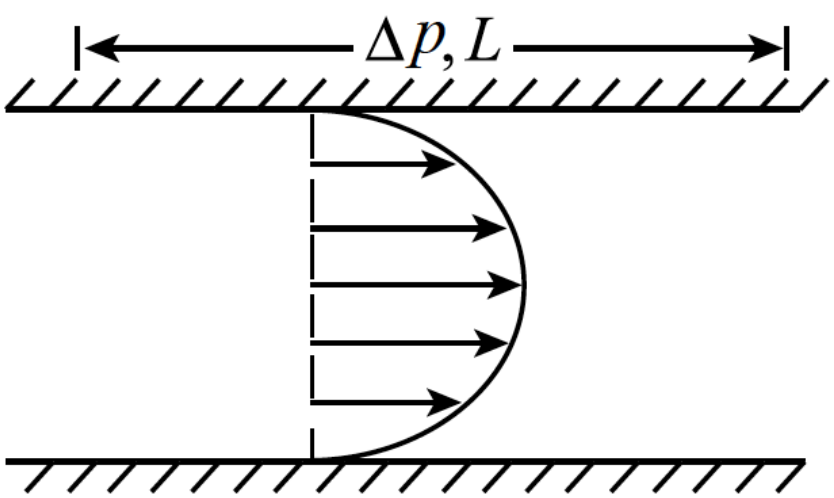
Flow: Cylindrical Tube



Poiseuille flow

Hydrodynamic resistance

$$R_H = \frac{\Delta p}{Q} = \frac{8\mu L}{\pi a^4}$$

recall: $-\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\Delta p}{L}$

Electrical analogy: resistance in fluid channels depends on the **fourth** power of the radius rather than the **second** power in the electrical case.
…………………small matters!

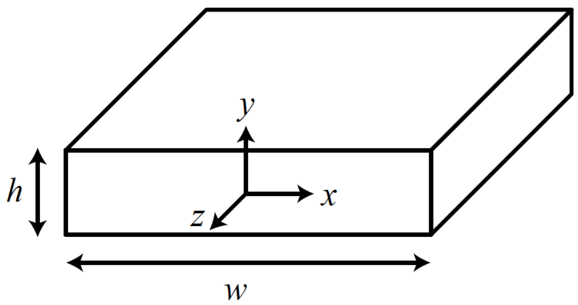
Scaling on physical & dimensional arguments…

- Can arrive at: $u \sim a^2 \Delta p/(\mu L)$ and $Q \sim ua^2 \sim a^4 \Delta p/(\mu L)$ from dimensional analysis.

- Fluid motion arises as a balance between the pressure drop driving motion and the frictional (viscous) resistance from the bounding walls. This balance applies **independent** of the **cross-sectional channel shape.**

Stone, Howard A. "Introduction to fluid dynamics for microfluidic flows." In CMOS Biotechnology, pp. 5-30. Springer US, 2007.

Flow: Rectangular



Average velocity: $k = \frac{h^2}{12}$

Hydrodynamic Resistance: $R_H = \frac{12\mu L}{wh^3}$

Parabolic velocity distribution:

$$u(y) = \frac{\Delta p}{2\mu L}\left[\left(\frac{h}{2}\right)^2 - y^2\right]$$

Approximate flow rate (channel of width w)

$$Q = w\int_0^h u(y)\mathrm{d}y = \frac{wh^3\Delta p}{12\mu L}$$

Parabolic velocity distribution (known analytically in terms of a Fourier series):

$$u(x,y) = \frac{\Delta p}{2\mu L}\left[\left(\left[\frac{h}{2}\right]^2 - y^2\right) - \sum_{n=0}^{\infty} a_n \cos\left(\frac{\lambda_n y}{h/2}\right)\cosh\left(\frac{\lambda_n x}{h/2}\right)\right]$$

where: $\lambda_n = \frac{(2n+1)\pi}{2}$ and, from the no-slip boundary conditions: $a_n = \frac{h^2(-1)^n}{\lambda_n^3 \cosh(\lambda_n w/h)}$

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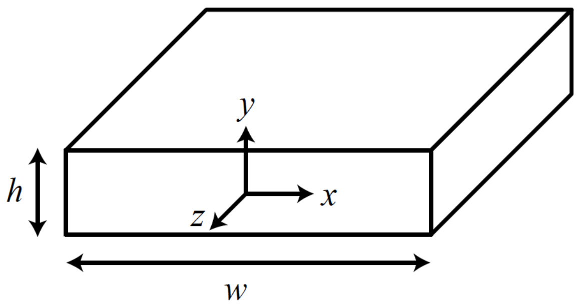
Flow: Rectangular



Parabolic velocity distribution:

$$u(y) = \frac{\Delta p}{2\mu L}\left[\left(\frac{h}{2}\right)^2 - y^2\right]$$

Average velocity: $k = \frac{h^2}{12}$

Approximate flow rate (channel of width w)

$$Q = w\int_0^h u(y)\mathrm{d}y = \frac{wh^3\Delta p}{12\mu L}$$

Hydrodynamic Resistance: $R_H = \frac{12\mu L}{wh^3}$

Corresponding flow rate

$$Q = 4\int_0^{w/2}\int_0^{h/2} u(x,y)\mathrm{d}y\mathrm{d}x$$

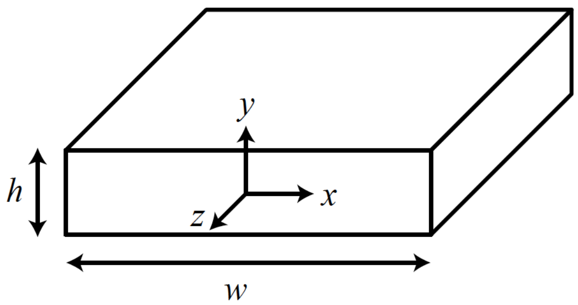
$$Q = \frac{wh^3\Delta p}{12\mu L}\left[1 - 6\left(\frac{h}{w}\right)\sum_{n=0}^{\infty}\lambda_n^{-5}\tanh\left(\frac{\lambda_n w}{h}\right)\right]$$

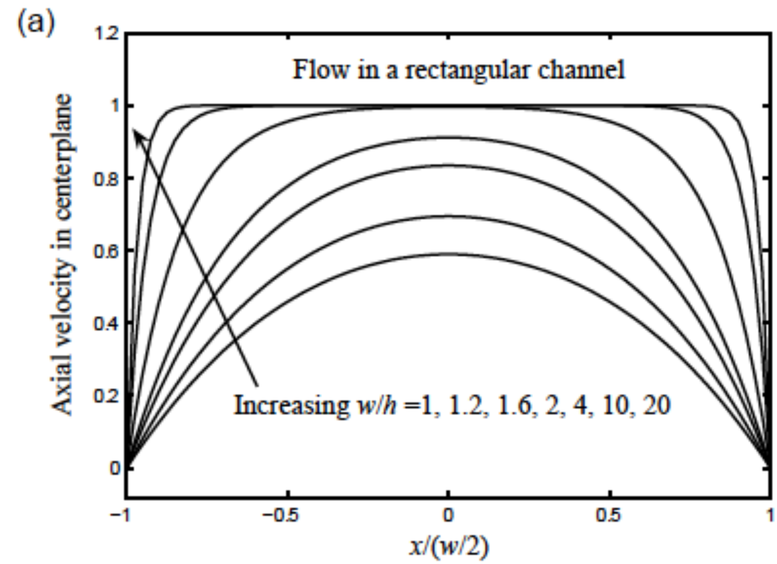
where: $\lambda_n = \frac{(2n+1)\pi}{2}$

Flow rate is nearly linear in aspect ratio h/w

Stone, Howard A. "Introduction to fluid dynamics for microfluidic flows." In CMOS Biotechnology, pp. 5-30. Springer US, 2007.

Flow: Rectangular



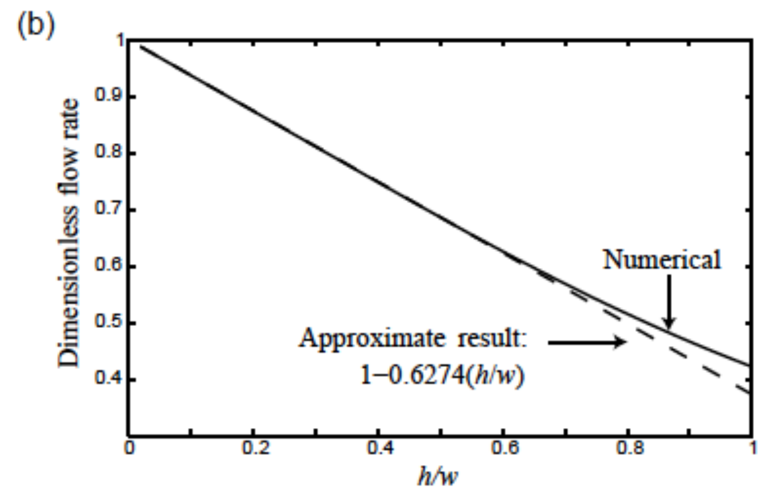


Flow rate in a rectangular channel

$$Q = \frac{wh^3 \Delta p}{12\mu L}\left[1 - 6\left(\frac{h}{w}\right)\sum_{n=0}^{\infty}\lambda_n^{-5}\tanh\left(\frac{\lambda_n w}{h}\right)\right]$$

Approximate dimensionless flow rate

$$\underbrace{\frac{12\mu L Q}{wh^3 \Delta p}}_{\text{dimensionless flow rate}} \approx 1 - \frac{6(2^5)}{\pi^5}\underbrace{\frac{h}{w}}_{\text{aspect ratio}}$$



Stone, Howard A. "Introduction to fluid dynamics for microfluidic flows." In CMOS Biotechnology, pp. 5-30. Springer US, 2007.

Workout Problem

While developing a microfluidic chip for your research, you decide a scale model would help you gain intuition about flow characteristics at the microscale. While most scale models are smaller that the original, your model will be 100 times larger (so that you can see it without a microscope).

	Viscosity	**Density**
Honey	8750 cP	1.42 g/mL
Olive oil	81 cP	0.92 g/mL
Water	1.025 cP @ 25°C	1 g/mL
Molasses	50,000 cP	1.50 g/mL

For the microfluidic chip, the flow conditions are: fluid velocity 1 mm/s, channel diameter 100 µm, and the liquid is water at T = 25°C. The Reynolds number for both systems is to be duplicated, and the microchannel is hemicylindrical (use the hydraulic diameter). For the scale model, which of the condiments listed above from your kitchen would you choose as the flow liquid, if the velocity in the model is:

a.) 1 mm/s
b.) 60 mm/s
c.) 340 mm/s

Recall: hydraulic diameter, D_H, relates flow in noncircular channels to those in a circular channel:

$$D_H = \frac{4A}{P}$$

http://www-bsac.eecs.berkeley.edu/projects/ee245/Homework/

Workout Problem

Hydraulic diameter of hemicylindrical channel:

$$D_H = \frac{4A}{P} = \frac{2\pi r^2}{\pi r + 2r} = 61.1\ \mu\text{m}$$

Reynold's number of microfluidic chip:

$$\mathscr{R} = \frac{\rho U D_H}{\mu} = \left(1000\ \text{kg/m}^3\right)\left(0.001\ \text{m/s}\right)\left(61.1\ \times 10^{-6}\text{m}\right)\left(1.025 \times 10^{-3}\ \text{Pa}\cdot\text{s}\right)^{-1}$$

$$\mathscr{R} = 0.0596$$

a. Fluid flowing at 1 mm/s in scaled up model

Use Olive Oil. $\mathscr{R} = \left(920\ \text{kg/m}^3\right)\left(0.001\ \text{m/s}\right)\underbrace{\left(6.11\ \times 10^{-3}\text{m}\right)}_{\text{100 times larger}}\left(81 \times 10^{-3}\ \text{Pa}\cdot\text{s}\right)^{-1} = 0.0694$

b. Fluid flowing at 60 mm/s in scaled up model

Use Honey. $\mathscr{R} = \left(1420\ \text{kg/m}^3\right)\underbrace{\left(0.06\ \text{m/s}\right)}_{\text{60 times larger}}\left(6.11\ \times 10^{-3}\text{m}\right)\left(8750 \times 10^{-3}\ \text{Pa}\cdot\text{s}\right)^{-1} = 0.0595$

c. Fluid flowing at 340 mm/s in scaled up model

Use Molasses. $\mathscr{R} = \left(1500\ \text{kg/m}^3\right)\underbrace{\left(0.34\ \text{m/s}\right)}_{\text{340 times larger}}\left(6.11\ \times 10^{-3}\text{m}\right)\left(50\ \text{Pa}\cdot\text{s}\right)^{-1} = 0.0620$

http://www-bsac.eecs.berkeley.edu/projects/ee245/Homework/

Confined Fluid Flow: Microfluidics and Capillarity

Reynolds Number: Inertia vs. Viscous effects

- Review of characteristic flows…

Péclet Number: Transport phenomena in a continuum

- Diffusion, separation, and mixing…

Geometric confinement: Controlling and manipulating fluid flow

- Microfluidic fabrication, valving, pumping…

Capillary Number: Viscosity vs. Surface tension

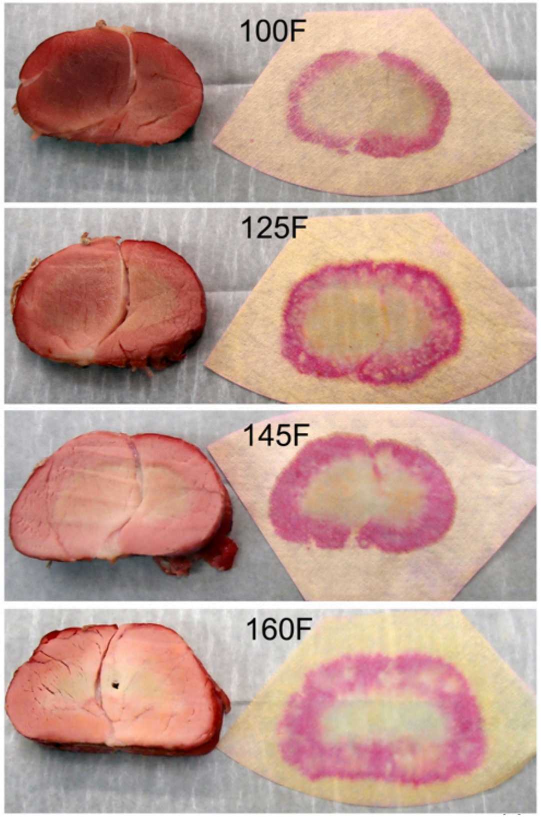
- Droplet formation, capillary rise, elasticity…

Fluid Behavior

Convection

Convection current as hot water mixes with cold water

Diffusion



Diffusion of salt ions while curing pork

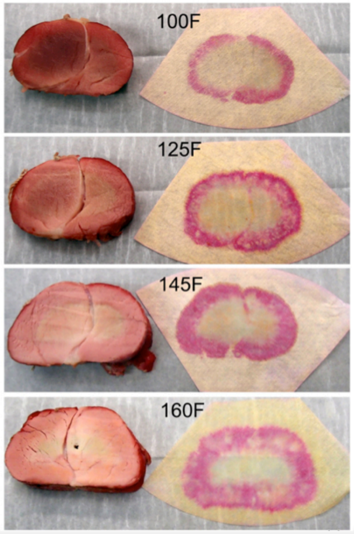
Péclet Number: convection/diffusion

http://quarkyscience.ca/project-of-the-month/convection-demonstration/
http://www.genuineideas.com/ArticlesIndex/diffusion.html

Fluid Behavior

Convection

Diffusion



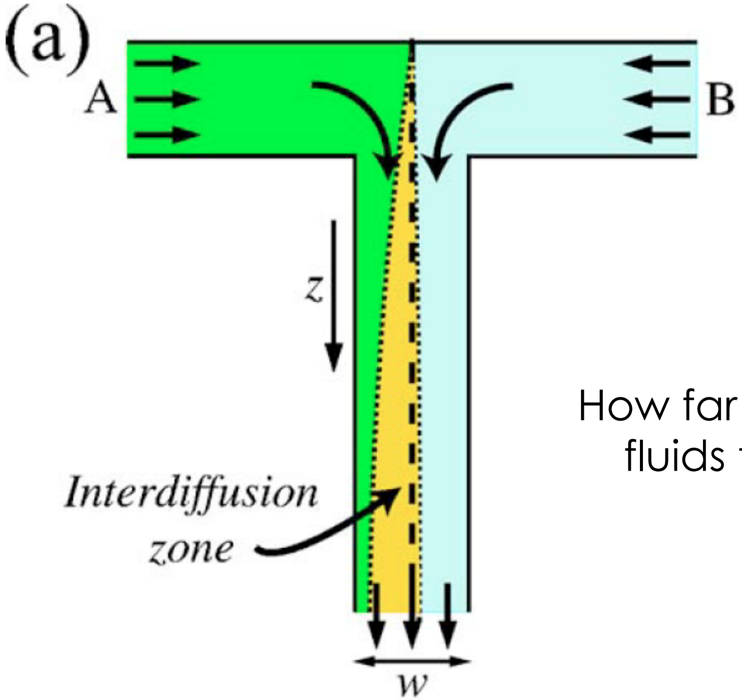
Péclet Number: convection/diffusion

High Reynolds: eddies chaotically stretch and fold fluid elements.

- Turbulent mixing & thermal convection move fluids.

Low Reynolds: mixing occurs by diffusion only

- Diffusion is remarkably slow – long mixing times.



T-Junction in which two fluids are injected and flow alongside each other

How far down the channel must the fluids flow before the channel is homogenized?

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
http://quarkyscience.ca/project-of-the-month/convection-demonstration/
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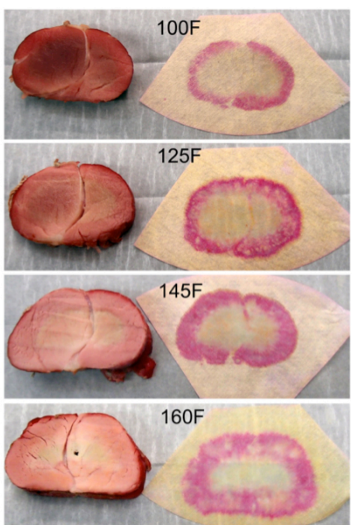
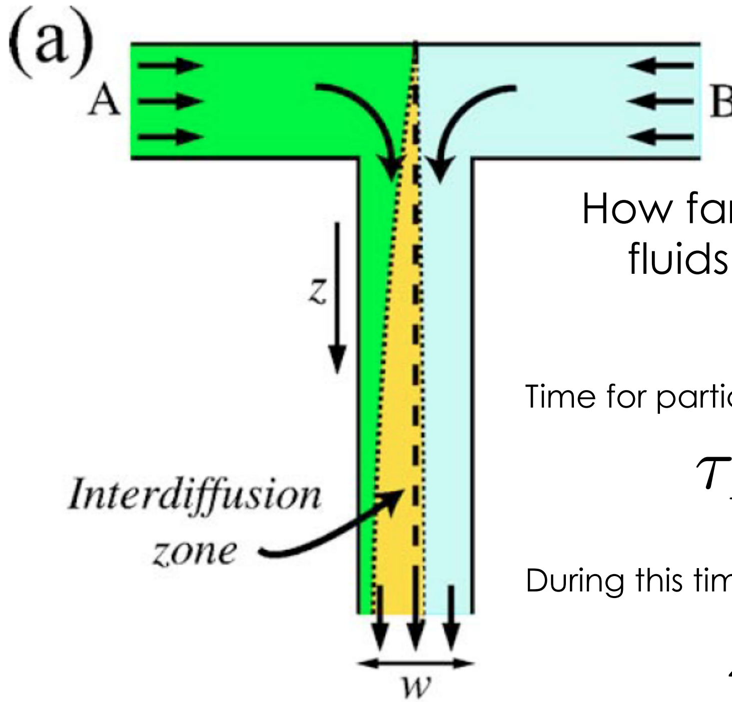
Fluid Behavior

Convection

Diffusion



Péclet Number: convection/diffusion



How far down the channel must the fluids flow before the channel is homogenized?

Time for particles/molecules to diffuse across channel:

$$\tau_D \sim w^2/D$$

During this time, the stripe moves down the channel:

$$Z \sim U_0 w^2/D$$

Number of channel widths required for complete mixing:

$$\frac{Z}{w} \sim \frac{U_0 w}{D} \equiv \mathscr{P}$$

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
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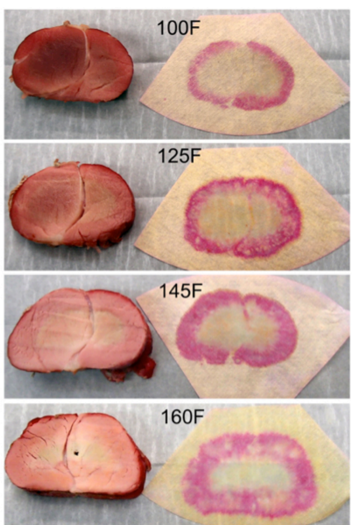
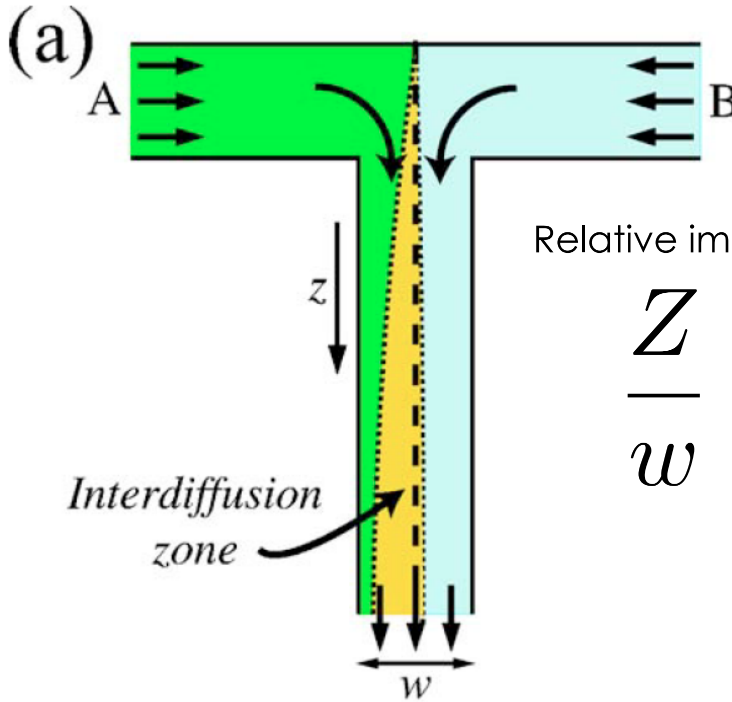
Convection

Diffusion



Péclet Number: convection/diffusion



Relative importance: **convection** to **diffusion**:

$$\frac{Z}{w} \sim \frac{U_0 w}{D} \equiv \mathscr{P}$$

Consider a **small protein** flowing with liquid:

- Typical size: 5 nm
- Diffusion constant: 40 μm²/s
- e.g. Channel width: 100 μm
- e.g. Flow velocity: 100 μm/s

$\mathscr{P} \sim 250$ channel widths

$L_0 \sim 2.5$ cm

$\tau_D \sim 4$ min

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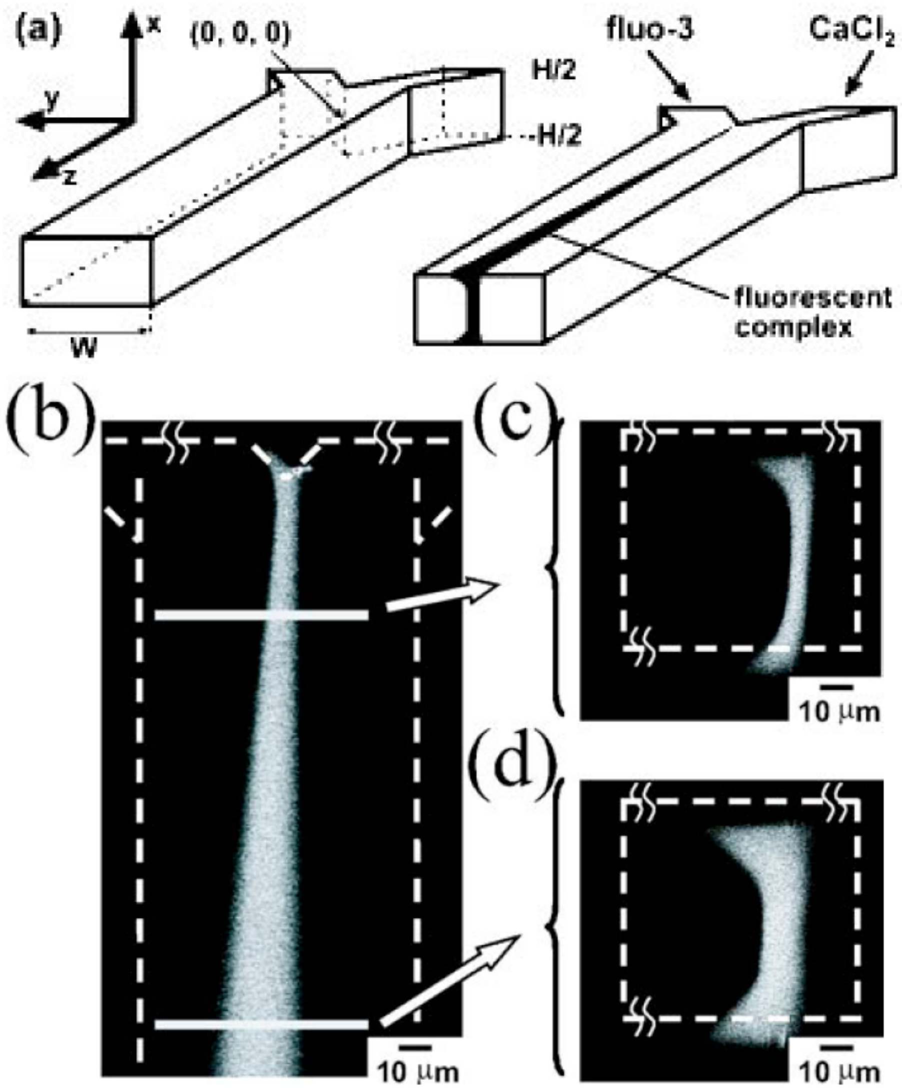
BOSTON UNIVERSITY

Sensing & Filtering



Parallel Laminar Flows

Pressure-driven laminar flow of two adjacent miscible streams.

- Solution on left contains **calcium**.
- Solution on right contains calcium-dependent **fluorophore**, Fluo-3.
- In water, Fluo-3 and calcium form a fluorescent complex at a **diffusion limited rate**.

Convective-diffusion equation:

$$\mathscr{P}\mathbf{u}\cdot\nabla c = \nabla^2 c \cong \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) c$$

For high Péclet number flows, $\mathscr{P} \gg z/H \gg 1$

- Diffusive broadening down the channel depends on both y & x

Away from boundaries, the interfacial region:

$$\underbrace{U_m}_{\text{Maximum velocity}} \frac{\partial c}{\partial z} = D\nabla^2 c \longrightarrow \delta_y(z) \sim \left(\frac{Dz}{\underbrace{U_a}_{\text{Average velocity}}}\right)^{1/2}$$

Ismagilov, Rustem F., Abraham D. Stroock, Paul JA Kenis, George Whitesides, and Howard A. Stone. "Experimental and theoretical scaling laws for transverse diffusive broadening in two-phase laminar flows in microchannels." Applied Physics Letters 76, no. 17 (2000): 2376-2378.

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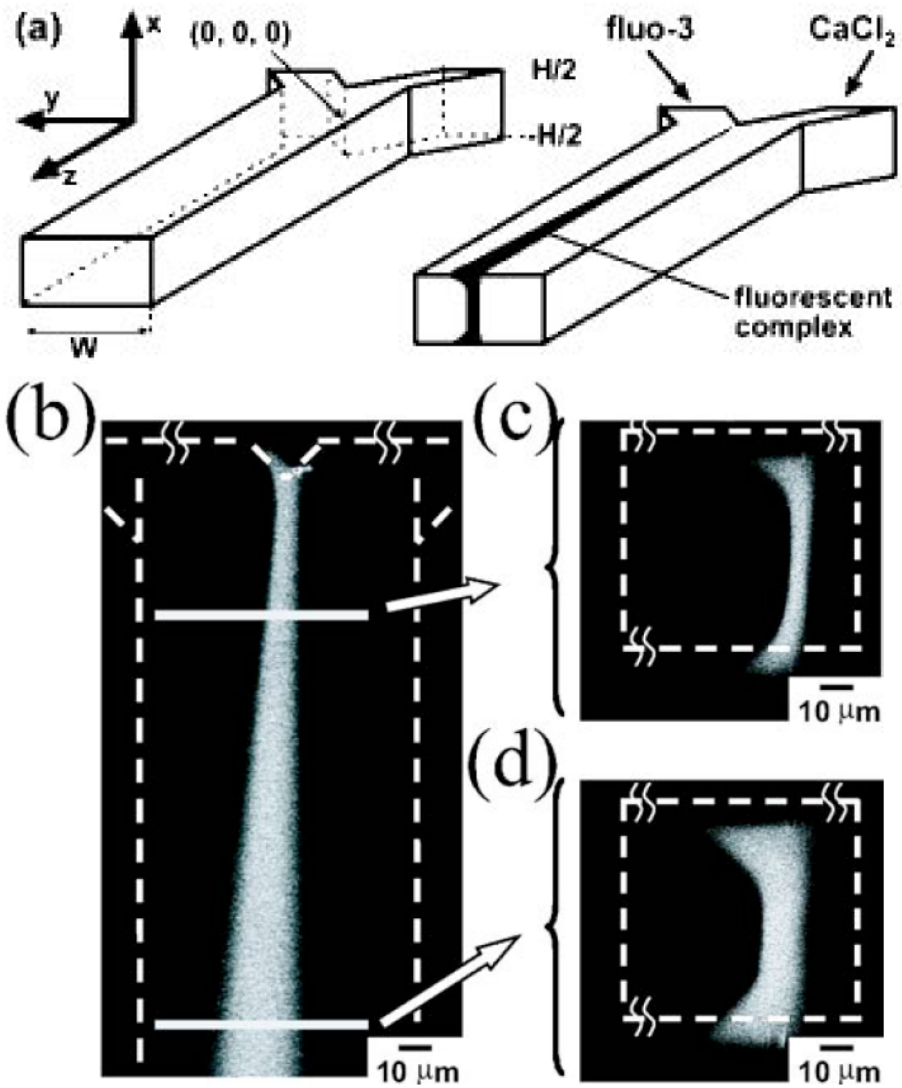
Sensing & Filtering



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Away from boundaries, the interfacial region:

$$\delta_y \sim t^{1/2} \sim z^{1/2}$$

Lévêque Problem: Near from boundaries, velocity varies **linearly** with distance into the channel.

$$\underbrace{\delta_x \sim (D_m t)^{1/2}}_{\text{Thickness of diffusion boundary layer}}$$

Diffusion across a linear flow field

$$t(z) \sim z/(G\delta_x)$$

$$\text{Shear rate: } G = \frac{\partial u_z}{\partial x}$$

Diffusion is the only time scale in both x & y:

$$\delta_y \sim \delta_x \sim \left(\frac{zD_m}{G}\right)^{1/3}$$

Ismagilov, Rustem F., Abraham D. Stroock, Paul JA Kenis, George Whitesides, and Howard A. Stone. "Experimental and theoretical scaling laws for transverse diffusive broadening in two-phase laminar flows in microchannels." Applied Physics Letters 76, no. 17 (2000): 2376-2378.
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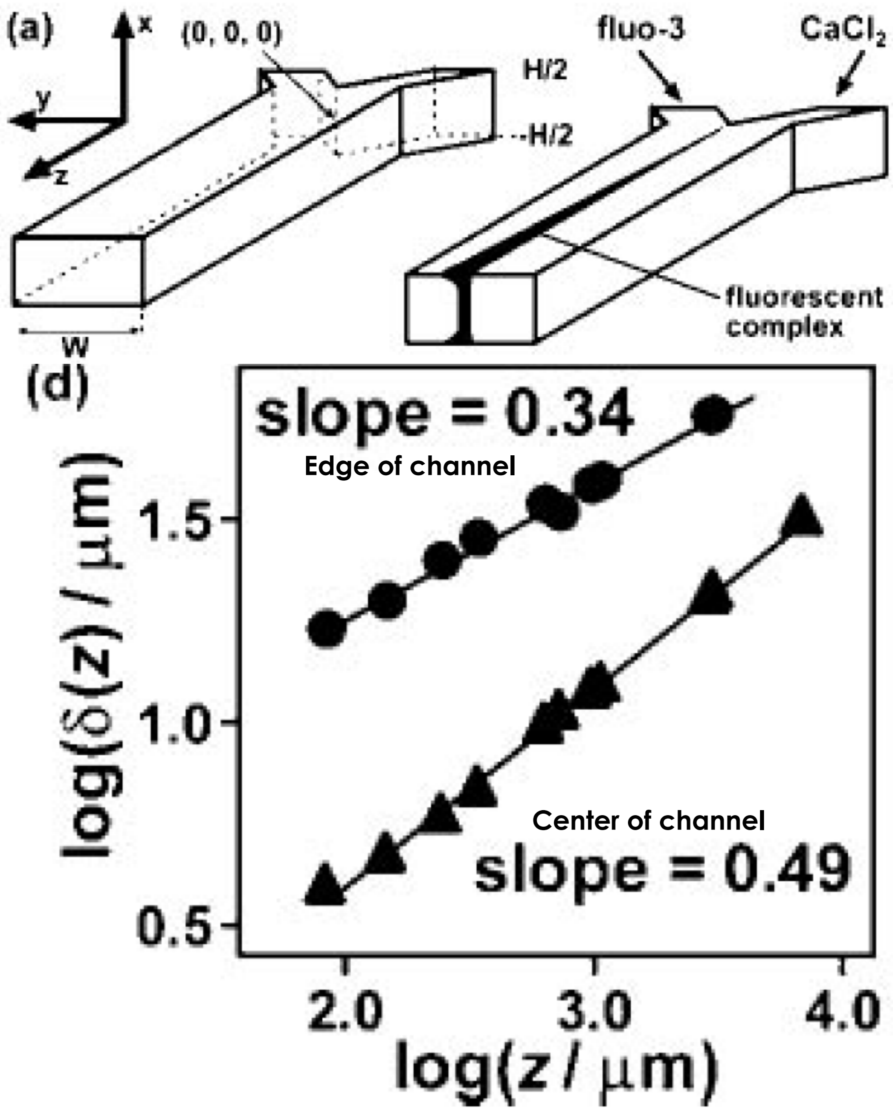
Sensing & Filtering



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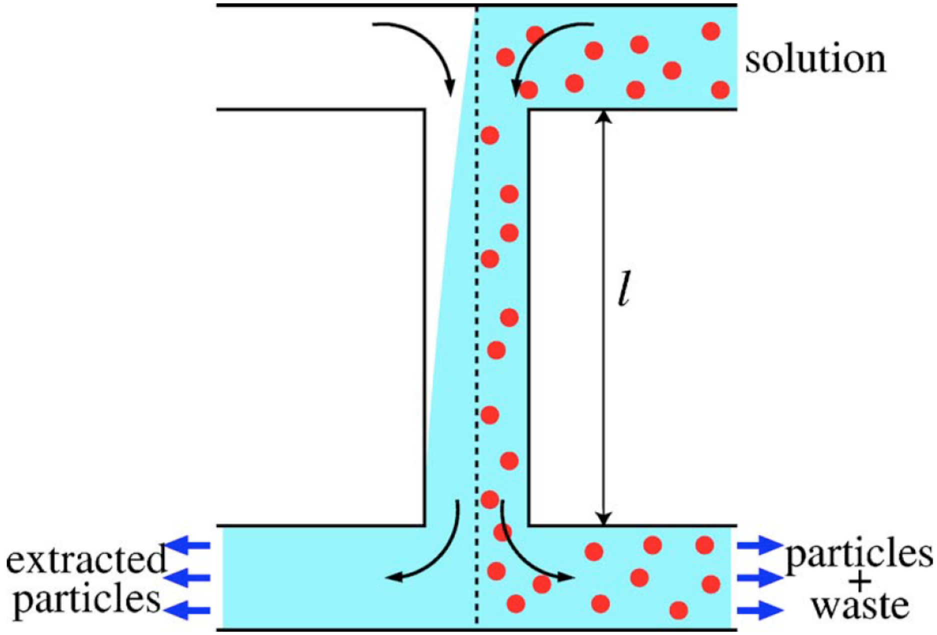
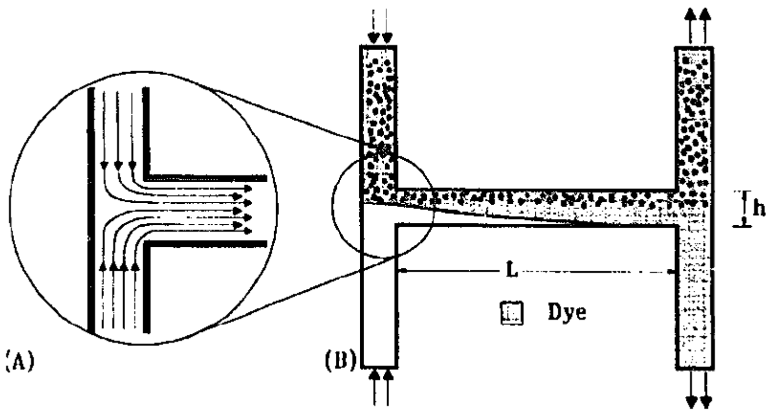
Stone, Howard A., Abraham D. Stroock, and Armand Ajdari. "Engineering flows in small devices: microfluidics toward a lab-on-a-chip." Annu. Rev. Fluid Mech. 36 (2004): 381-411.

Sensing & Filtering

H Filter





Parallel Laminar Flows

Filter particles by size without a membrane.

- Pressure-driven laminar flow of two adjacent miscible streams.
- One stream is a dilute solution of different sized particles.
- Each particle has its own **diffusivity** and **Péclet number**.

Péclet number determines the channel length required for each component to **diffuse across the channel width**.

H filter works best when one Péclet number is large and the other is small.

- e.g. small particles diffuse across channel, large ones do not.

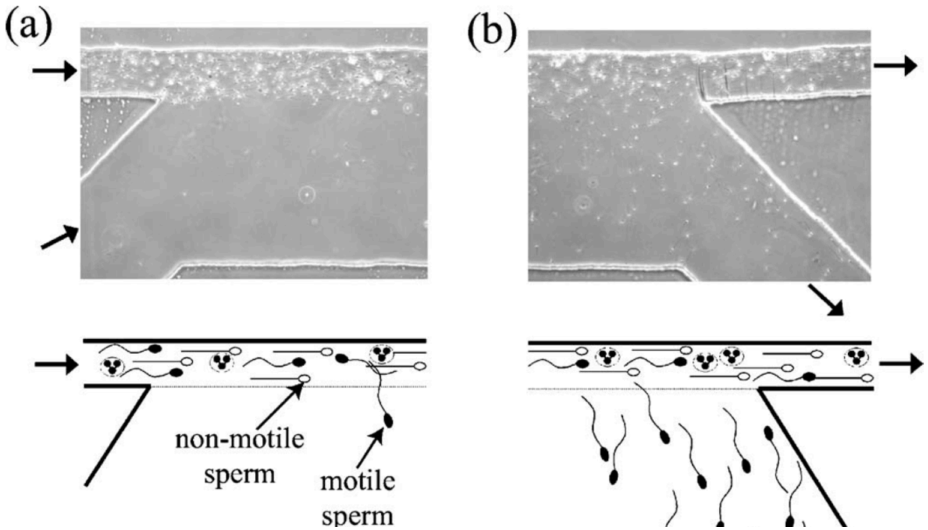
Brody, James P., Paul Yager, Raymond E. Goldstein, and Robert H. Austin. "Biotechnology at low Reynolds numbers." Biophysical journal 71, no. 6 (1996): 3430-3441.

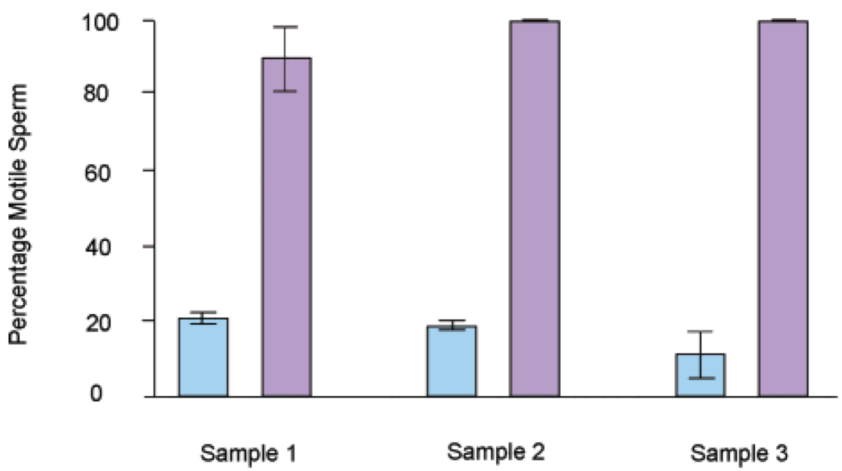
Brody, James P., and Paul Yager. "Diffusion-based extraction in a microfabricated device." Sensors and Actuators A: Physical 58, no. 1 (1997): 13-18.

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Sensing & Filtering

H Filter - Variant





Separation by Péclet number

- Components to be separated spread across channel at different rates.
- e.g. Separation of motile vs. nonmotile sperm.

Motile sperm **swim rapidly** and **randomly** to fill the channel, as compared to nonmotile sperm.

- Nonmotile sperm spread by diffusion alone.

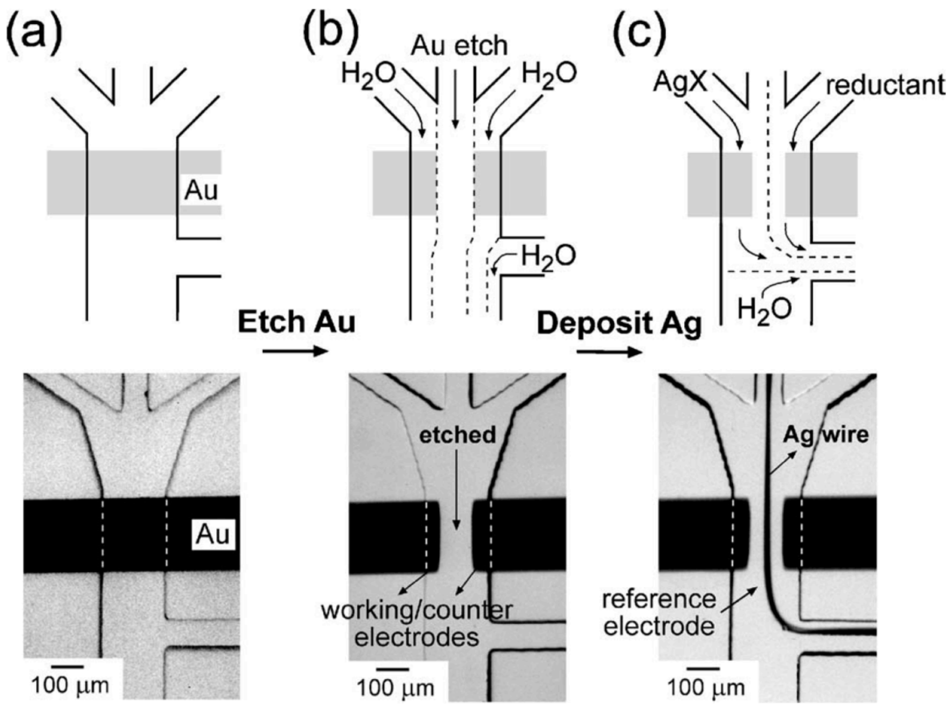
Motile sperm purity at inlet (blue) vs. motile sperm purity at outlet (purple)

Cho, Brenda S., Timothy G. Schuster, Xiaoyue Zhu, David Chang, Gary D. Smith, and Shuichi Takayama. "Passively driven integrated microfluidic system for separation of motile sperm." Analytical Chemistry 75, no. 7 (2003): 1671-1675.

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Fabrication using multiple laminar streams

…beating diffusion…



Fabrication of a three electrode system

- Multiple streams & large Péclet number.
- Minimal mixing over large distances.

Nonmixing, **high Péclet flows** can fabricate/ etch structures within microchannels.

e.g. Parallel streams **selectively etch** a gold electrode

- A silver wire is formed by a precipitation reaction between the two streams.
- Result: three-electrode system.

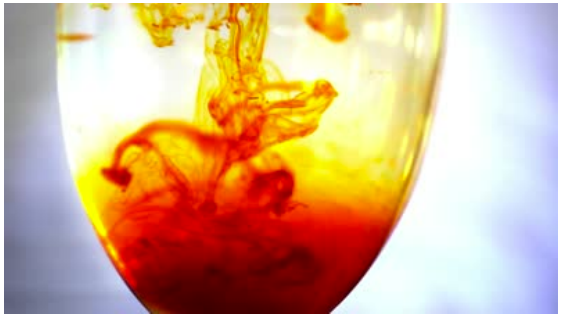
Kenis, Paul JA, Rustem F. Ismagilov, and George M. Whitesides. "Microfabrication inside capillaries using multiphase laminar flow patterning." Science 285, no. 5424 (1999): 83-85.

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Mixing

Rapid mixing means doing better than diffusion.

Why mix?
- Study chemical reaction kinetics…
- Probe protein folding…



Fluid **stirring** will **stretch** and **fold** inhomogenous fluid elements until **mixing** occurs.
- Mixing: diffusive migration across streamlines.
- Stirring motions reduce distance over which mixing must occur.

General design principles for mixing:

Dispersion of tracers occurs first by **convective stretching** with the fluid, followed by **diffusive homogenization**.

Tracers: an object transported by and diffuses into the fluid,
e.g. dyes, analyte molecules, proteins, cells, salt, or heat.

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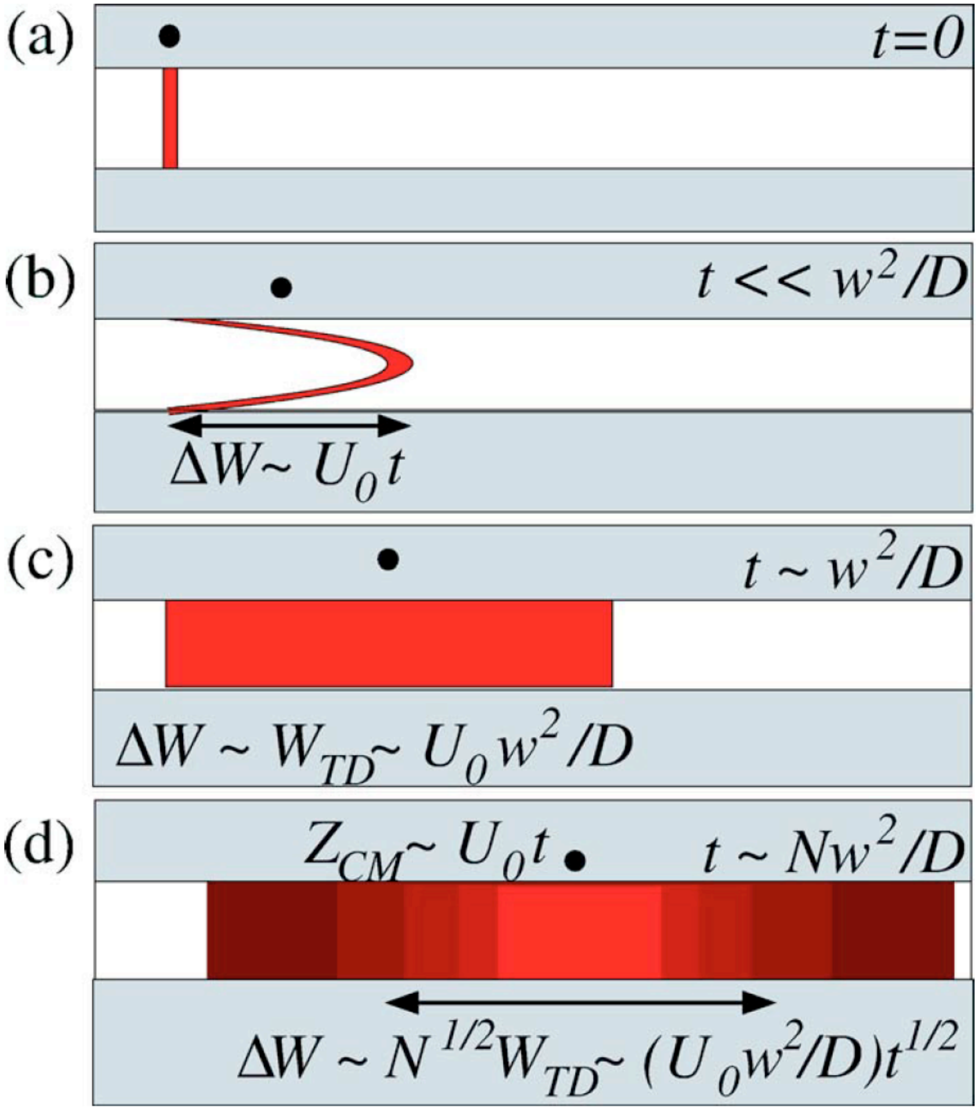
Mixing



Taylor Dispersion

Thin stripe of tracers spans circular channel

- Channel radius: *w*

Pressure-driven (Poiseuille) flow stretches strip into a parabola with profile:

$$u_z = U_0 \left(1 - \frac{r^2}{w^2}\right)$$

Molecular diffusion across the channel smears the parabola into a plug of width:

$$\Delta W \sim W_{TD} \sim U_0 \frac{w^2}{D}$$

Each stripe is **convectively stretched** then **diffusively smeared**

After *N* time steps of $t \sim Nw^2/D$, the initial stripe grows into a Gaussian with width:

$$\langle W^2 \rangle^{1/2} \sim N^{1/2} W_{TD} \sim \left(\frac{U_0^2 w^2}{D} t\right)^{1/2}$$

Taylor, Geoffrey. "Dispersion of soluble matter in solvent flowing slowly through a tube." In Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, vol. 219, no. 1137, pp. 186-203. The Royal Society, 1953.

Taylor, Geoffrey. "Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion." Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 225, no. 1163 (1954): 473-477.

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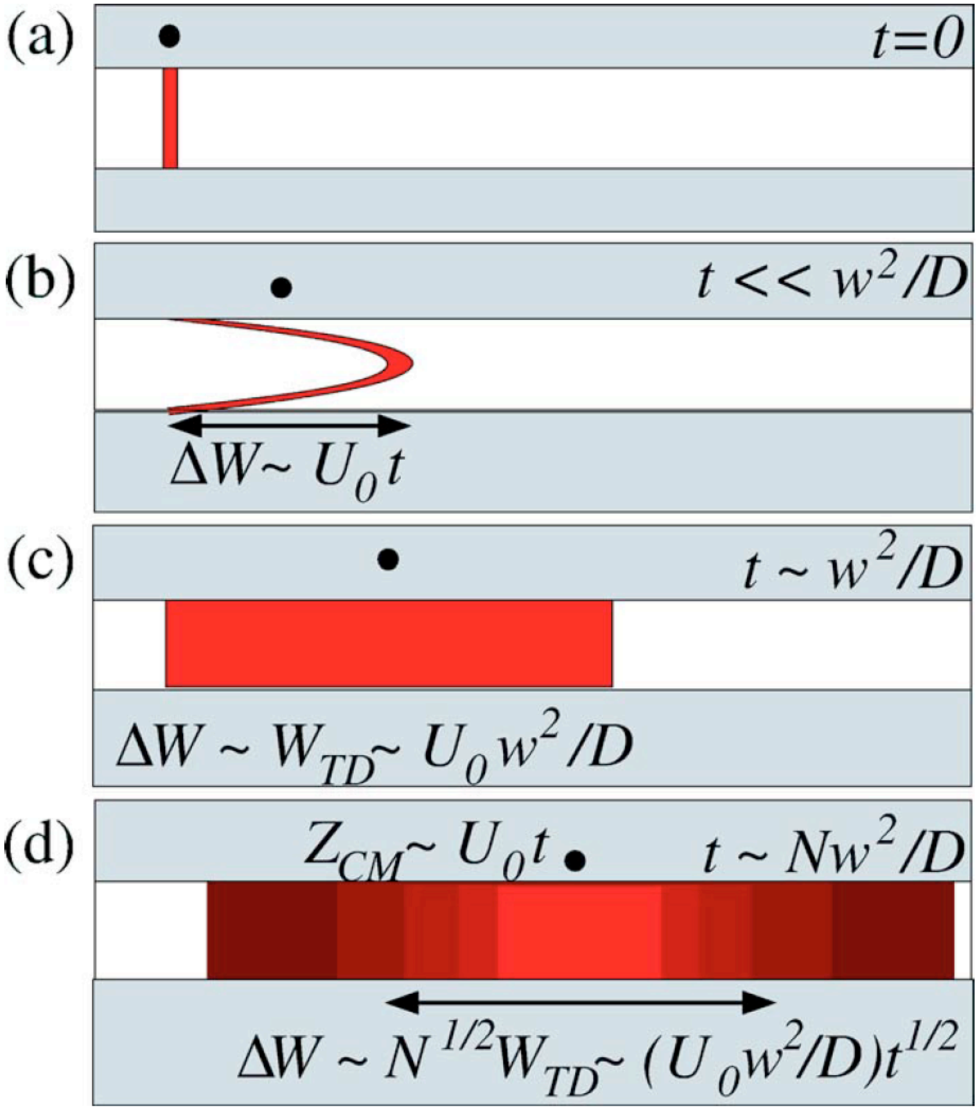
Mixing



Taylor Dispersion

Initial stripe grows to a Gaussian with width:

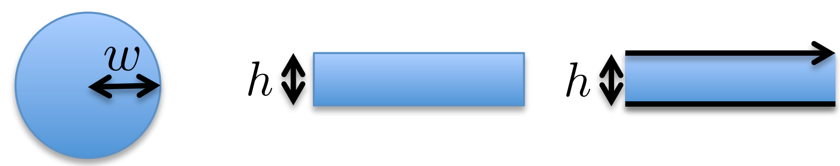
$$\langle W^2 \rangle^{1/2} \sim N^{1/2} W_{TD} \sim \left(\frac{U_0^2 w^2}{D} t \right)^{1/2}$$

The **tracer distribution grows** diffusively with an effective long-time **axial diffusivity**:

$$D_z \sim \frac{U_0^2 w^2}{D} \sim \mathscr{P}^2 D$$

Axial diffusivity occurs in addition to molecular diffusivity.

Relevant examples for specific geometries:



$$D_z = \frac{U_0^2 w^2}{48D} \qquad D_z = \frac{U_0^2 h^2}{210D} \qquad D_z = \frac{2U_0^2 h^2}{15D}$$

Taylor, Geoffrey. "Dispersion of soluble matter in solvent flowing slowly through a tube." In Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, vol. 219, no. 1137, pp. 186-203. The Royal Society, 1953.

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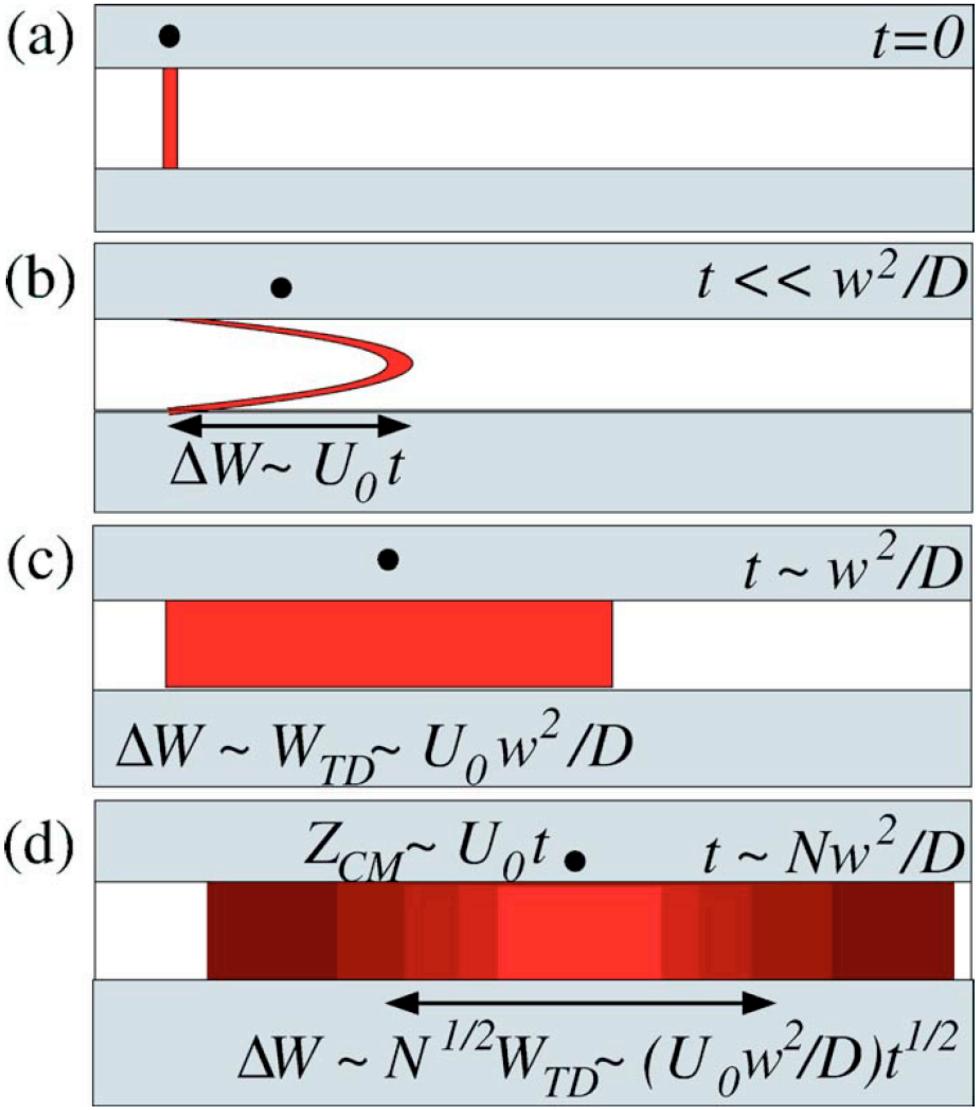
Mixing



Taylor Dispersion

$$D_z \sim \frac{U_0^2 w^2}{D} \sim \mathscr{P}^2 D$$

Convective stretching enhances axial dispersivity

Taylor dispersivity is only valid at

- Long time scales: $t \gg w^2/D$
- Downstream lengths: $L \gg \mathscr{P} w$

Taylor dispersion acts in direction of flow.

- Not observed in T sensors or H filters

Taylor, Geoffrey. "Dispersion of soluble matter in solvent flowing slowly through a tube." In Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, vol. 219, no. 1137, pp. 186-203. The Royal Society, 1953.

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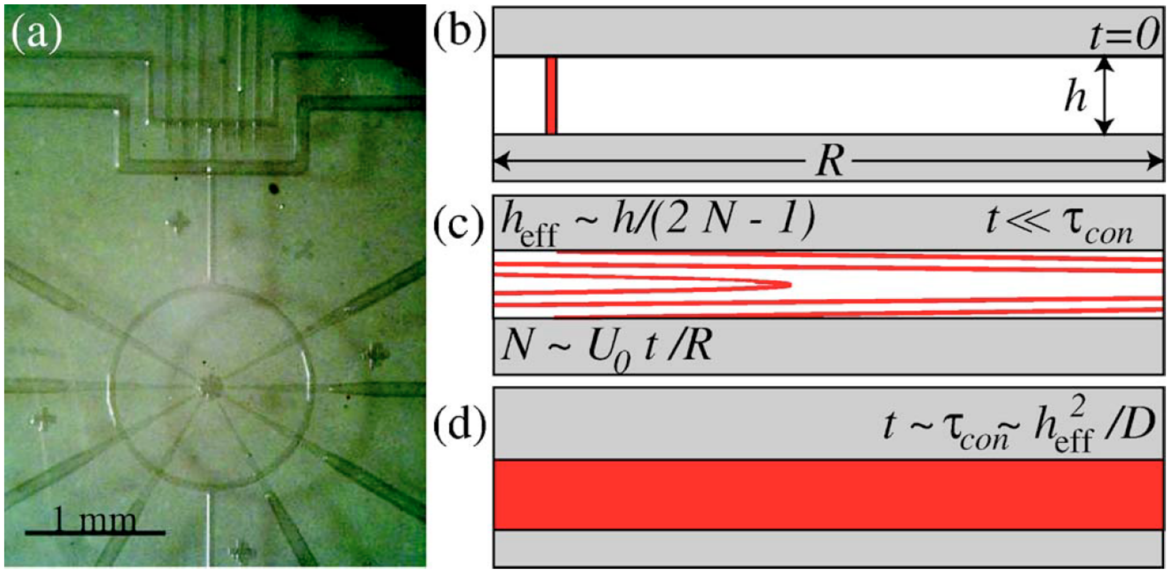
Mixing

Rotary Mixer



Fluid pumped around a circular channel:

- radius R
- height h
- average velocity U_0.

(i.) Diffusion-dominated

$$\mathscr{P} = U_0 h/D \ll 1$$

- Mixing occurs when tracers diffuse around the circumference of the ring, at time:

$$\tau_R \sim \frac{(2\pi R^2)}{D} = \left(\frac{2\pi R}{h}\right)^2 \tau_D$$

- Independent of Péclet number.

(ii.) Taylor dispersion-mediated

$$1 \ll \mathscr{P} \ll 2\pi R/h$$

- Axial spreading increases diffusivity with Taylor dispersivity:

$$\tau_{TD} \sim \frac{(2\pi R^2)}{D_z} = \frac{D(2\pi R)^2}{U_0^2 h^2} \sim \frac{\tau_R}{\mathscr{P}^2}$$

- Dominant when molecules diffuse over h before convection.

(iii.) Convectively Stirred

$$\mathscr{P} \gg 2\pi R/h$$

- At high flow rates, tracer stripes **fold** into themselves before molecules diffuse across channel.

$$\tau_{\rm con} \sim \mathscr{P}^{-2/3}\left(\frac{\pi R}{h}\right)^{2/3}\tau_D$$

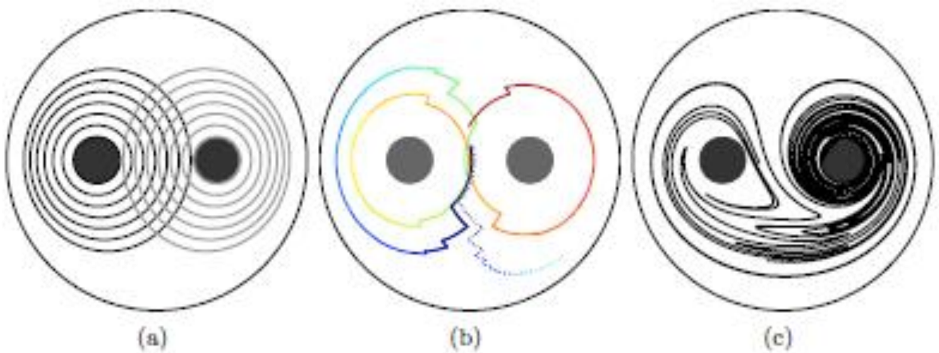
$$\sim \mathscr{P}^{-2/3}\left(\frac{h}{\pi R}\right)^{4/3}\tau_R$$

H.P. Chou, M.A. Unger, and S. Quake, Biomed. Microdevices, **3**, 323, (2001).
Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Mixing

Chaotic advection

Blinking vortex flow



Even simple **Stokes flows** can have **chaotic streamlines** that exponentially **stretch** and **fold**.

Steady, incompressible **two-dimensional flows** are integrable and **cannot exhibit chaotic trajectories**.

Steady **three-dimensional flows** can have chaotic streamlines (as can unsteady two dimensional flows).

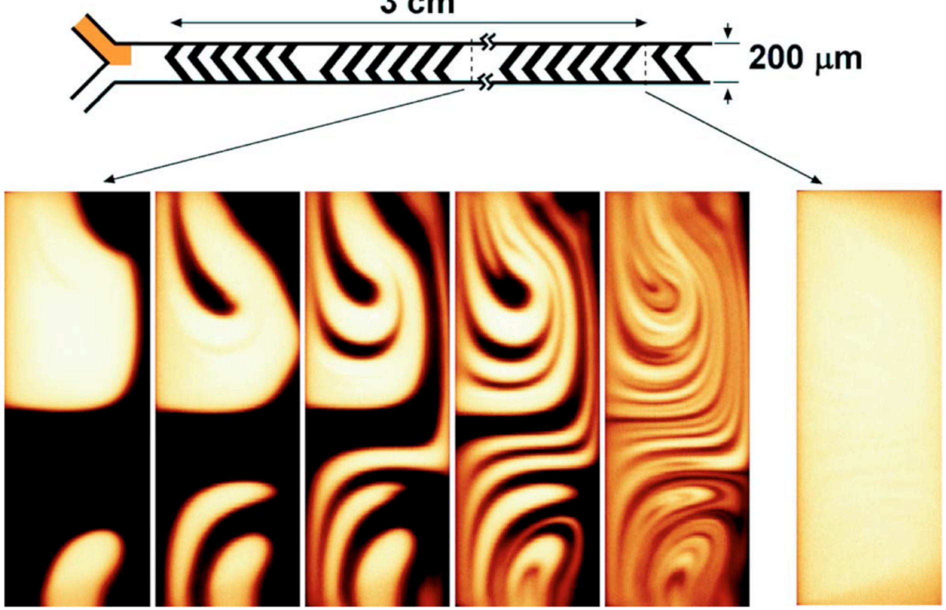
Can occur in droplets by superposing two simple flow fields

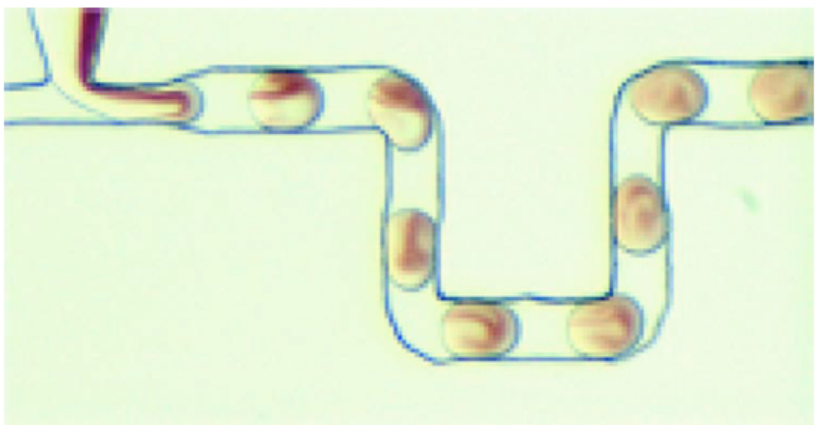
- (e.g. sedimenting drop in a shear flow)

Aref, Hassan. "Stirring by chaotic advection." Journal of fluid mechanics 143 (1984): 1-21.
Aref, Hassan. "The development of chaotic advection." Physics of Fluids, 14,4 (2002): 1315-1325.
http://upload.wikimedia.org/wikipedia/en/b/b8/Blinking_vortex_flow.jpg
Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Mixing

Continuous-flow staggered herringbone mixer





Chaotic advection

Even simple **Stokes flows** can have **chaotic streamlines** that exponentially **stretch** and **fold**.

The **staggered herringbone mixer** is a chaotic mixer for continuous flow systems – independent of inertia.

- **Asymmetric grooves** in channel walls induce axially modulated **secondary flow**.
- Counter-rotating fluid rolls.
- Asymmetry is periodically reversed so that the distance between stripes halves with each cycle – **exponential stretching/folding**.

Aref, Hassan. "Stirring by chaotic advection." Journal of fluid mechanics 143 (1984): 1-21.
Aref, Hassan. "The development of chaotic advection." Physics of Fluids, 14,4 (2002): 1315-1325.
Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Mixing

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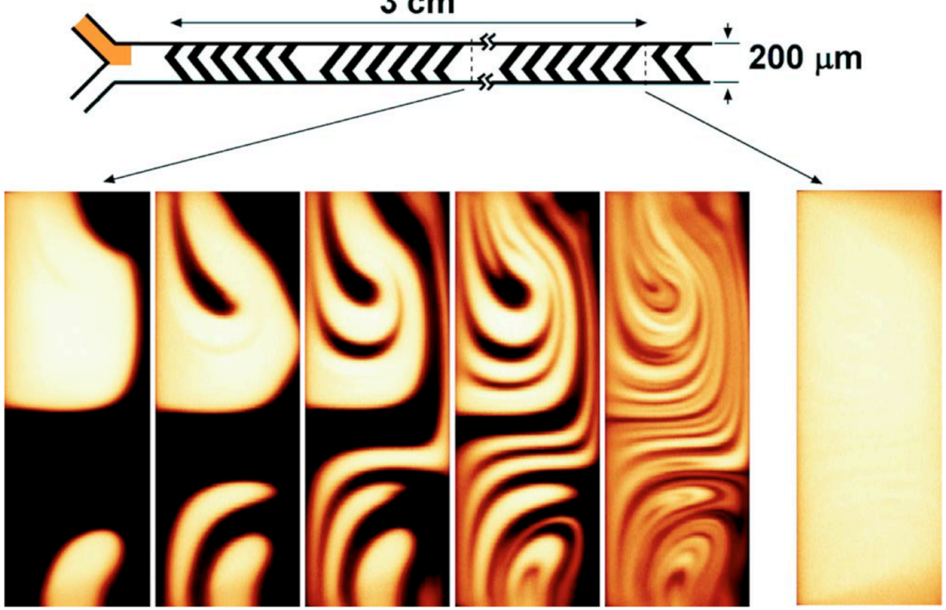
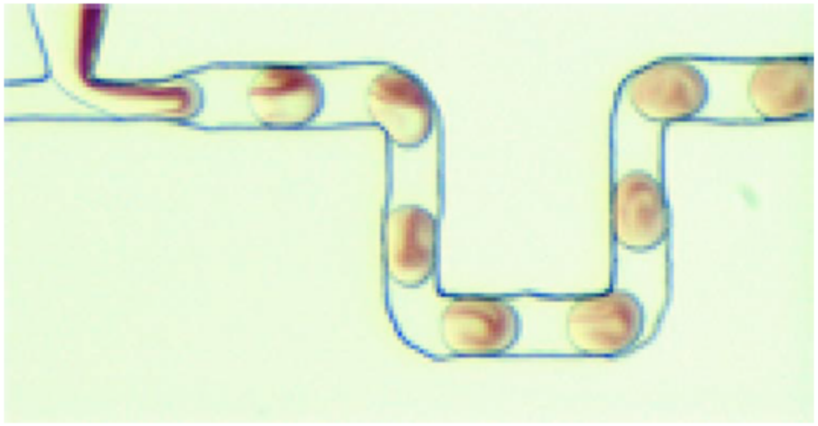




Chaotic advection

Asymmetry pattern on channel causes exponential stretching/folding.

After N cycles of time: $\tau_{\text{cyc}} \sim N L_{\text{cyc}}/U$

Stripes separated by distance: $h_{\text{eff}} \sim h/2^N$

Time to diffuse between stripes: $\tau_D \sim h_{\text{eff}}^2/D$

Mixing: $\tau_{\text{cyc}} \sim \tau_D$

Therefore, the number of cycles:

$$N_{\text{chaotic}} \sim \ln \mathscr{P}$$

Time for chaotic mixing:

$$\tau_{\text{chaotic}} \sim \frac{L_{\text{cyc}}}{h} \frac{\ln \mathscr{P}}{\mathscr{P}} \tau_D$$

Aref, Hassan. "Stirring by chaotic advection." Journal of fluid mechanics 143 (1984): 1-21.
Aref, Hassan. "The development of chaotic advection." Physics of Fluids, 14,4 (2002): 1315-1325.
Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Confined Fluid Flow: Microfluidics and Capillarity

Reynolds Number: Inertia vs. Viscous effects

- Review of characteristic flows…

Péclet Number: Transport phenomena in a continuum

- Diffusion, separation, and mixing…

Geometric confinement: Controlling and manipulating fluid flow

- Microfluidic fabrication, valving, pumping…

Capillary Number: Viscosity vs. Surface tension

- Droplet formation, capillary rise, elasticity…

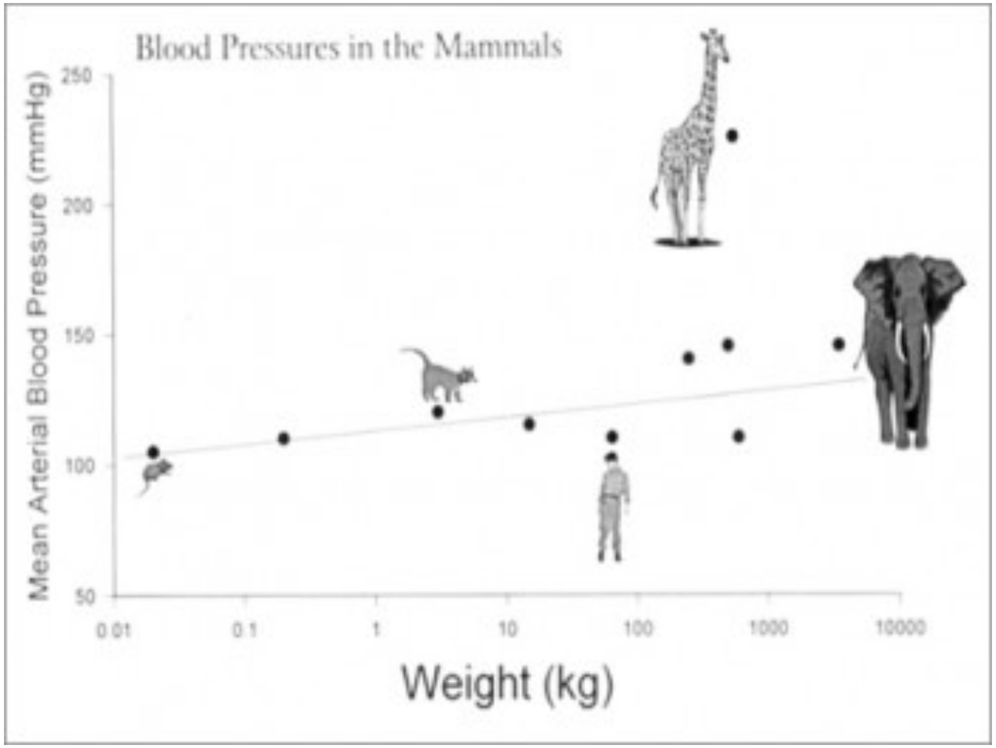
Geometric Confinement:

Control & Manipulation of Fluid Flow

A Giraffe's Jugular

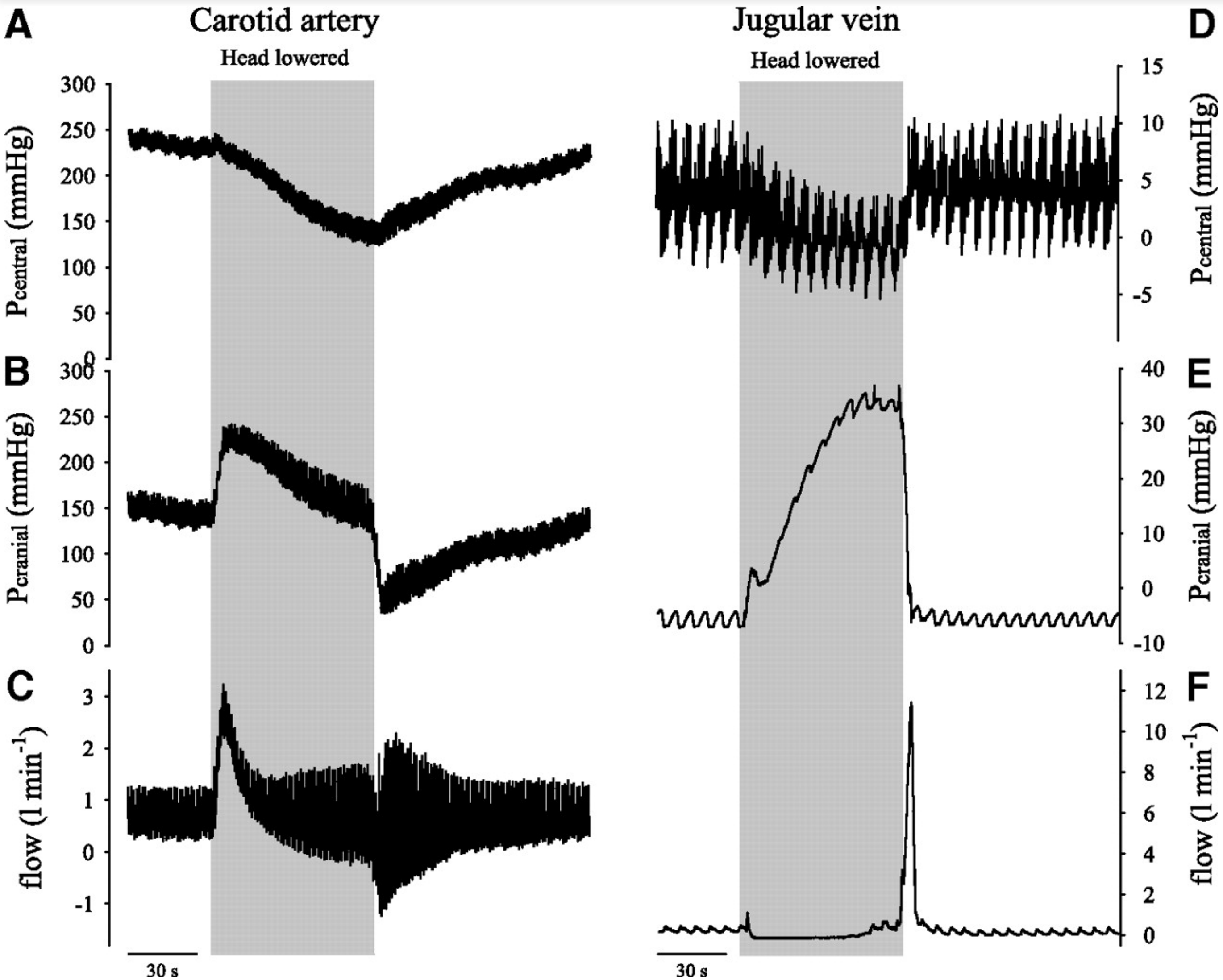
http://upload.wikimedia.org/wikipedia/commons/e/e6/Giraffes_at_west_midlands_safari_park.jpg

A Giraffe's Jugular



Top giraffe: http://cdn.asiancorrespondent.com/wp-content/uploads/2011/11/Giraf2-349x248.jpg
Bottom giraffe: http://www.onekind.org/uploads/a-z/az_giraffe1.jpg

A Giraffe's Jugular



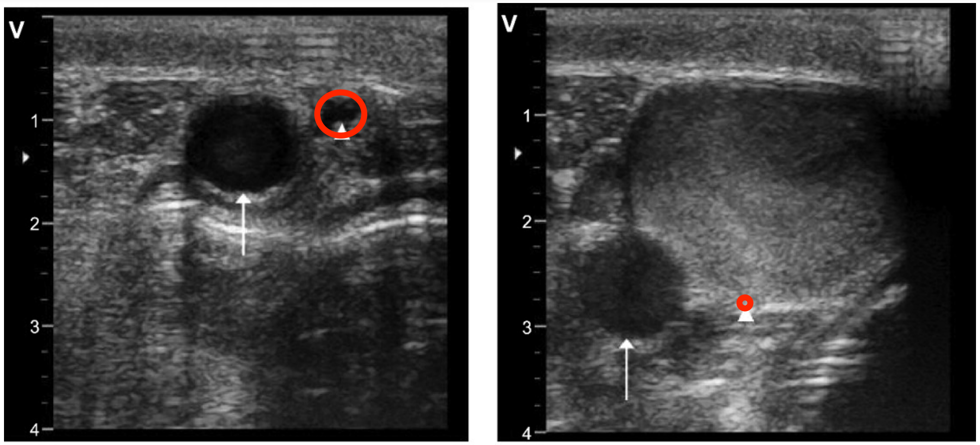
Arterial and venous pressure decrease upon lowering the head.

Rise in cranial jugular pressure coincides with cessation of jugular flow.

Jugular venous flow ceased for ~30s when lowered.

Brøndum, E., et al. "Jugular venous pooling during lowering of the head affects blood pressure of the anesthetized giraffe." American Journal of Physiology-Regulatory, Integrative and Comparative Physiology 297.4 (2009): R1058-R1065.

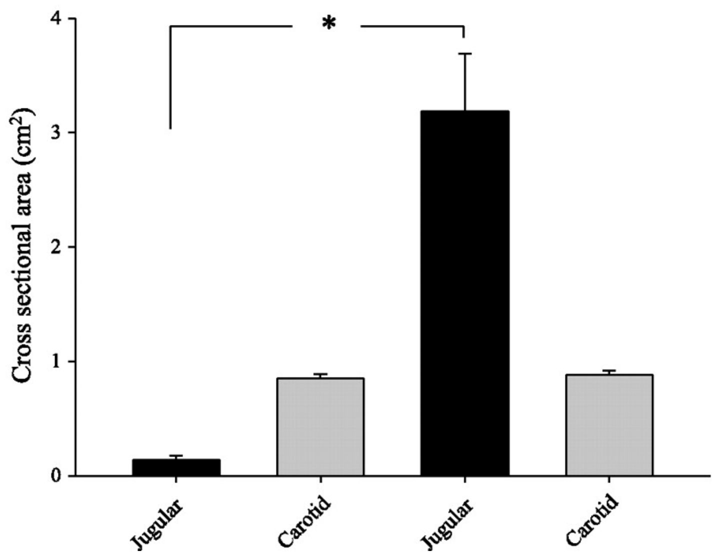
A Giraffe’s Jugular

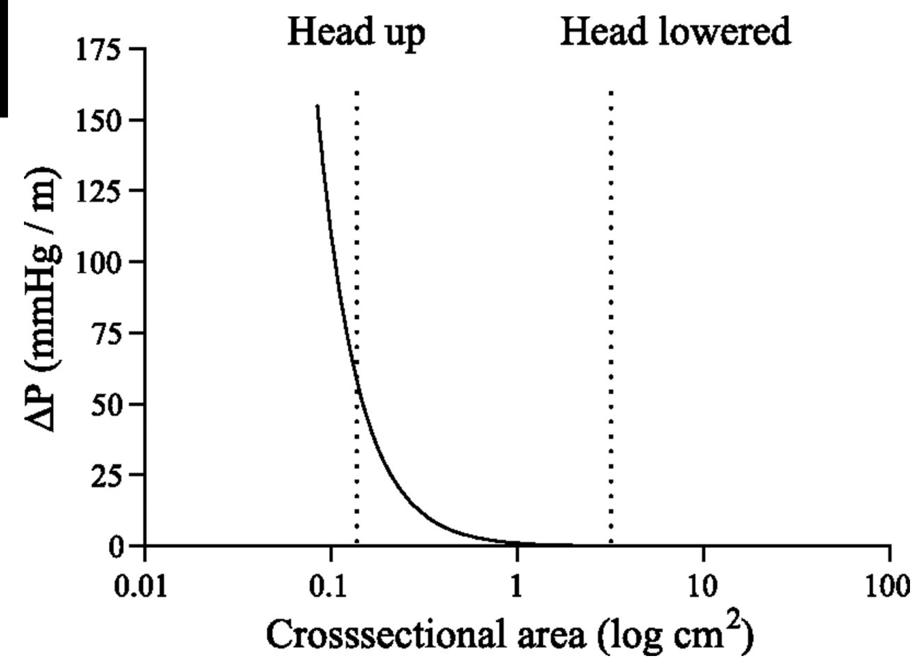


Upright position

Head lowered

Jugular vein’s cross sectional area changes upon lifting/lowering of the Giraffe's head.





Brøndum, E., et al. "Jugular venous pooling during lowering of the head affects blood pressure of the anesthetized giraffe." American Journal of Physiology-Regulatory, Integrative and Comparative Physiology 297.4 (2009): R1058-R1065.

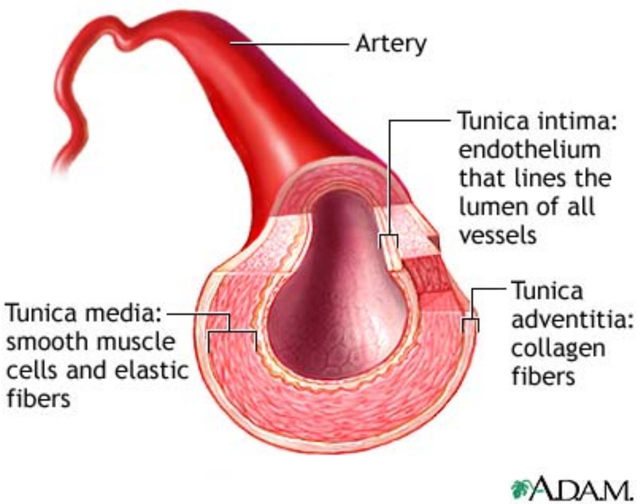
A Giraffe's Jugular

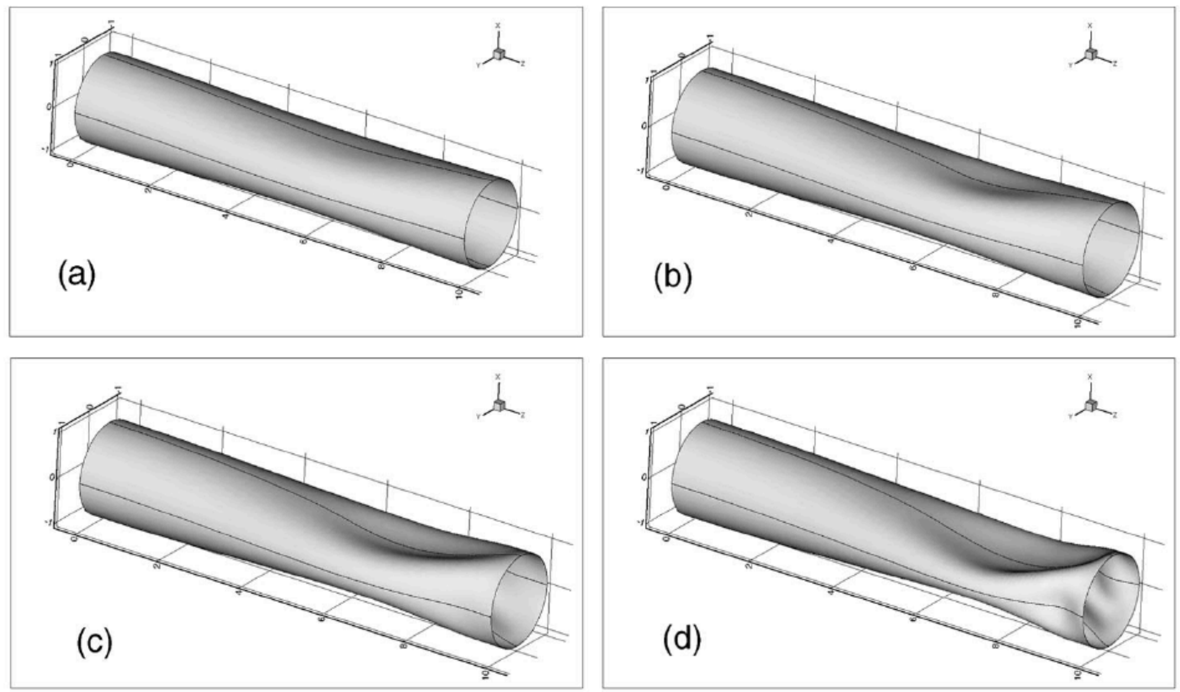
Saara Tuulia, http://factloops.blogspot.com/2012/01/short-animations-on-a-brief.html

Flexible Tubes

Almost all **vessels** carrying **fluids** with the body are **flexible**.

Fluid-structure interactions between **internal flow** and tube **deformation** often **dictate** a vessel's **biological function** or dysfunction.





Grotberg, James B., and Oliver E. Jensen. "Biofluid mechanics in flexible tubes." Annual Review of Fluid Mechanics 36.1 (2004): 121.

Flexible Tubes

Starling Resistor

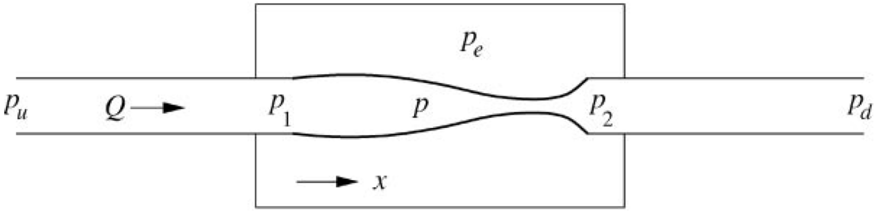
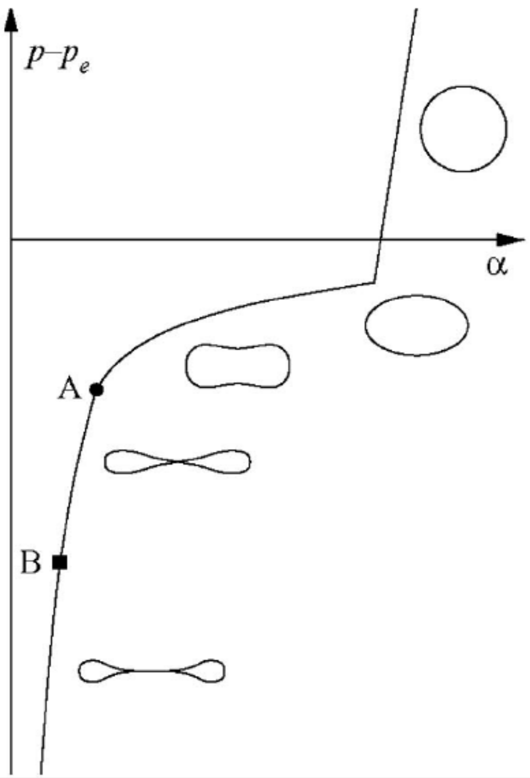
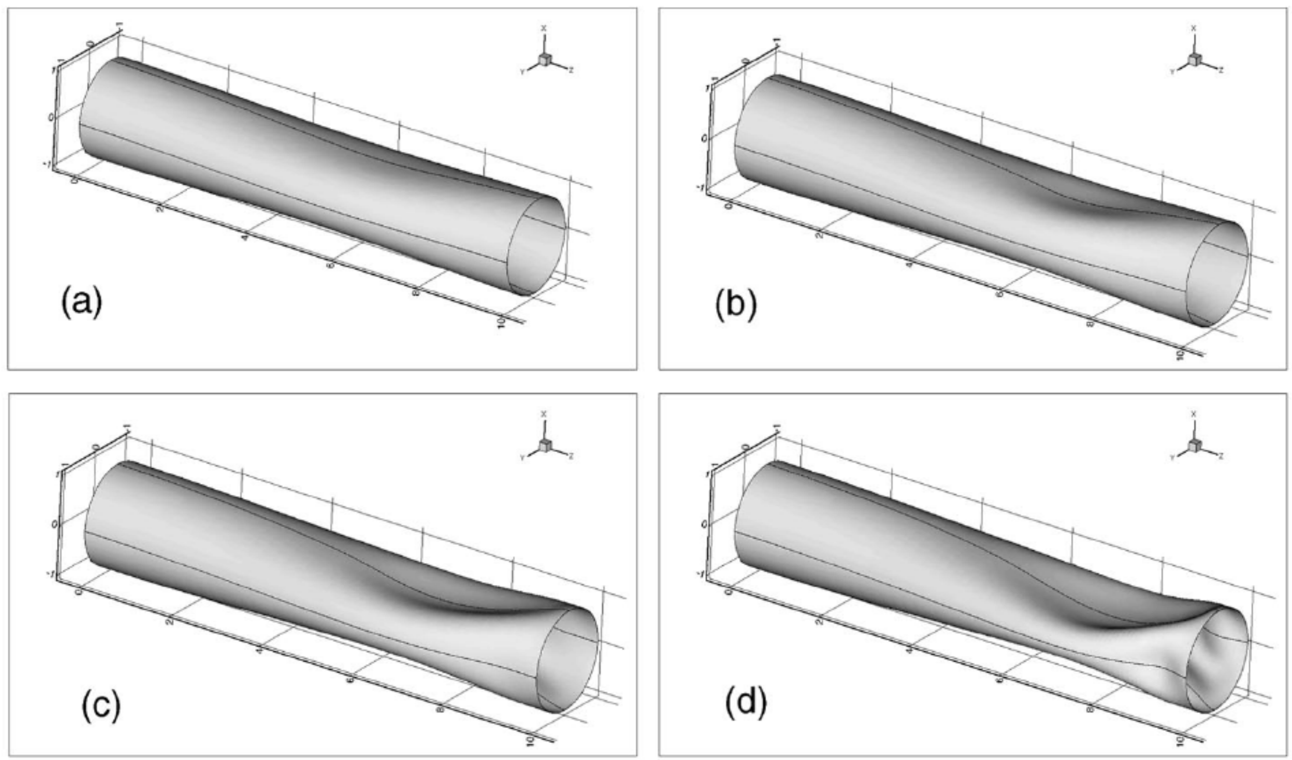


Figure 1 A Starling Resistor: a collapsible tube is mounted between two rigid tubes and is enclosed in a chamber held at pressure p_e. Flow with volume flux Q is driven by the imposed pressure drop $p_u - p_d$.





Grotberg, James B., and Oliver E. Jensen. "Biofluid mechanics in flexible tubes." Annual Review of Fluid Mechanics 36.1 (2004): 121.

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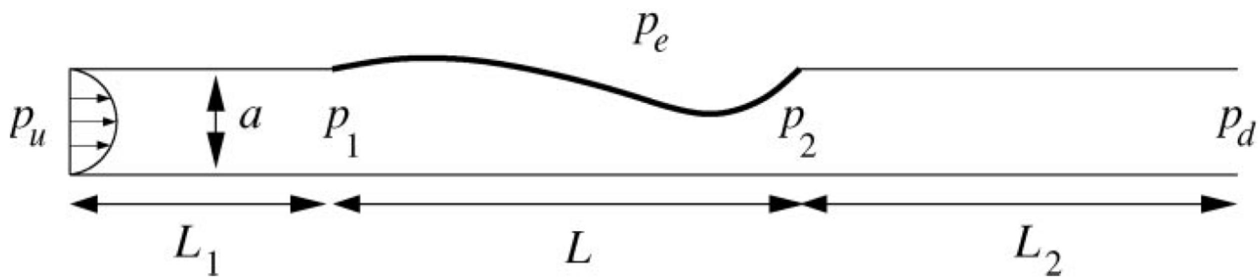
Flexible Tubes

Starling Resistor (1D Model)



Mass conservation:

$$\frac{\partial \alpha}{\partial t} + \frac{\partial (u\alpha)}{\partial x} = 0$$

$\alpha =$ tube cross sectional area

Momentum conservation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) \approx \underbrace{-\frac{\partial p}{\partial x}}_{\text{pressure}} - \underbrace{R(u,\alpha)}_{\substack{\text{viscous resistance}\\ \text{per unit length}}}$$

Pressure-area relation:

$$p - p_e = \underbrace{\mathcal{P}(\alpha)}_{\text{compliance}} - \underbrace{T\frac{\partial^2 \alpha}{\partial x^2}}_{\substack{\text{longitudinal}\\ \text{tension}}} + \underbrace{D\frac{\partial \alpha}{\partial t}}_{\substack{\text{viscous}\\ \text{damping}}} + \underbrace{M\frac{\partial^2 \alpha}{\partial t^2}}_{\text{inertia}}$$

(Very) Reduced model

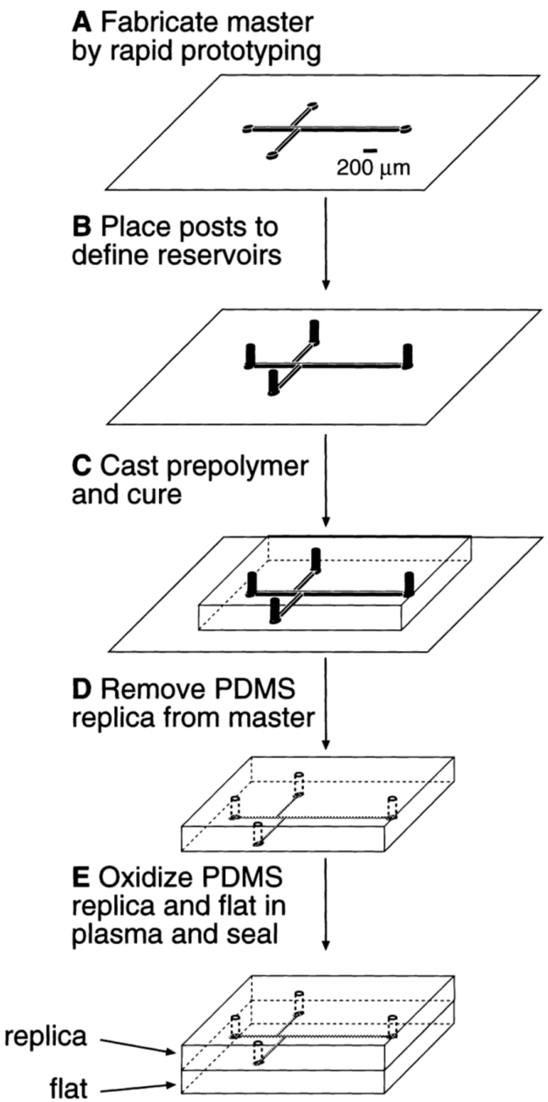
- "Bench-top" model for a deformable airway
- Consider airways as single, compliant tube
- Prone to instabilities – best with low Reynolds

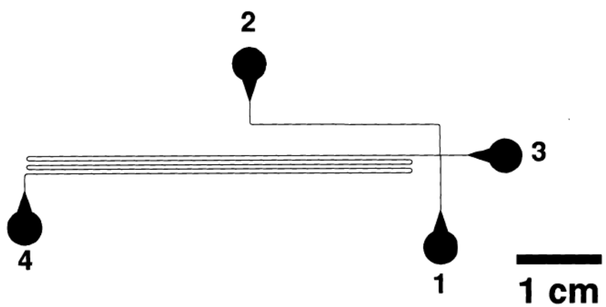
Compliance of tube gives rise to **fluid-structure interactions**

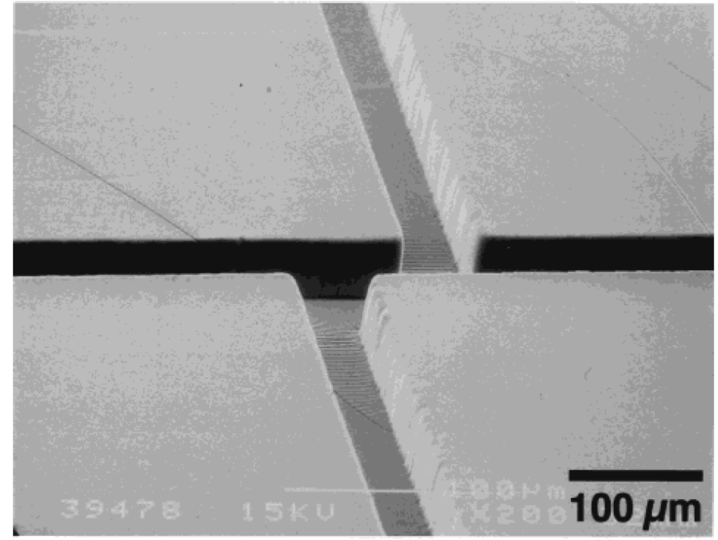
O.E. Jensen, "Flows through deformable airways" (2002), see: http://www.biomatematica.it/urbino2002/programmi/oejnotes.pdf
Grotberg, James B., and Oliver E. Jensen. "Biofluid mechanics in flexible tubes." Annual Review of Fluid Mechanics 36.1 (2004): 121.

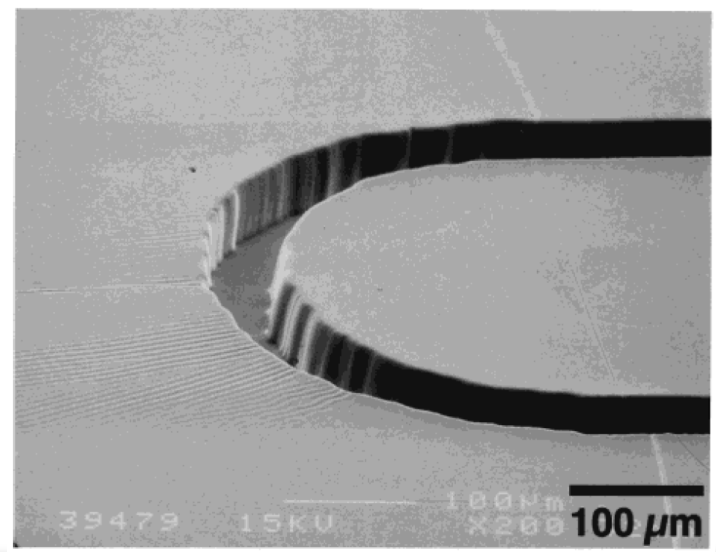
How do we translate these ideas to microfluidic devices?

Fabrication









Duffy, David C., et al. "Rapid prototyping of microfluidic systems in poly(dimethylsiloxane)." Analytical chemistry 70.23 (1998): 4974-4984.

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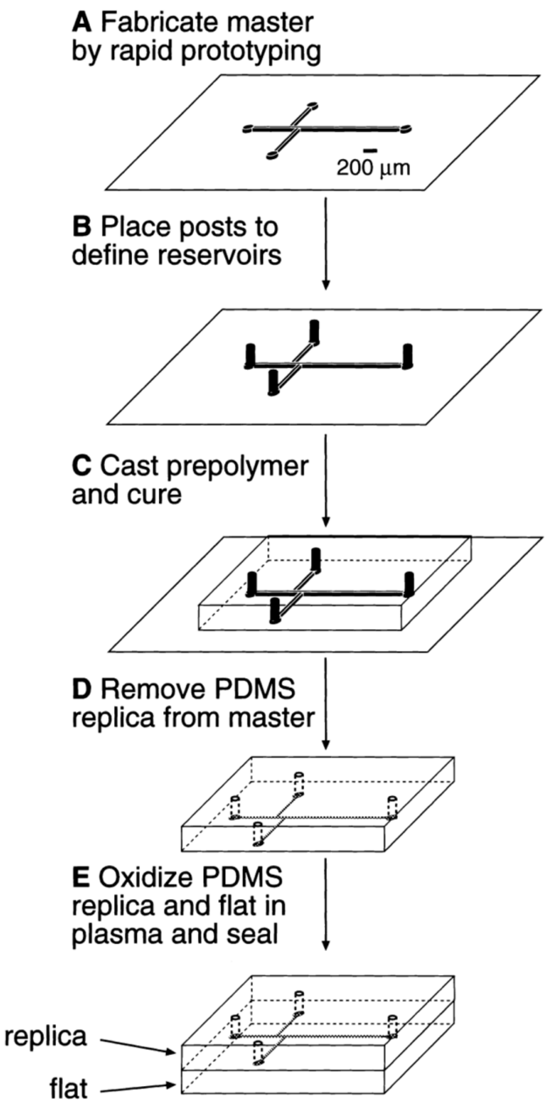
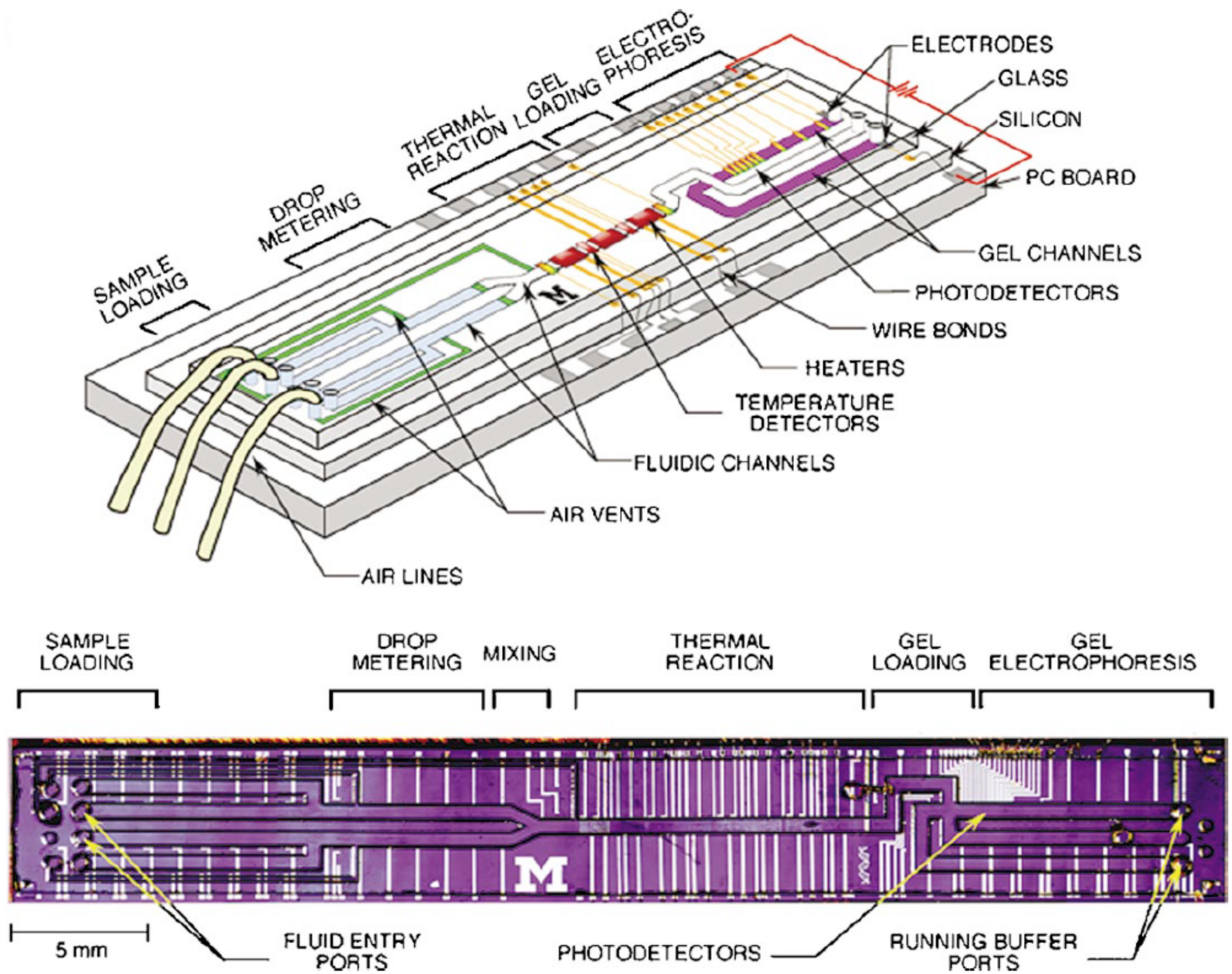
Fabrication





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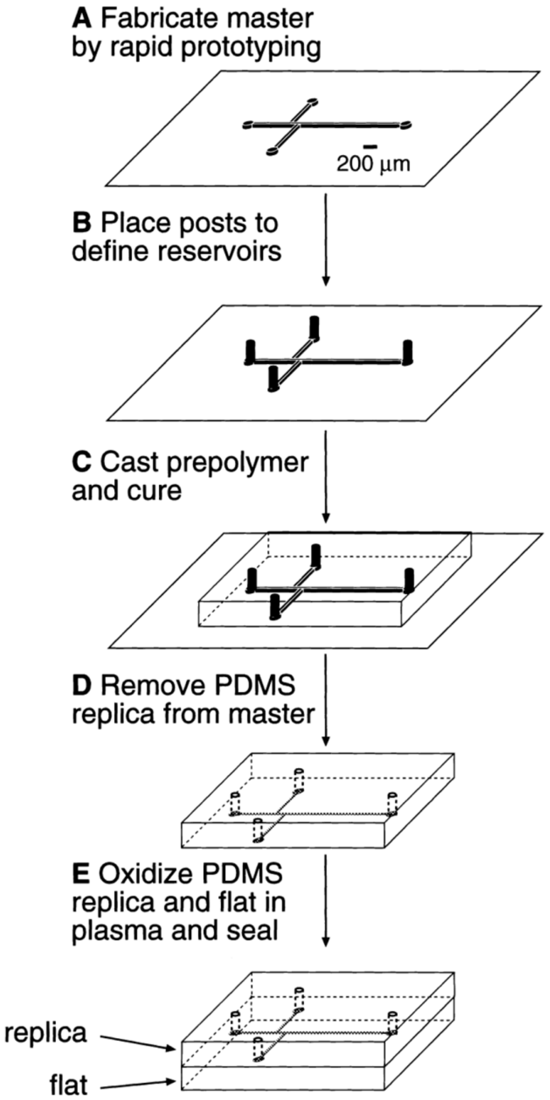
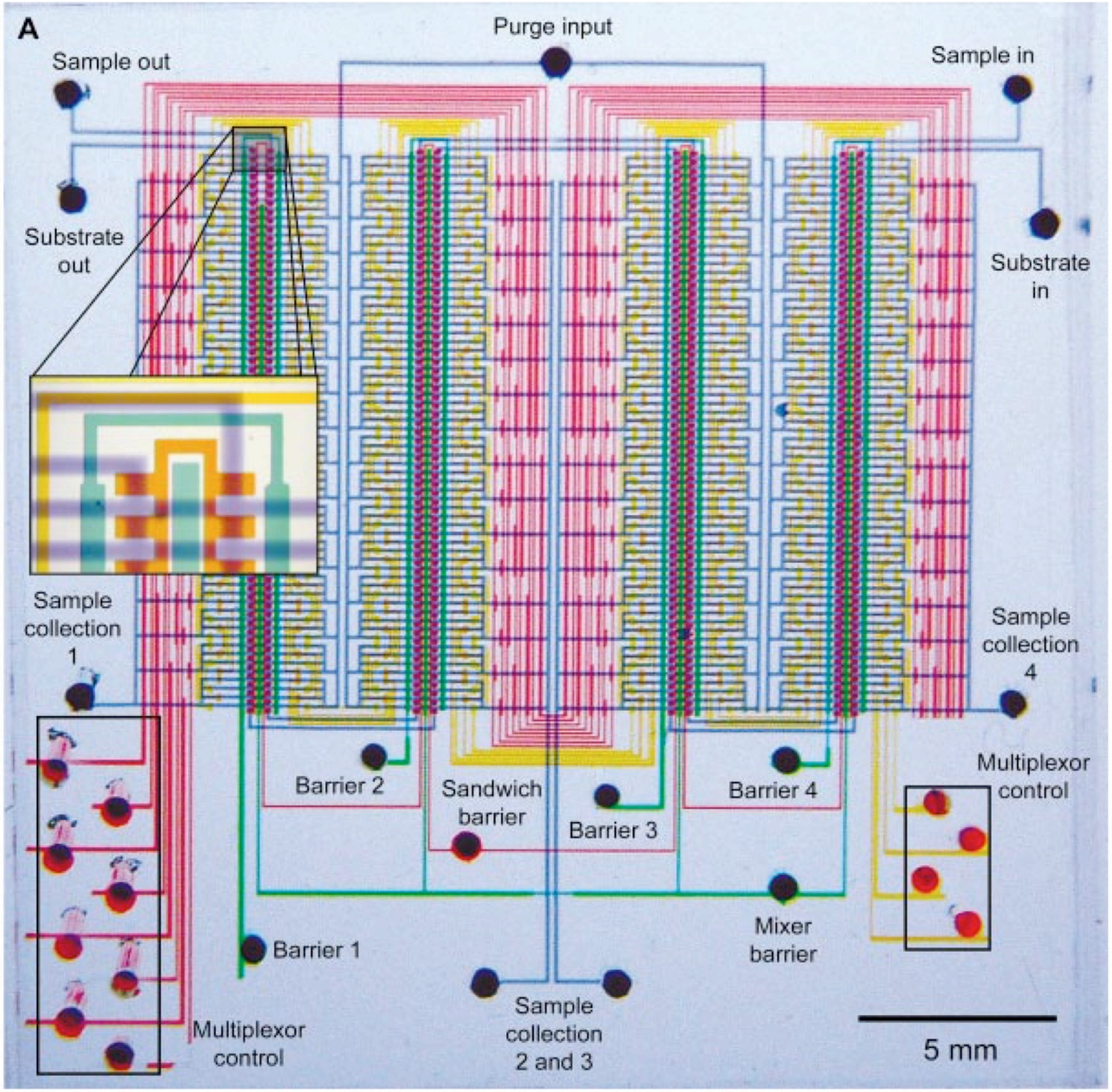
M.A. Burns, C.H. Mastrangelo, T.S. Sammarco, F.P. Man, J.R. Webster, et al., "Microfabricated structures for integrated DNA analysis." Proc. Natl. Acad. Sci. USA, 93:5556-5561, (1996).

Fabrication





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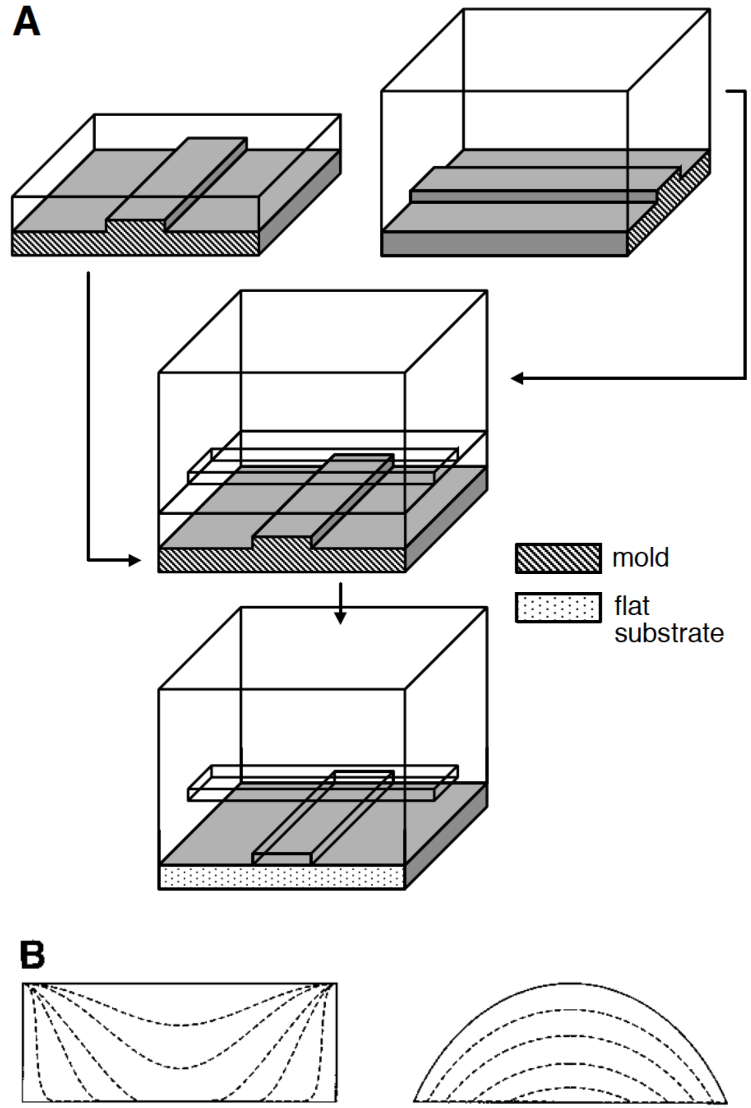
T. Thorsen, S.J. Maerkl, S.R. Quake, "Microfluidic large-scale integration." Science, 298:580-584, (2002).

Pneumatic Valves

"Quake" valve

- Bilayer microfluidic chip.
- Thin film separating two flow channels.
    - One channel: Fluid
    - One channel: Air (controlling)
- Pressurized air deflects the thin film and closes the fluid channel.



Unger, Marc A., Hou-Pu Chou, Todd Thorsen, Axel Scherer, and Stephen R. Quake. "Monolithic microfabricated valves and pumps by multilayer soft lithography." Science 288, no. 5463 (2000): 113-116.

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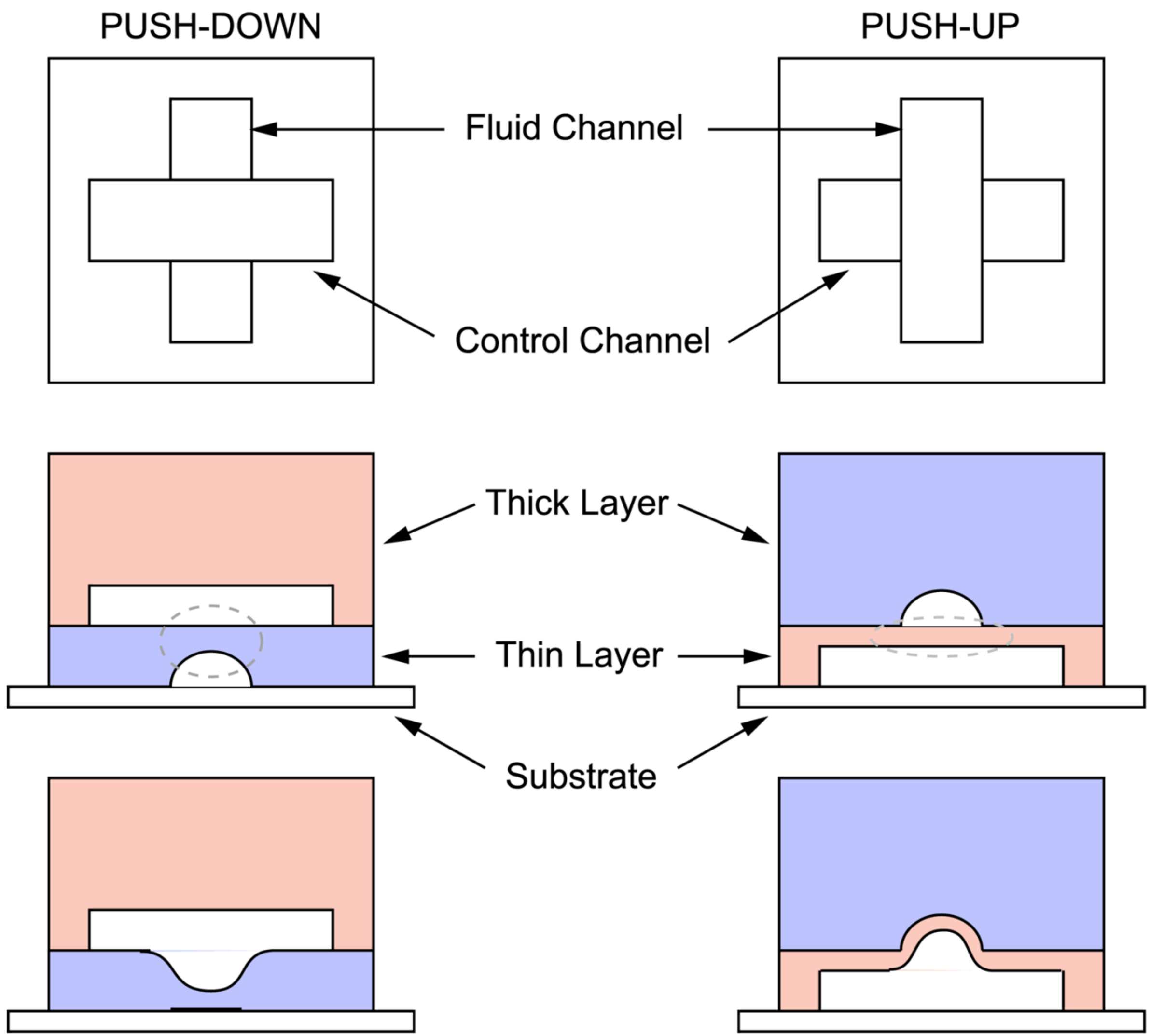
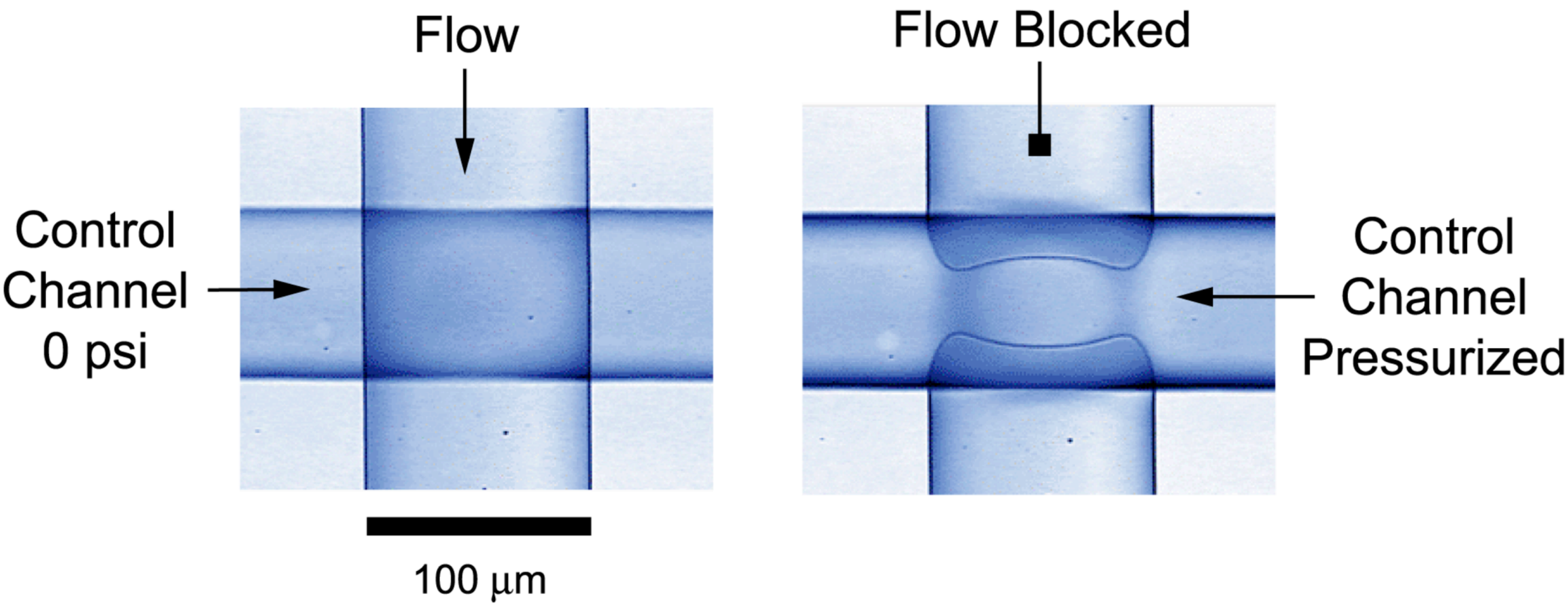


Pneumatic Valves

“Quake” valve



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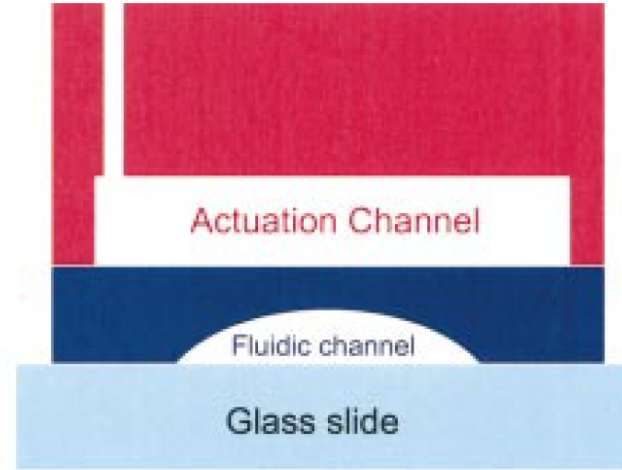
Pneumatic Valves

"Quake" valve

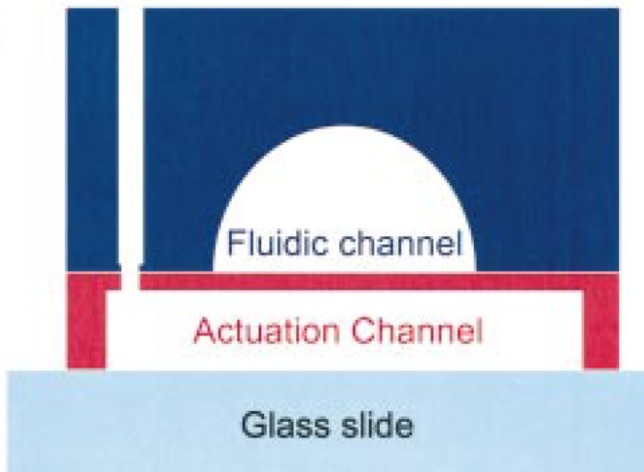
Push-down Valves

- Control lines above flow channels.
- Pressure flattens membrane valve **down** to seal.
- Suitable for low aspect ratio (1:10) & shallow (~10um) channels.
    - Flow geometry: 100um x 13um
    - Control geometry: 100um x 10-25um
- Applications: where fluid flow must be in contact with substrate (spotting DNA, patterned substrate, etc.)



Push-up Valves

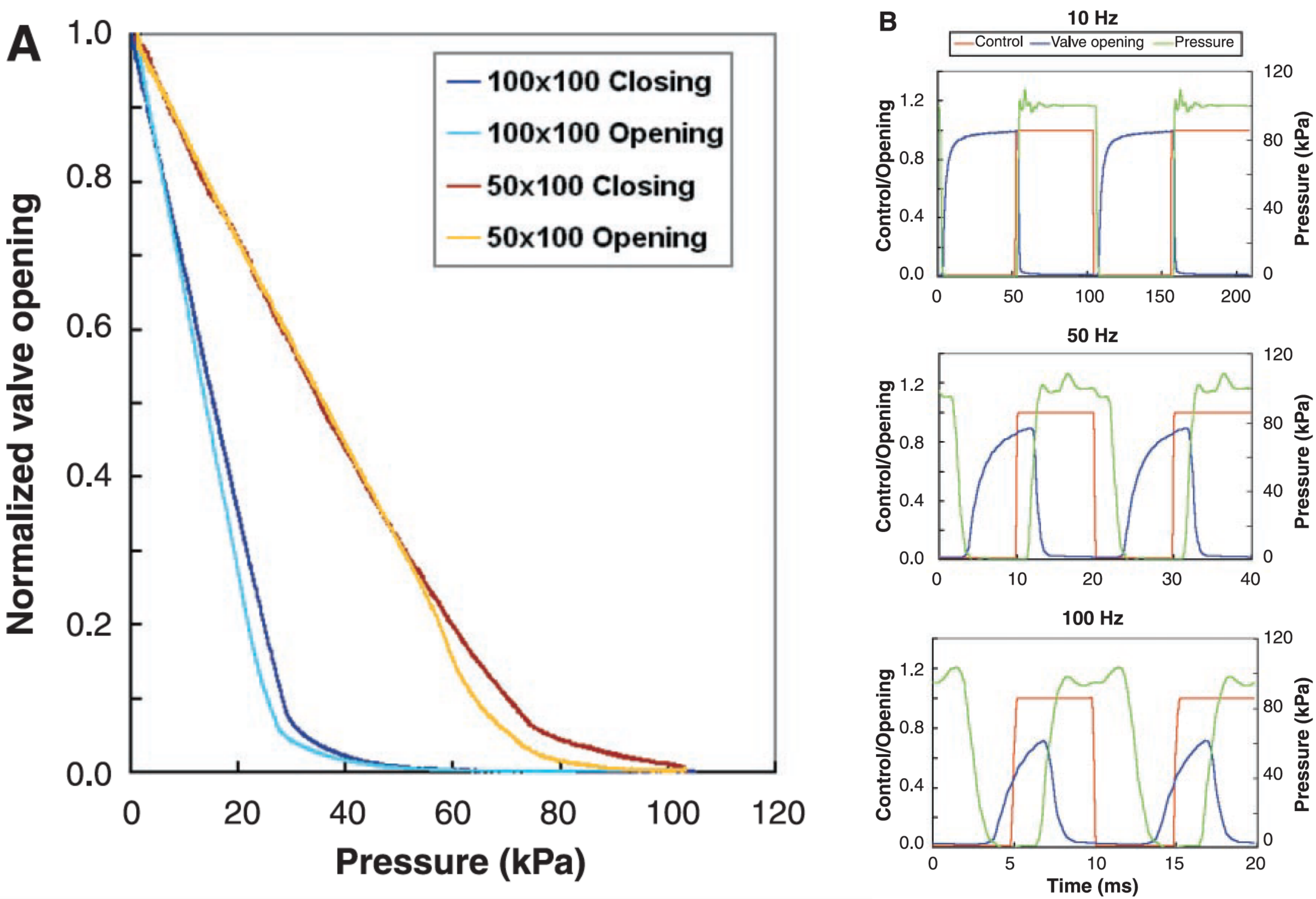
- Control lines belowflow channels.
- Pressure deflect membrane valve **up** to seal.
    - Flow geometry: 100um x 13um-50um
    - Control geometry: 100um x 10-25um
- Applications: suspension of large particles (eukaryotic cells, large beads, etc.)



Studer, Vincent, Giao Hang, Anna Pandolfi, Michael Ortiz, W. French Anderson, and Stephen R. Quake. "Scaling properties of a low-actuation pressure microfluidic valve." Journal of Applied Physics 95, no. 1 (2004): 393-398.

https://sharedfacilities.stanford.edu/service_center/show_external/22/microfluidics-foundry

Pneumatic Valves



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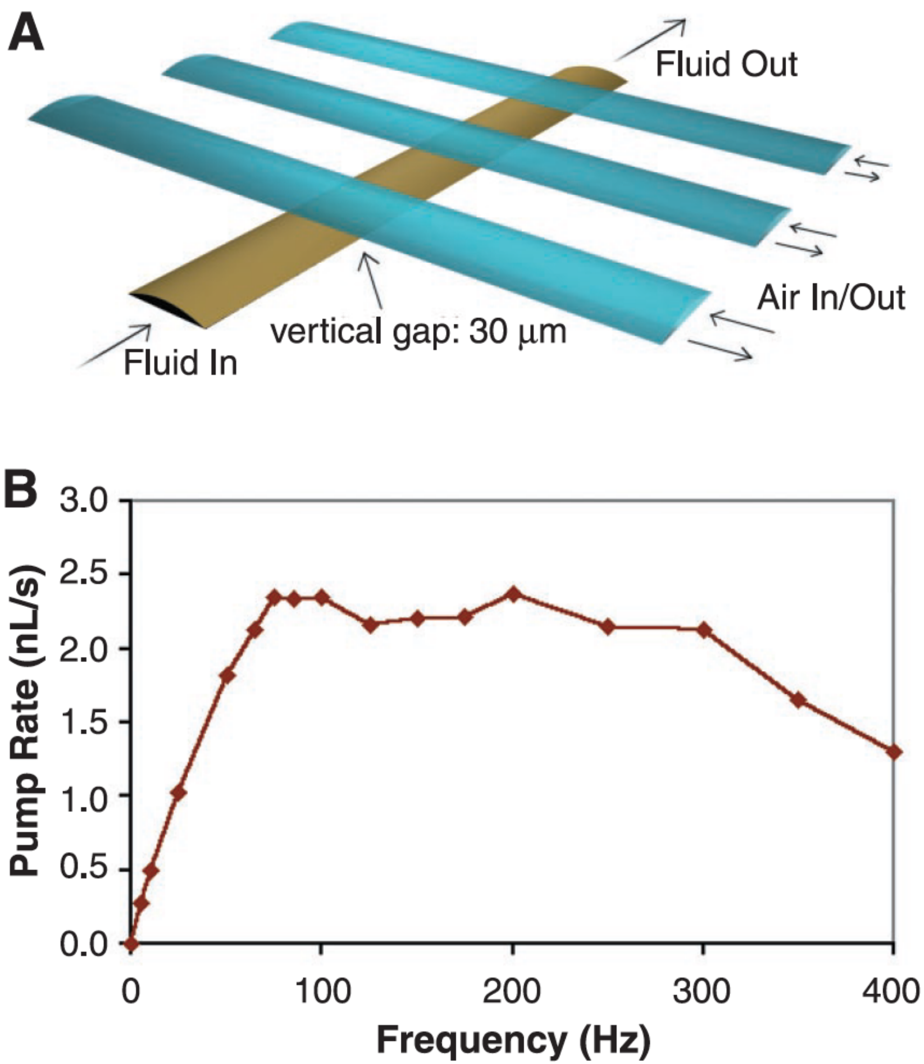
Pneumatic Pumps



Multiple control channels – alternating pressure in/out.

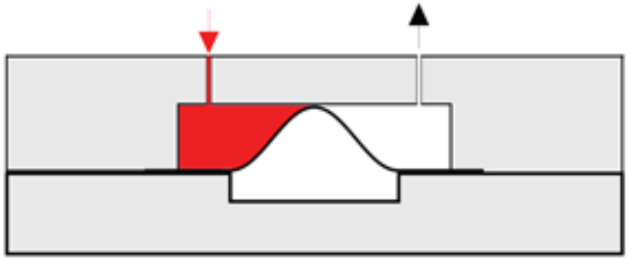
Peristaltic pumping:

- Frequency dependent flow rate

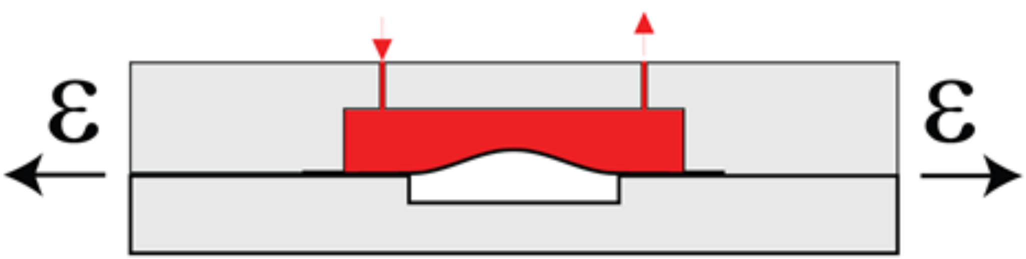
Unger, Marc A., Hou-Pu Chou, Todd Thorsen, Axel Scherer, and Stephen R. Quake. "Monolithic microfabricated valves and pumps by multilayer soft lithography." Science 288, no. 5463 (2000): 113-116.

Mechanical Valves

Flexible microfluidic device with single deformable arch.

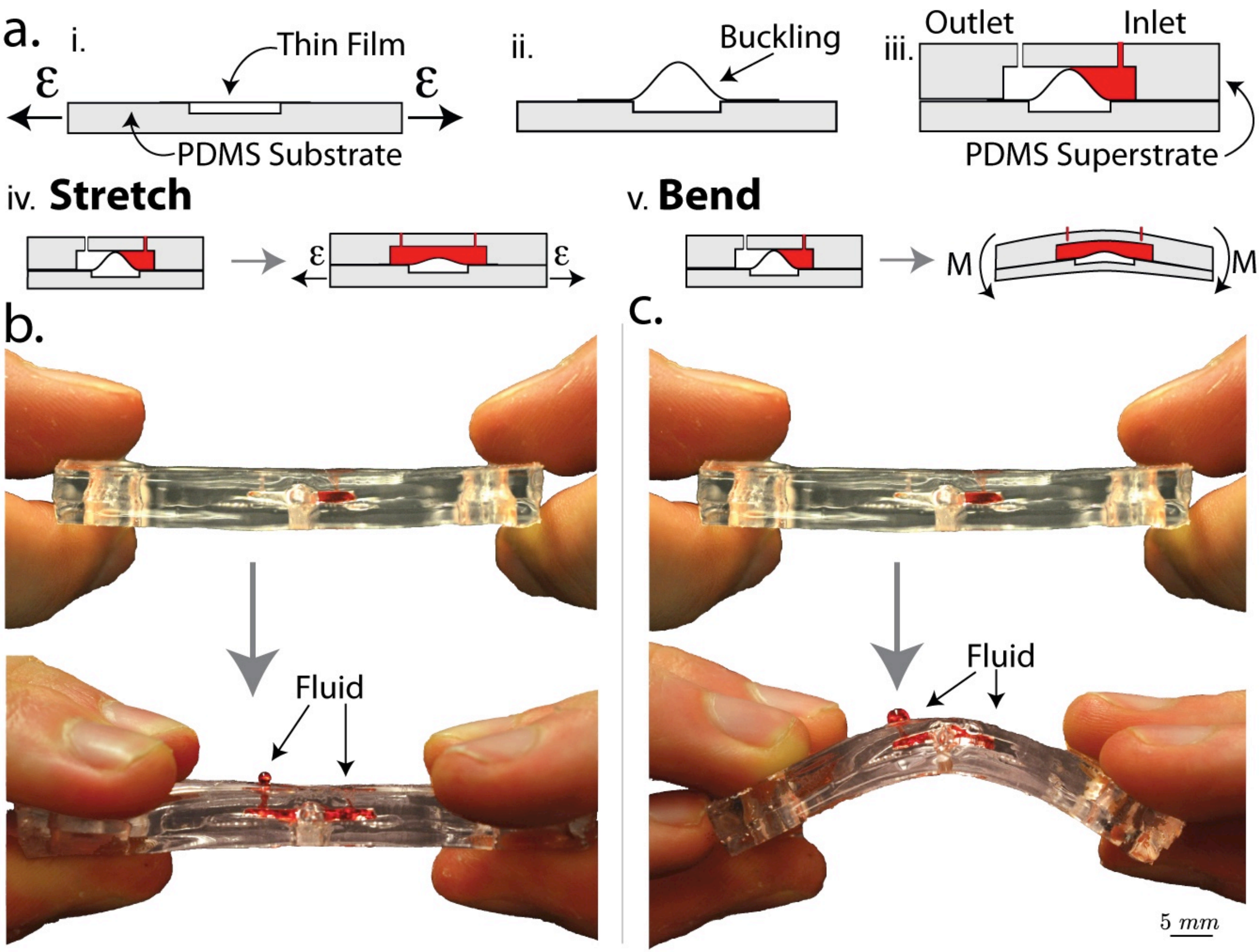


Stretching /bending the device reduces the arch height, partially opens the channel.



Douglas P. Holmes, Behrouz Tavakol, Guillaume Froehlicher, and Howard A. Stone. "Control and manipulation of microfluidic flow via elastic deformations." Soft Matter 9, no. 29 (2013): 7049-7053.

Mechanical Valves



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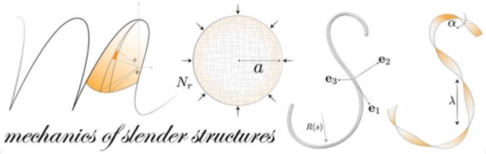
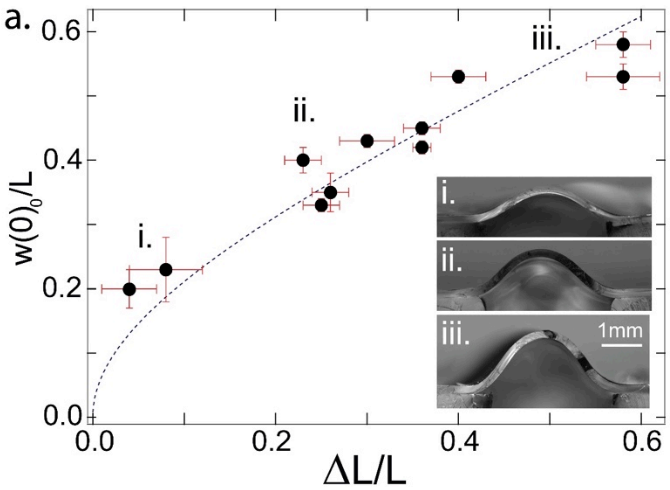


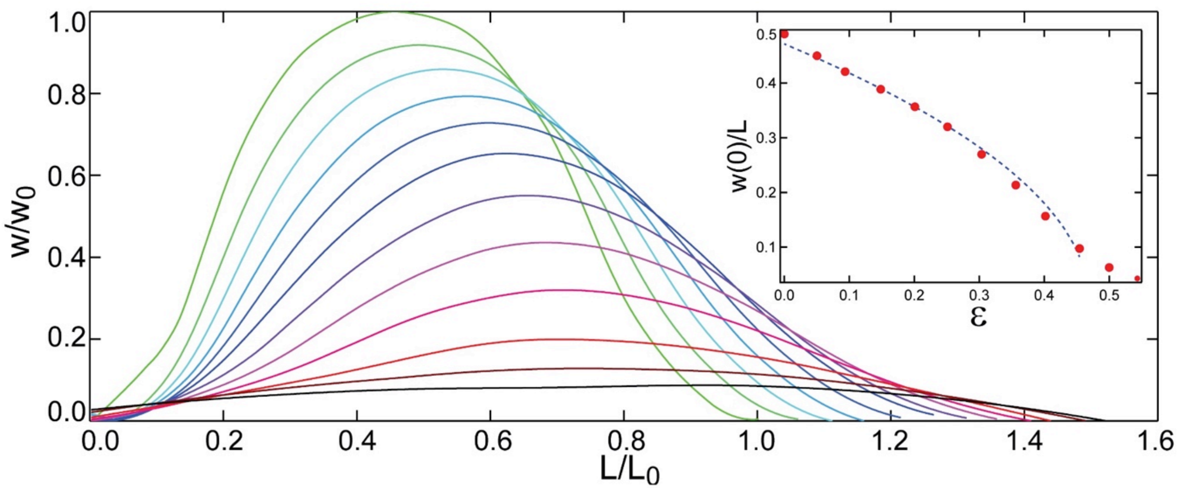
Mechanical Valves

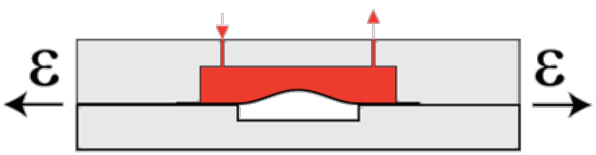


Buckled arch in microfluidic chamber.

Arch shape obtained from buckling analysis of elastica. Apex:

$$w(0) = \frac{2}{\pi}\sqrt{\Delta L_0 \left(L_0 + \Delta L_0\right)}$$





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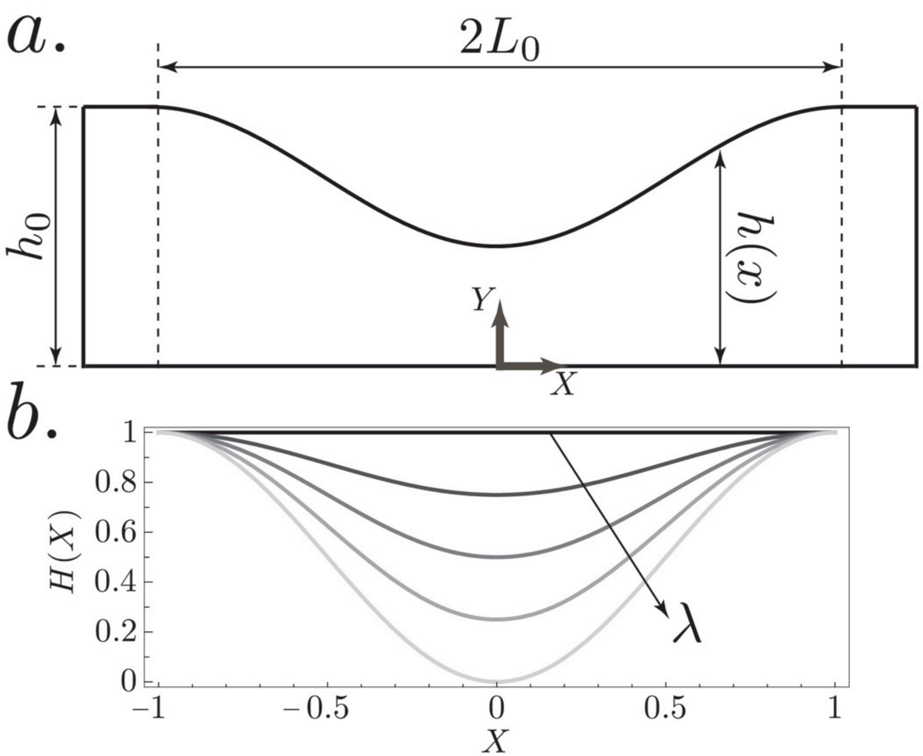
Mechanical Valves



Navier-Stokes Equations:

...conservation of momentum...

$$\overbrace{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right)}^{\text{Inertial acceleration}} = \overbrace{-\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}}^{\text{Forces}}$$

Continuity Equation:

...conservation of mass...

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

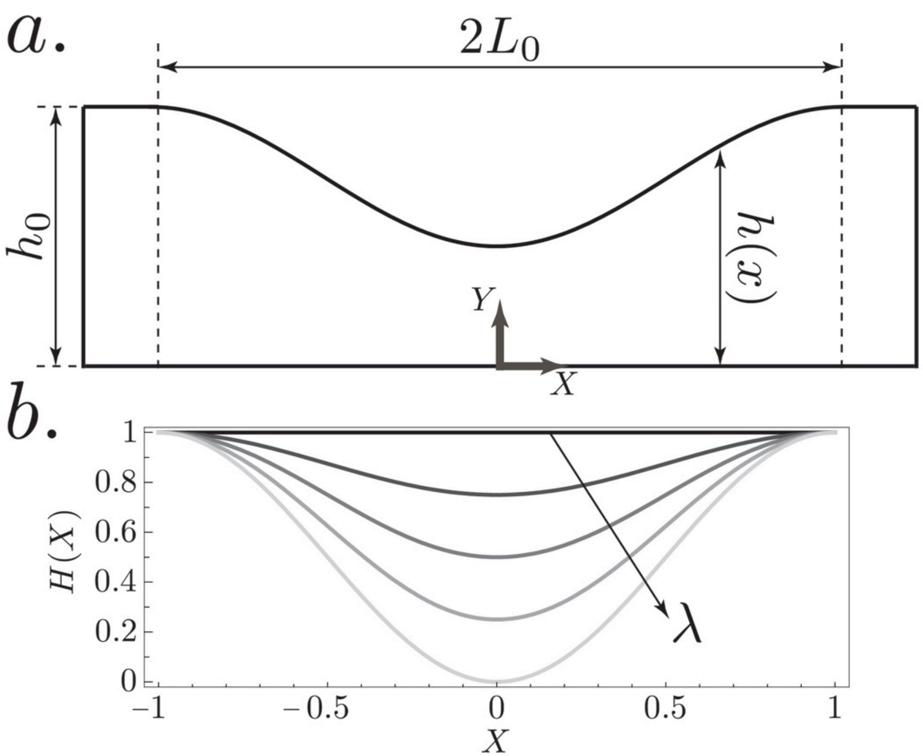
Stokes Equations:

$\mu\nabla^2\mathbf{u} = \nabla p$ Neglect inertia & body forces

$\nabla\cdot\mathbf{u} = 0$ Incompressible: $\delta\rho/\rho \ll 1$

B. Tavakol, G. Froehlicher, D.P. Holmes, and H.A. Stone. "Extended Lubrication Theory: Estimation of Fluid Flow in Channels with Variable Geometry," Under Review: Physics of Fluids, (2015).

Mechanical Valves



Shape of the impingement:

$$H(X) = 1 - \frac{\lambda}{2}\left[1 + \cos(\pi X)\right]$$

Stokes Equations:

$\mu \nabla^2 \mathbf{u} = \nabla p$ — Neglect inertia & body forces

$\nabla \cdot \mathbf{u} = 0$ — Incompressible: $\delta\rho/\rho \ll 1$

Dimensionless Parameters:

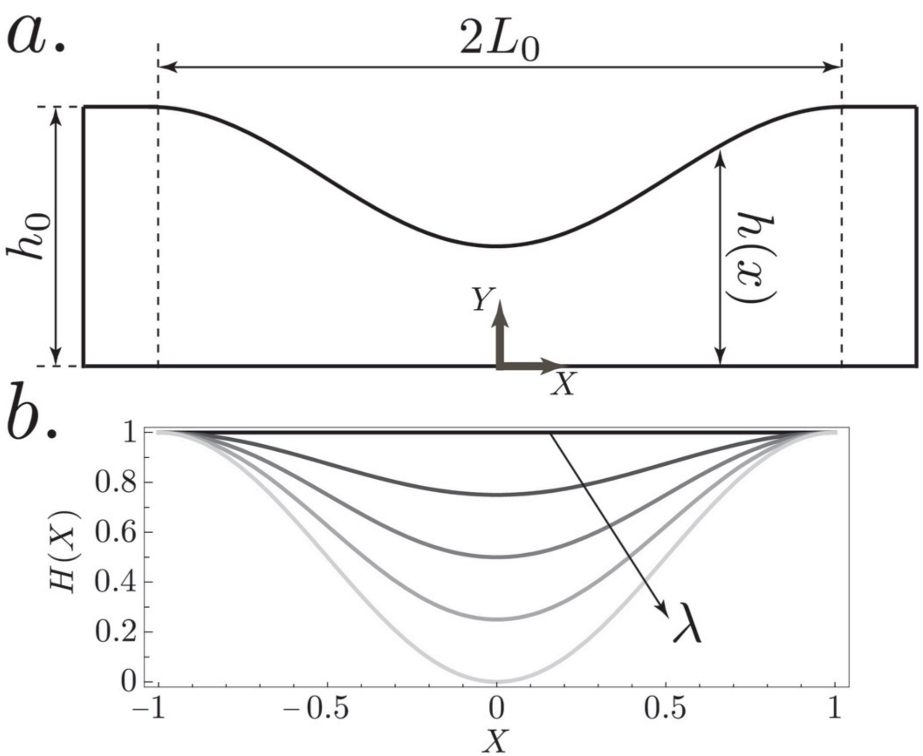
Lengths: $X = \dfrac{x}{L_0},\; Y = \dfrac{y}{h_0}$

Velocities: $U = \dfrac{u}{q_0/h_0},\; V = \dfrac{v}{q_0/L_0}$

Pressure: $P = \dfrac{p}{\Delta p} = \dfrac{p}{\mu q_0 L_0/h_0^3}$

B. Tavakol, G. Froehlicher, D.P. Holmes, and H.A. Stone. "Extended Lubrication Theory: Estimation of Fluid Flow in Channels with Variable Geometry," Under Review: Physics of Fluids, (2015).

Mechanical Valves



Shape of the impingement:

$$H(X) = 1 - \frac{\lambda}{2}\left[1 + \cos(\pi X)\right]$$

Dimensionless Equations:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

$$\delta^2 \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = \frac{\partial P}{\partial X}$$

$$\delta^4 \frac{\partial^2 V}{\partial X^2} + \delta^2 \frac{\partial^2 V}{\partial Y^2} = \frac{\partial P}{\partial Y}$$

Boundary Conditions:

$U = 0,\ V = 0 \text{ at } Y = 0 \text{ and } H(X)$ No slip

$\int_0^{H(X)} U(X,Y)\,\mathrm{d}Y = 1$ Total flow rate is prescribed

Perturbation Expansion:

$$U(X,Y;\delta) = U_0(X,Y) + \delta^2 U_2(X,Y) + \delta^4 U_4(X,Y) + \ldots$$

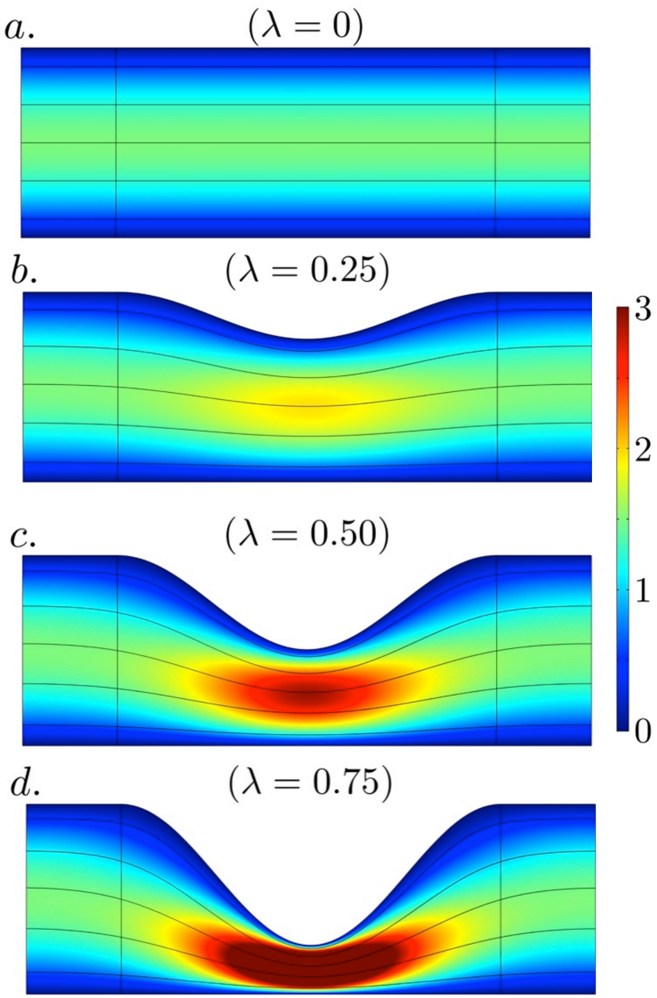
$$V(X,Y;\delta) = V_0(X,Y) + \delta^2 V_2(X,Y) + \delta^4 V_4(X,Y) + \ldots$$

$$P(X,Y;\delta) = P_0(X,Y) + \delta^2 P_2(X,Y) + \delta^4 P_4(X,Y) + \ldots$$

B. Tavakol, G. Froehlicher, D.P. Holmes, and H.A. Stone. "Extended Lubrication Theory: Estimation of Fluid Flow in Channels with Variable Geometry," Under Review: Physics of Fluids, (2015).

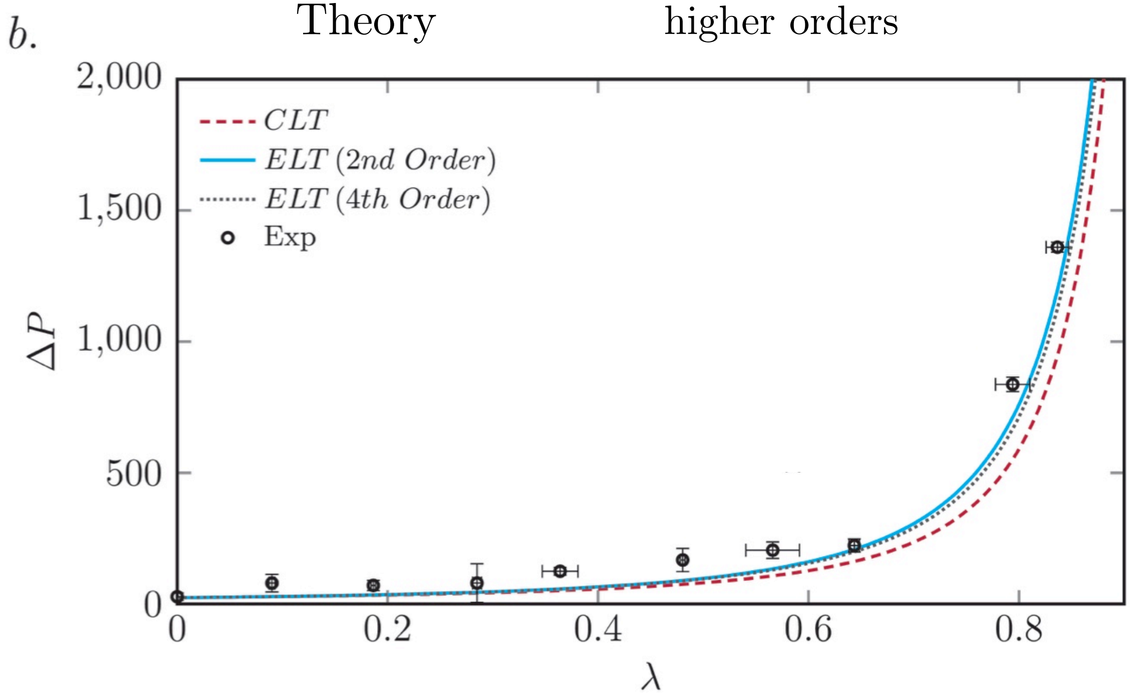
Extended Lubrication

Fluid flow through channels with variable **geometry**



Extended Lubrication Theory

$$\Delta P = \underbrace{\Delta P_0 \Big(1}_{\text{Lubrication Theory}} + \underbrace{\frac{4}{5}\lambda^2\delta^2 - \frac{64}{225}\lambda^4\delta^4 + \mathcal{O}(\delta^6)}_{\text{Perturbation to higher orders}} \Big)$$



B. Tavakol, G. Froehlicher, D.P. Holmes, and H.A. Stone. "Extended Lubrication Theory: Estimation of Fluid Flow in Channels with Variable Geometry," Under Review: Physics of Fluids, (2015).

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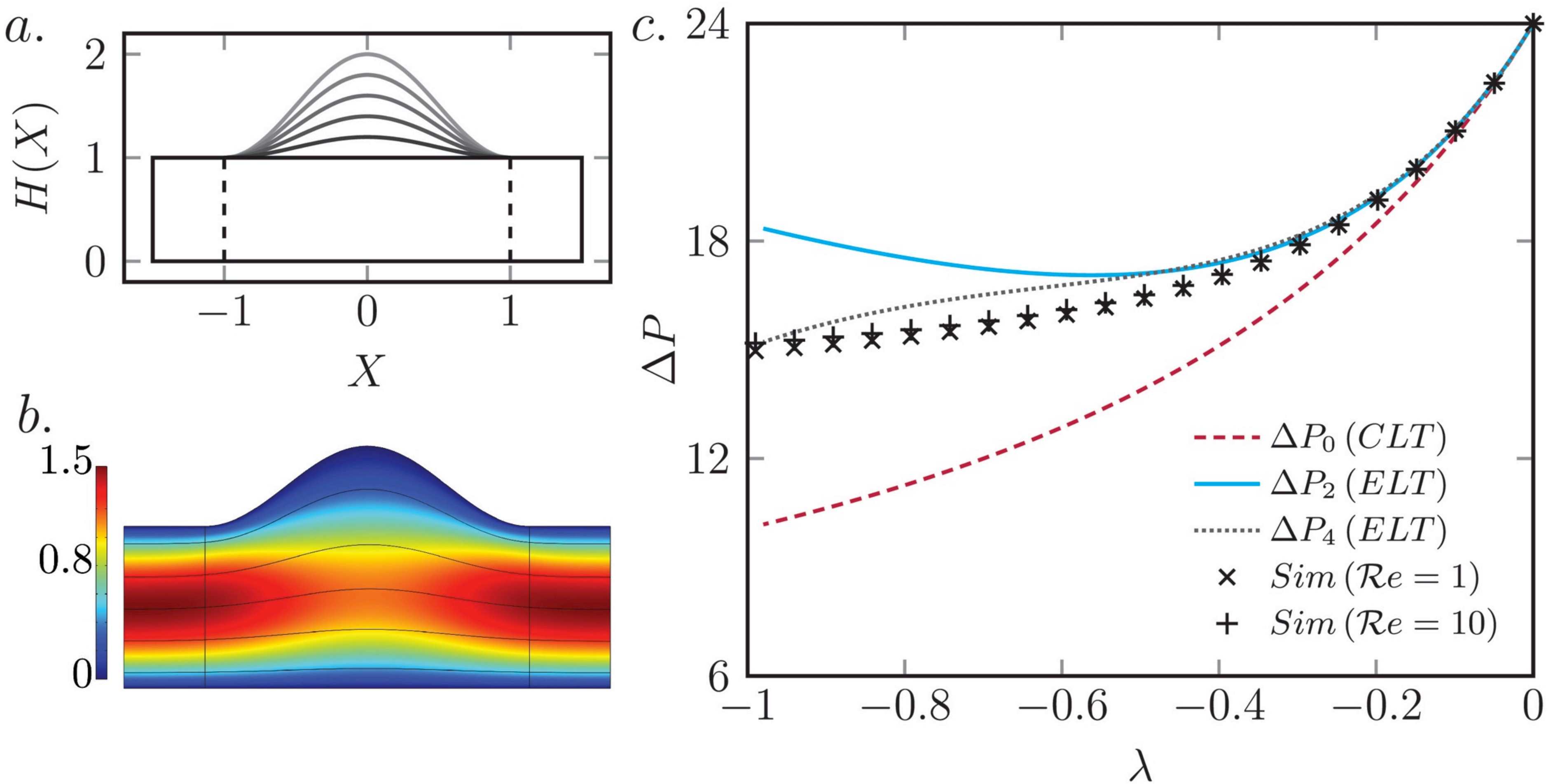
Extended Lubrication



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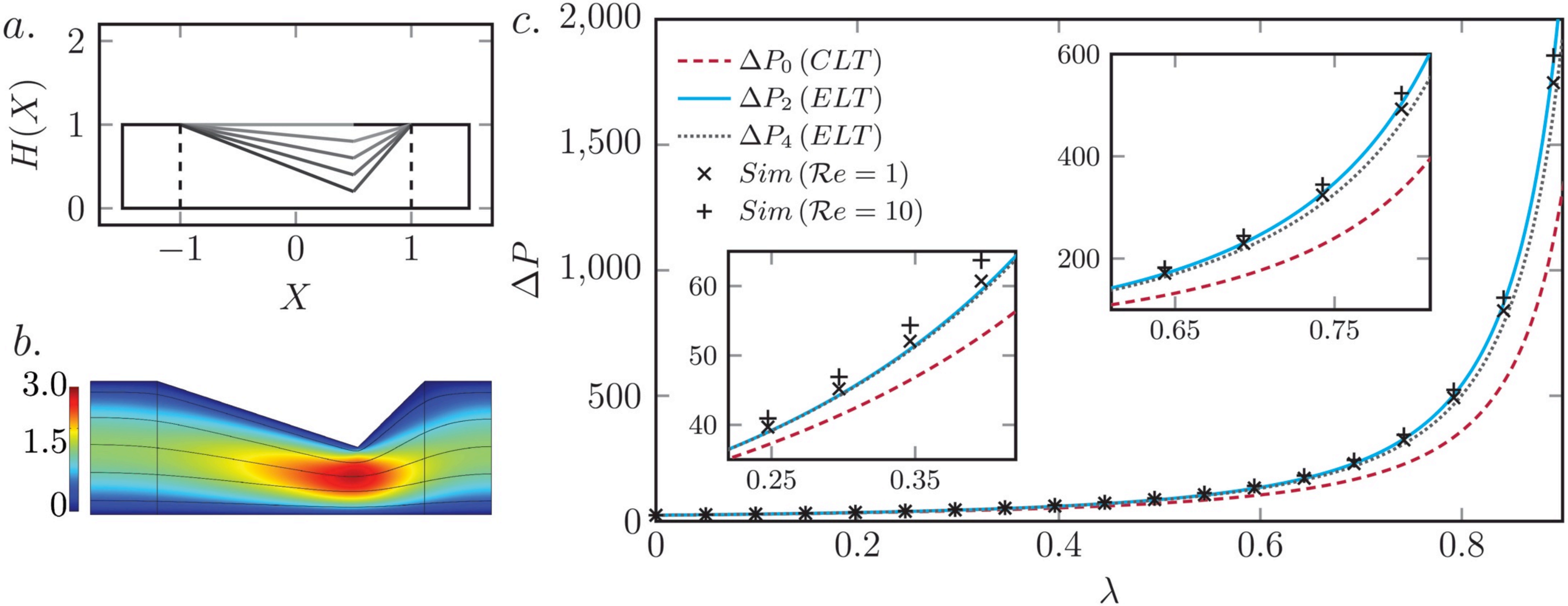
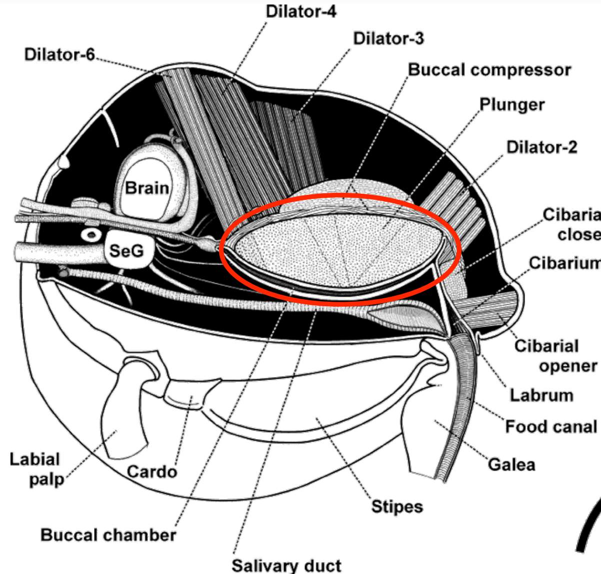
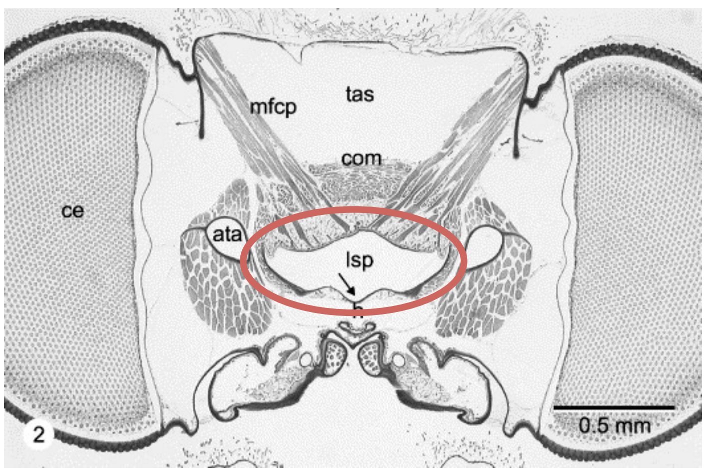
Extended Lubrication

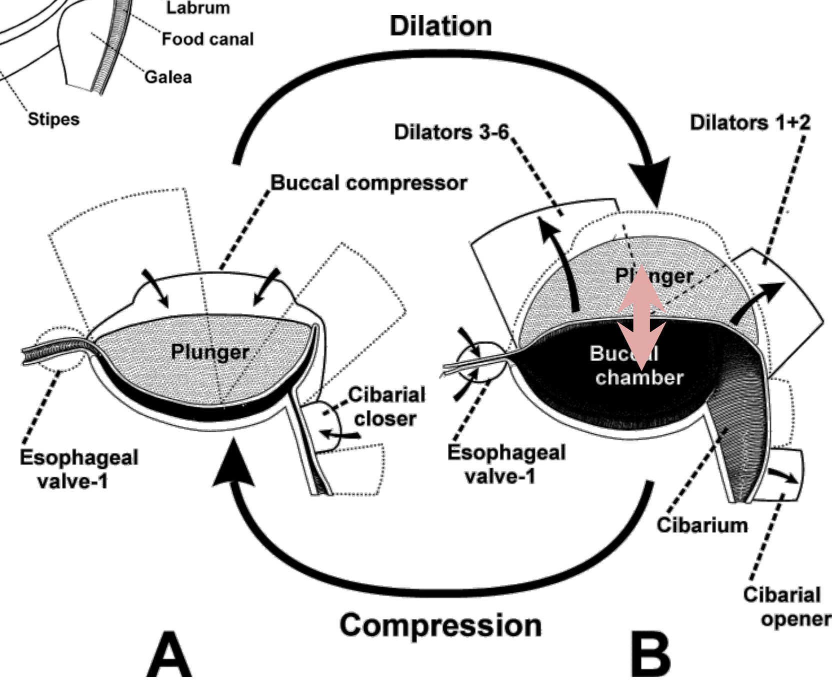


B. Tavakol, G. Froehlicher, D.P. Holmes, and H.A. Stone. "Extended Lubrication Theory: Estimation of Fluid Flow in Channels with Variable Geometry," Under Review: Physics of Fluids, (2015).

Bioinspiration







Large, **rapid**, **reversible** diaphragm **deformation** for fluid **pumping**.

Davis, N. T. and J. G. Hildebrand (2006). "Neuroanatomy of the sucking pump of the moth, Manduca sexta (Sphingidae, Lepidoptera)." Arthropod Structure & Development.

Eberhard, S. H. and H. W. Krenn (2005). "Anatomy of the oral valve in nymphalid butterflies and a functional model for fluid uptake in Lepidoptera." Zoologischer Anzeiger - A Journal of Comparative Zoology 243(4): 305-312.

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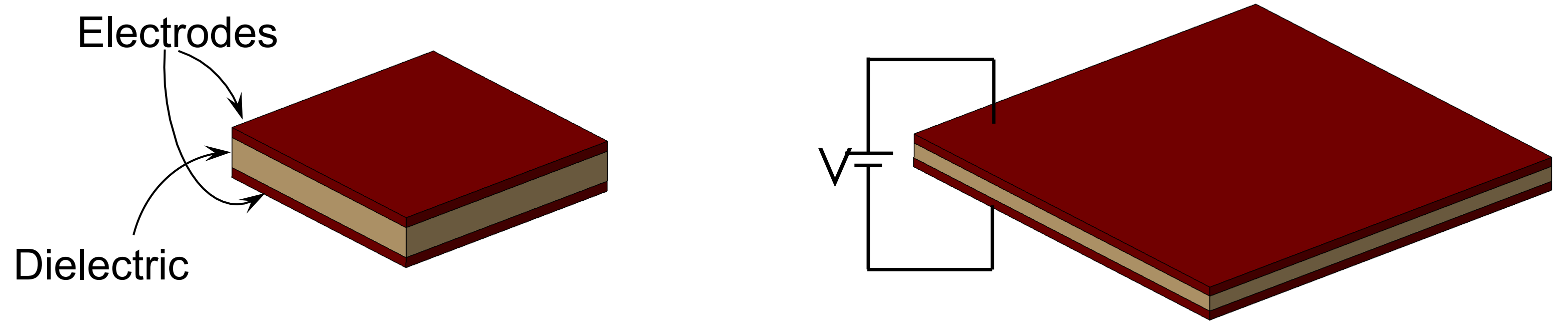
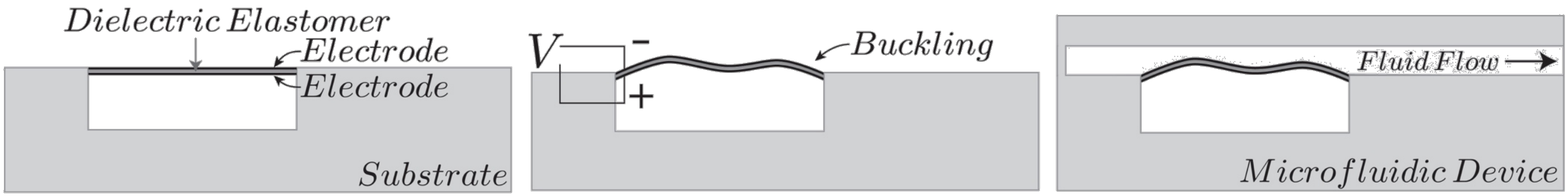
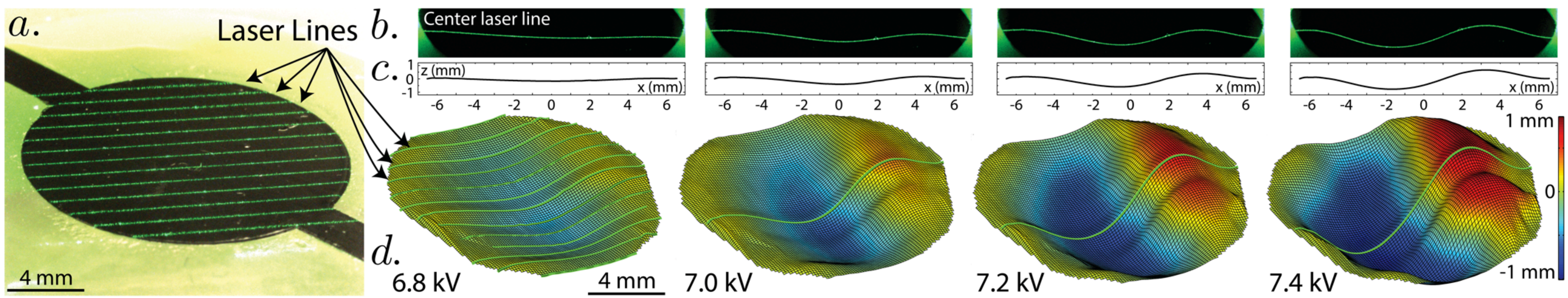
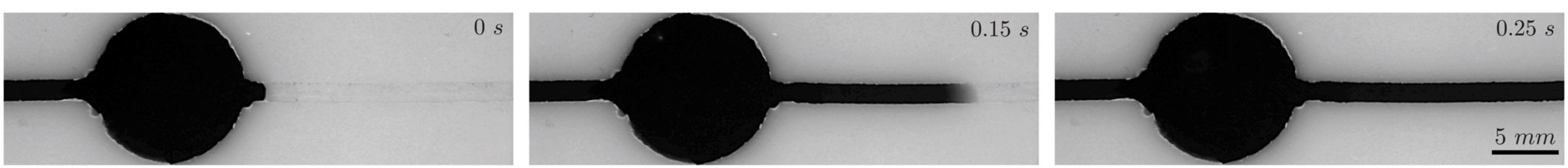
Electrical Valves



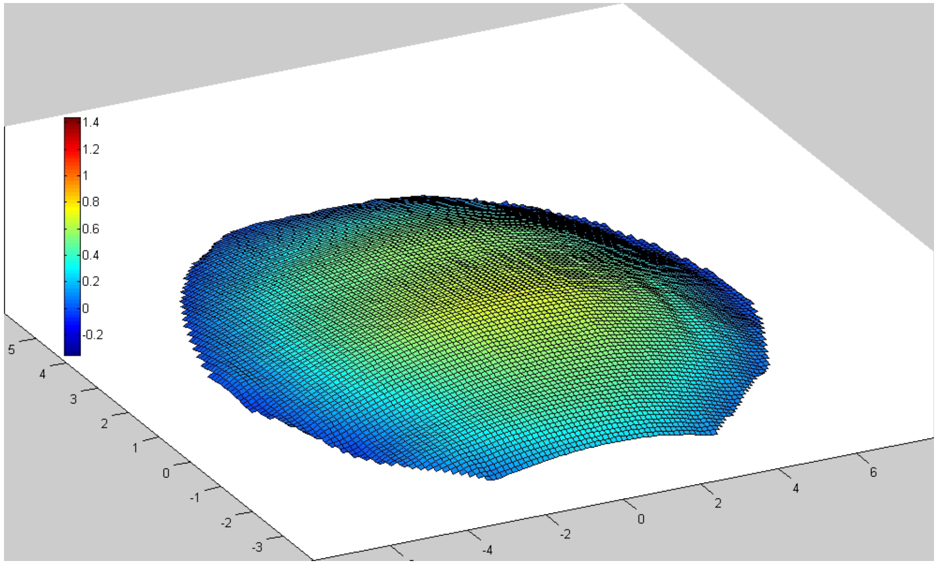


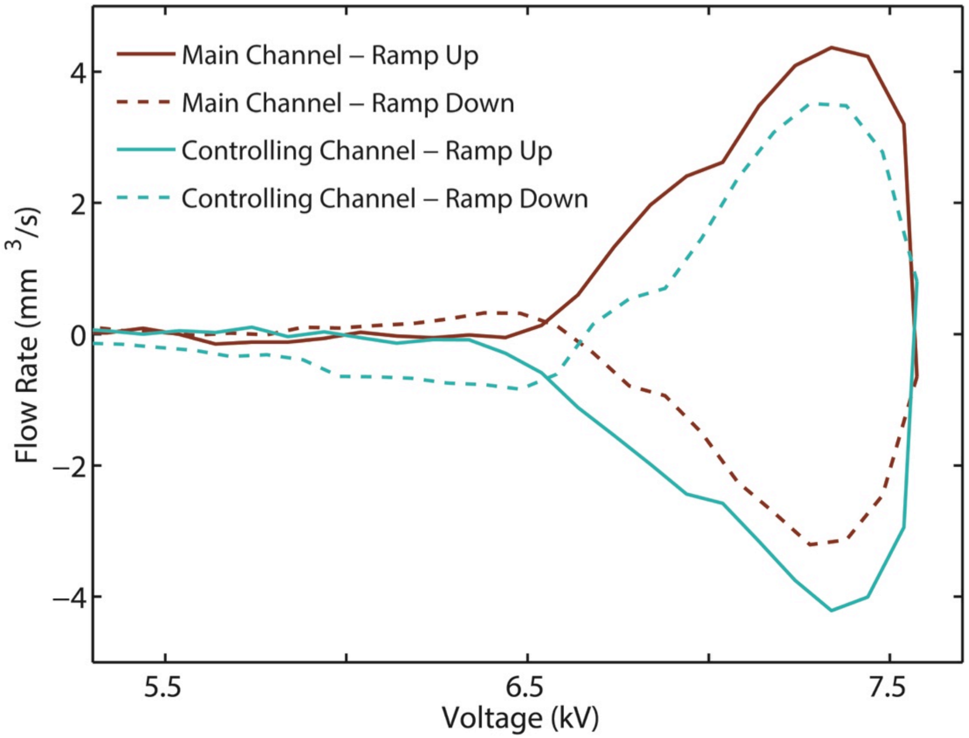




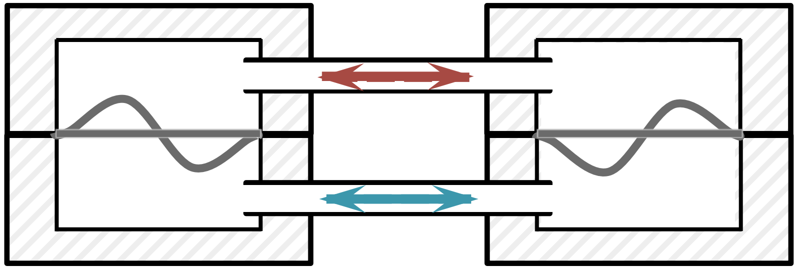
B. Tavakol, M. Bozlar, G. Froehlicher, H. A. Stone, I. Aksay, and D. P. Holmes, "Buckling Instabilities of Dielectric Elastomeric Plates for Flexible Microfluidic Pumps," Soft Matter, 10(27), 4789–4794, (2014).

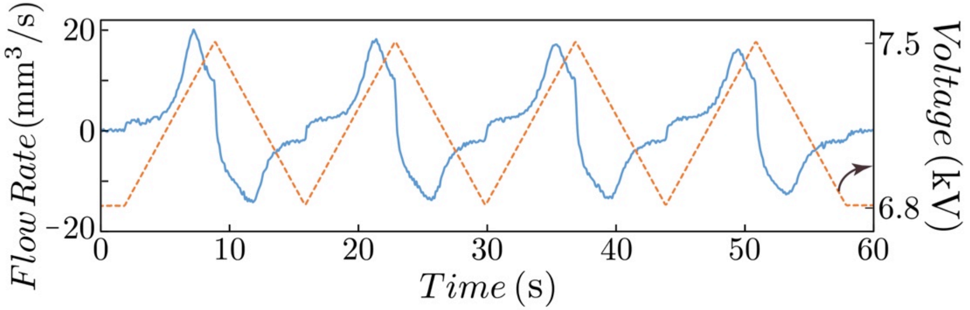
Electrical Valves





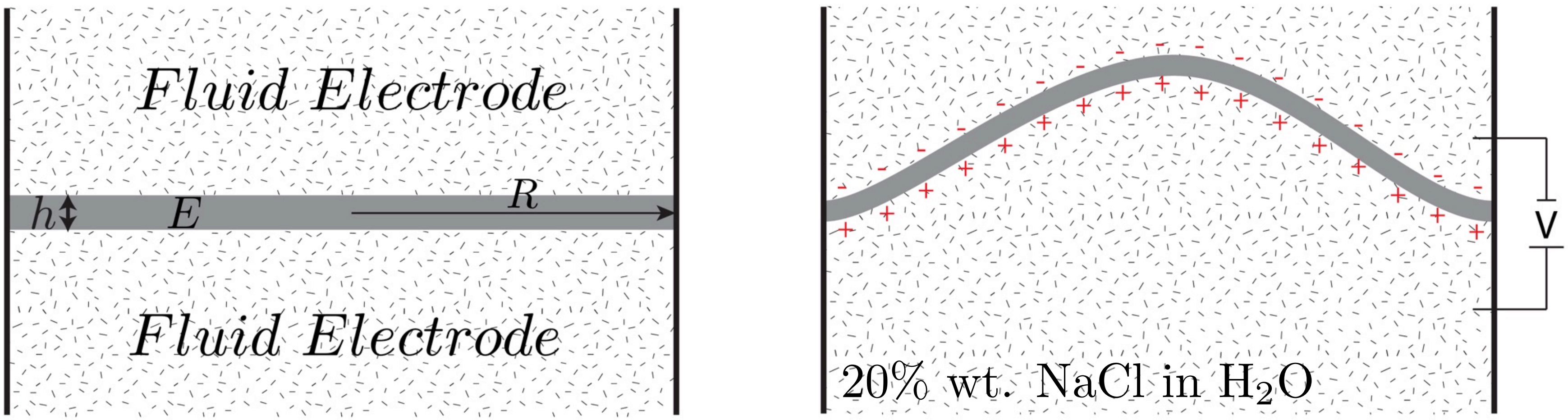
Coupled Chambers





B. Tavakol, M. Bozlar, G. Froehlicher, H. A. Stone, I. Aksay, and D. P. Holmes, "Buckling Instabilities of Dielectric Elastomeric Plates for Flexible Microfluidic Pumps," Soft Matter, 10(27), 4789–4794, (2014).

Fluid Electrodes



At what **voltage** will the plate **buckle**?

$$\varepsilon_r \approx \frac{\epsilon_0 \epsilon}{2E}\left(\frac{V}{h}\right)^2 \longrightarrow \sigma_r \approx \frac{\epsilon_0 \epsilon}{2(1-\nu^2)}\left(\frac{V}{h}\right)^2 \longrightarrow \sigma_c = \frac{kEh^2}{12(1-\nu^2)R^2}$$

Radial strain in a dielectric elastomer

Plane stress relation

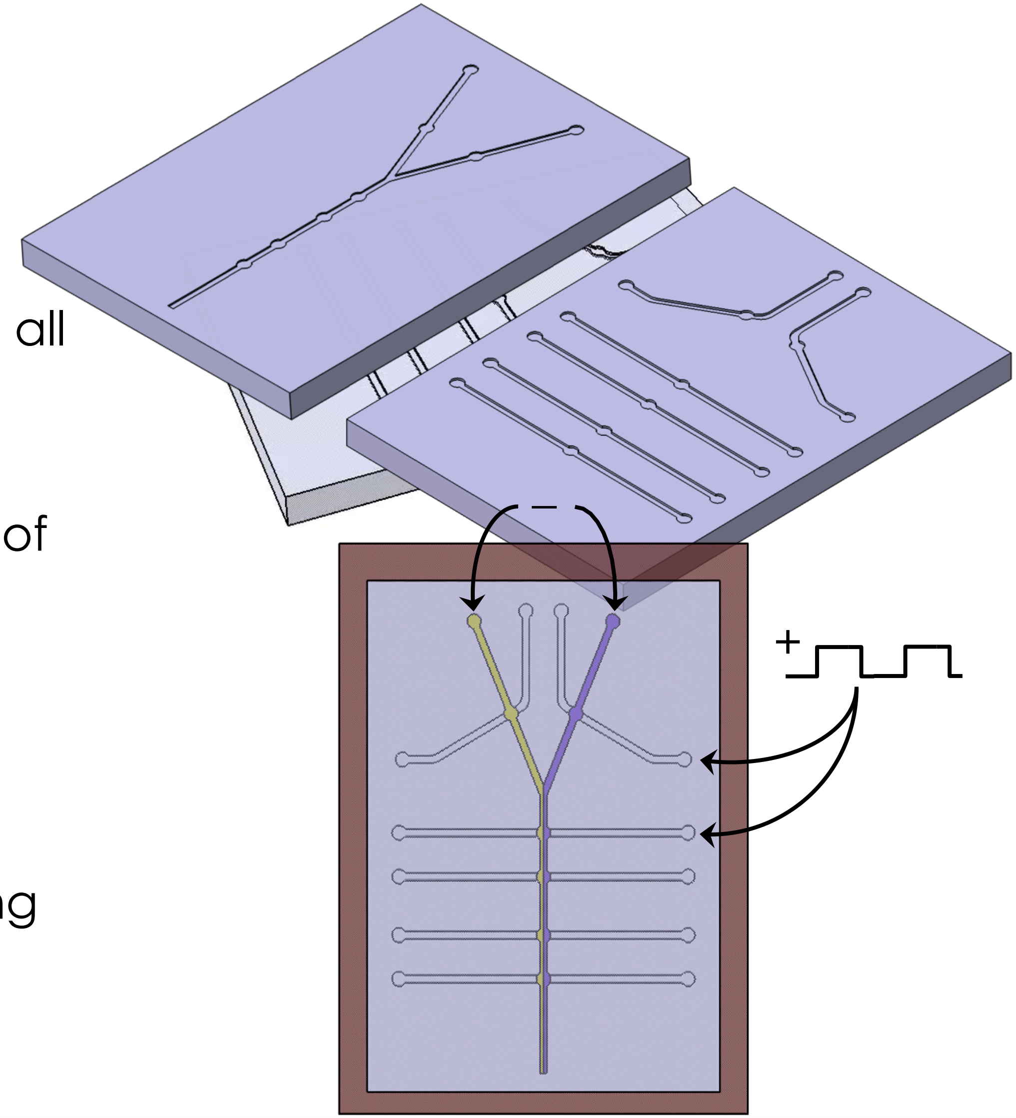
Linear stability analysis for a buckling plate

$$V_c \approx h^2 \left(\frac{kE}{6\epsilon_0 \epsilon R^2}\right)^{1/2}$$

B. Tavakol, A. Chawan, and D. P. Holmes, "Buckling Instability of Thin Films as a Means to Control or Enhance Fluid Flow within Microchannels," in preparation, (2015).

Microfluidic Fabrication

1. Prepare the **top** substrate with the **microfluidic** channel.
2. Prepare the **bottom** substrate with all **controlling** channels.
3. Make the thin, **dielectric film.**
4. Bond the substrates on both sides of the dielectric film.
   - Film will **buckle** at the intersection between channels.
5. Fill the channels with conductive fluids.
6. Apply a voltage to induce buckling



B. Tavakol, A. Chawan, and D. P. Holmes, "Buckling Instability of Thin Films as a Means to Control or Enhance Fluid Flow within Microchannels," in preparation, (2015).

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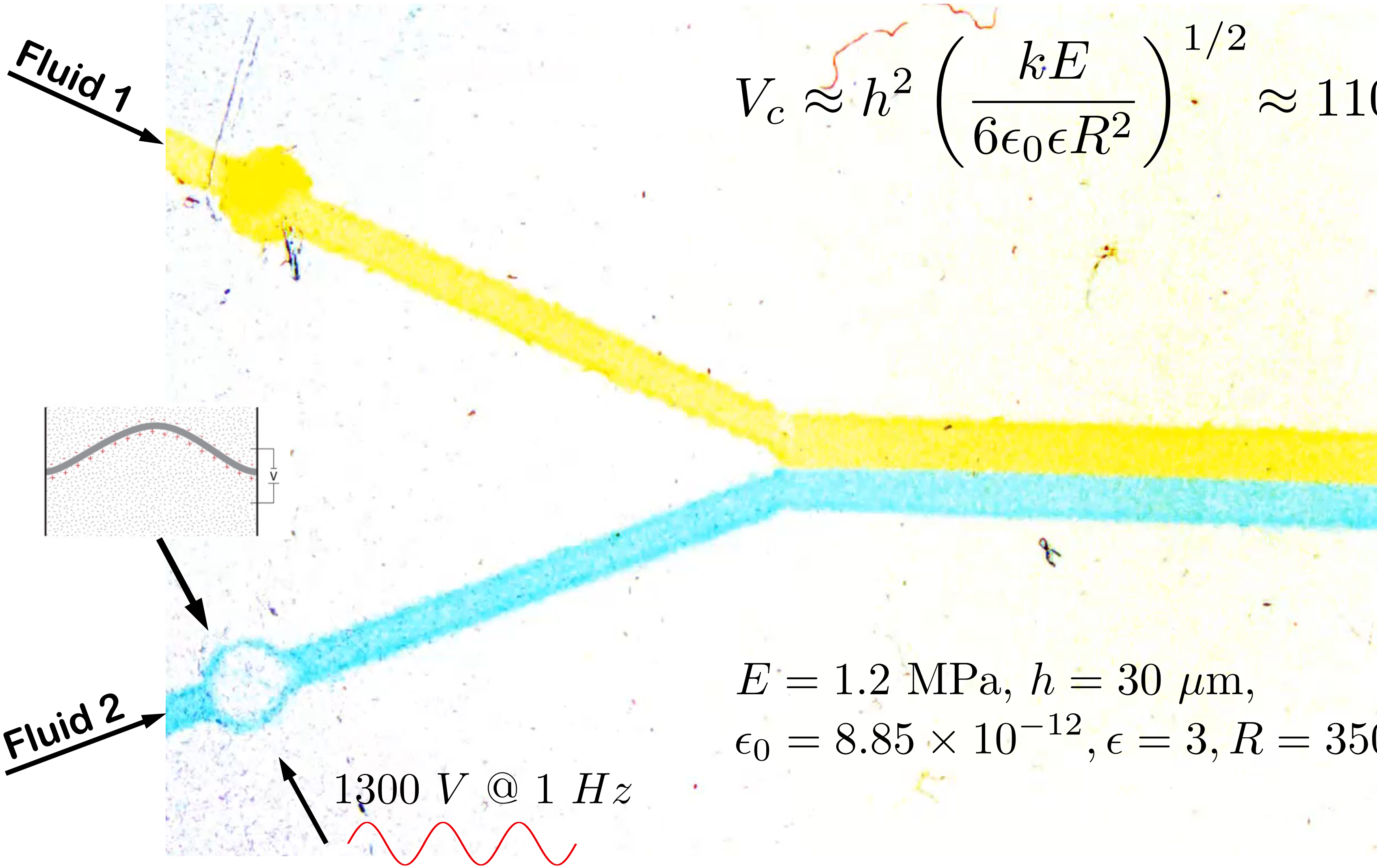
Valving

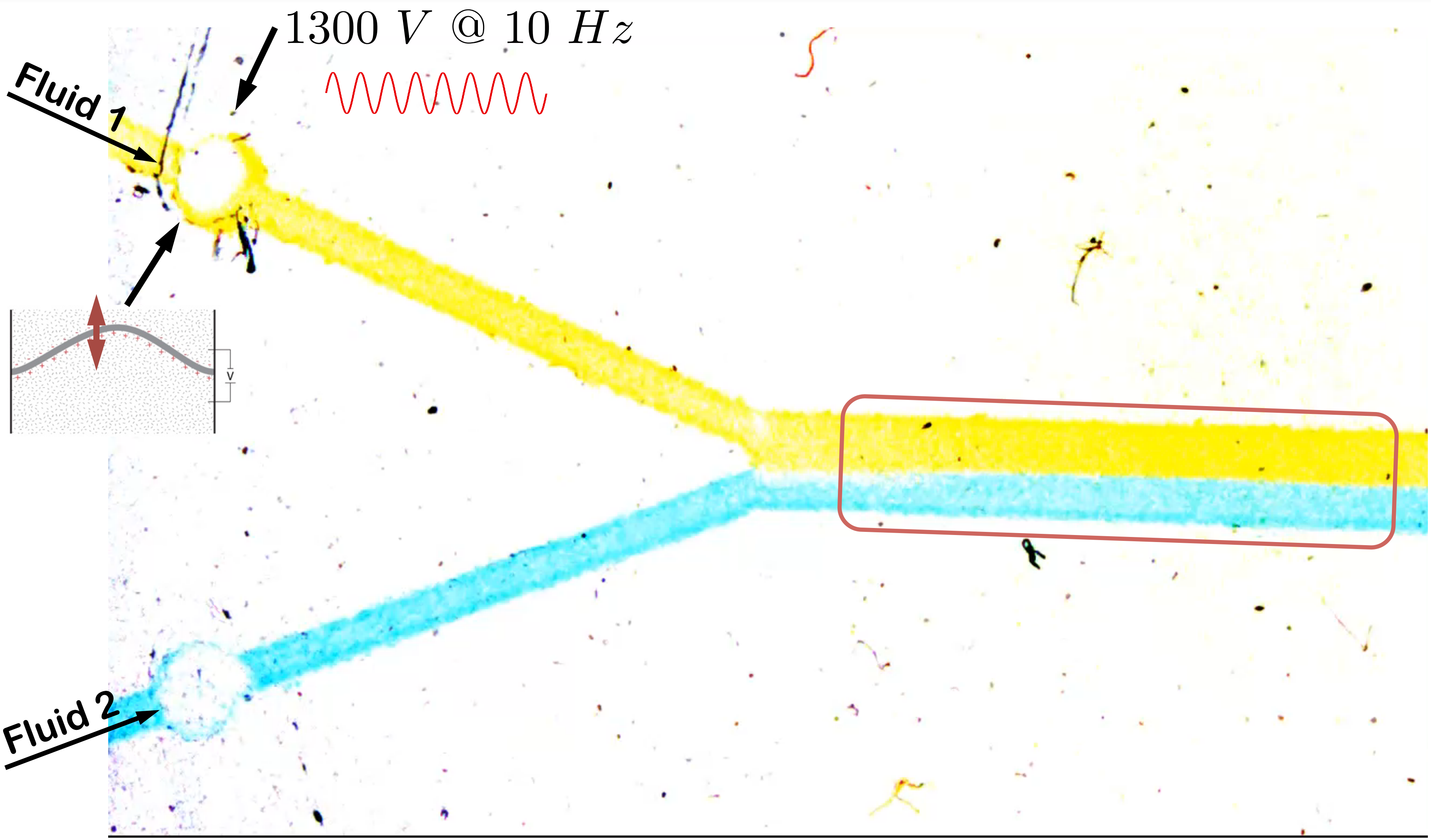
$$V_c \approx h^2 \left(\frac{kE}{6\epsilon_0 \epsilon R^2} \right)^{1/2} \approx 1100\ V$$



$E = 1.2$ MPa, $h = 30\ \mu$m,
$\epsilon_0 = 8.85 \times 10^{-12}, \epsilon = 3, R = 350\ \mu$m

B. Tavakol, A. Chawan, and D. P. Holmes, "Buckling Instability of Thin Films as a Means to Control or Enhance Fluid Flow within Microchannels," in preparation, (2015).

Pumping



B. Tavakol, A. Chawan, and D. P. Holmes, "Buckling Instability of Thin Films as a Means to Control or Enhance Fluid Flow within Microchannels," in preparation, (2015).

Confined Fluid Flow: Microfluidics and Capillarity

Reynolds Number: Inertia vs. Viscous effects
- Review of characteristic flows…

Péclet Number: Transport phenomena in a continuum
- Diffusion, separation, and mixing…

Geometric confinement: Controlling and manipulating fluid flow
- Microfluidic fabrication, valving, pumping…

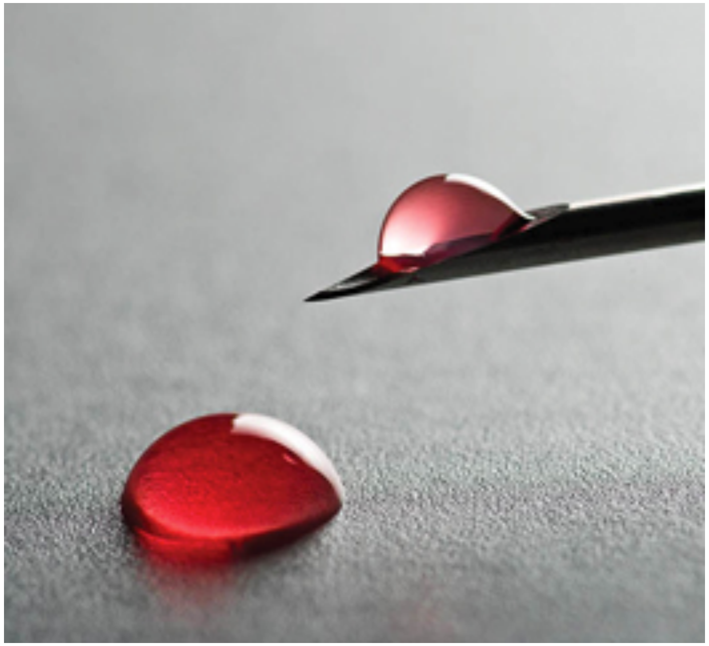
Capillary Number: Viscosity vs. Surface tension
- Droplet formation, capillary rise, elasticity…

Fluid Behavior

Viscous Forces

Interfacial Forces

Coiling honey



Wetting of water on a textured surface

Capillary Number: viscous/interfacial

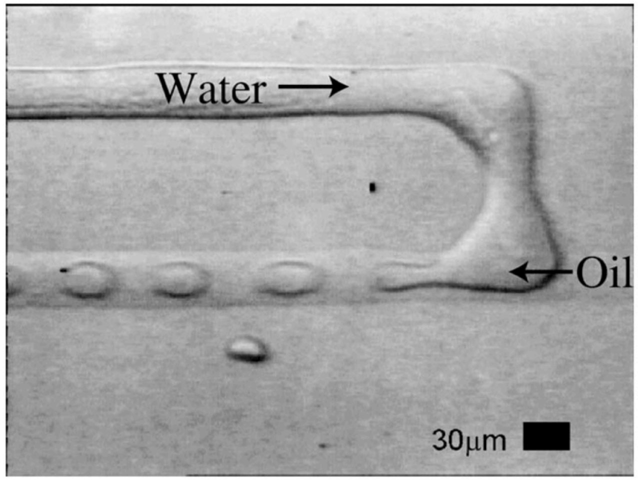
Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
Droplet: http://www.rycobel.be/en/technical-info/articles/1337/measuring-dynamic-absorption-and-wetting

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial



Monodisperse droplet generation

- Droplet emulsions in immiscible fluids
- Injection of water into stream of oil

Interfacial tension prevents the fluids from flowing alongside each other.

Surface tension acts to reduce the interfacial area.

$$\sigma_c \sim \gamma / R$$

Viscous stresses act to extend and drag the interface downstream.

$$\sigma_v \sim \mu U_0 / h$$

Characteristic droplet size:

$$R \sim \frac{\gamma}{\mu U_0} h = \frac{h}{\mathscr{C}}$$

Capillary number:

$$\mathscr{C} \equiv \frac{\mu U_0}{\gamma}$$

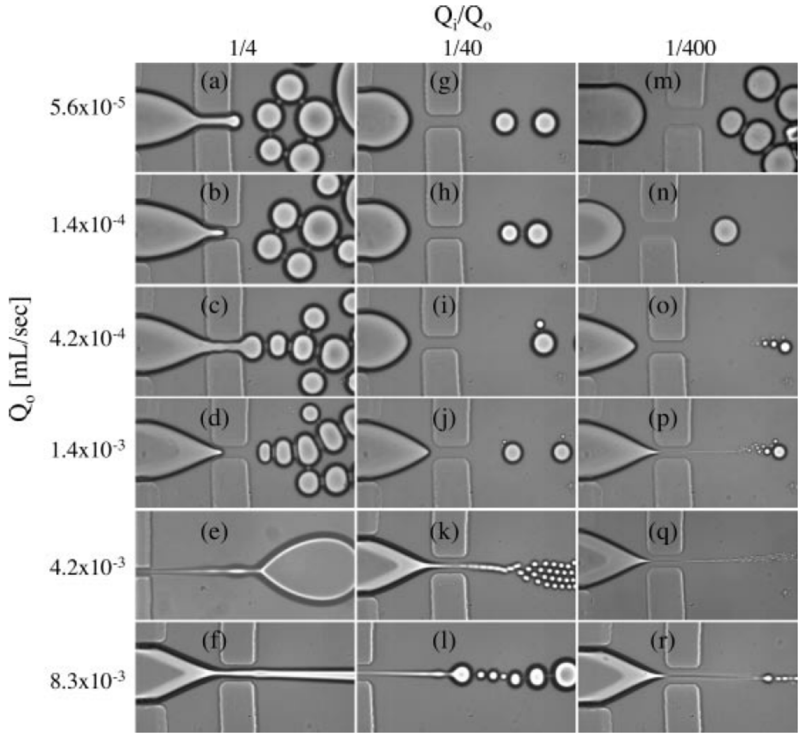
Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
Thorsen, Todd, et al. "Dynamic pattern formation in a vesicle-generating microfluidic device." Physical review letters 86.18 (2001): 4163.
Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
Droplet: http://www.rycobel.be/en/technical-info/articles/1337/measuring-dynamic-absorption-and-wetting

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial



Drop formation in a flow-focusing configuration

Water drops are formed in a silicone oil continuous phase.

Q_i is the flow rate of water
Q_o is the flow rate of oil

Characteristic droplet size:

$$R \sim \frac{\gamma}{\mu U_0} h = \frac{h}{\mathscr{C}}$$

Stone, Howard A., Abraham D. Stroock, and Armand Ajdari. "Engineering flows in small devices: microfluidics toward a lab-on-a-chip." Annu. Rev. Fluid Mech. 36 (2004): 381-411.

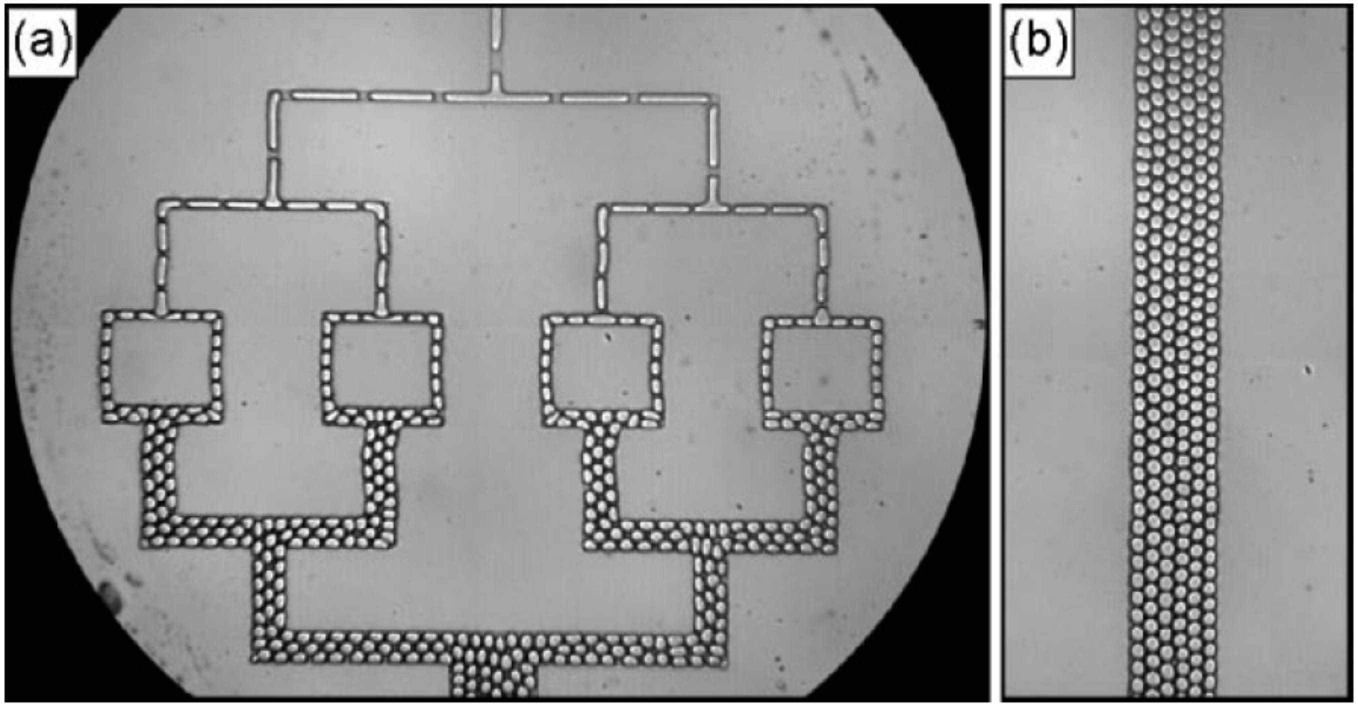
Anna et al. 2003

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial



Sequential drop breakup at T-junctions

- Generally, it is difficult to form small drops at high-volume fraction.
- This "passive" route allows size reduction after fabrication.
- The ratio of two daughter droplets depends on the lengths of the arms off the T-junction.

Stone, Howard A., Abraham D. Stroock, and Armand Ajdari. "Engineering flows in small devices: microfluidics toward a lab-on-a-chip." Annu. Rev. Fluid Mech. 36 (2004): 381-411.

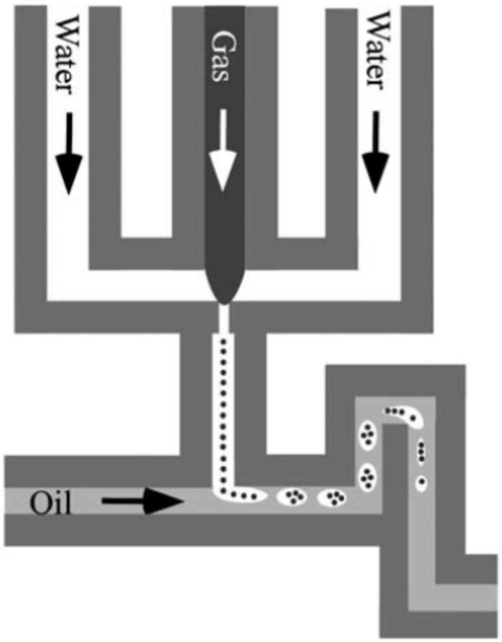
Link et al. 2003

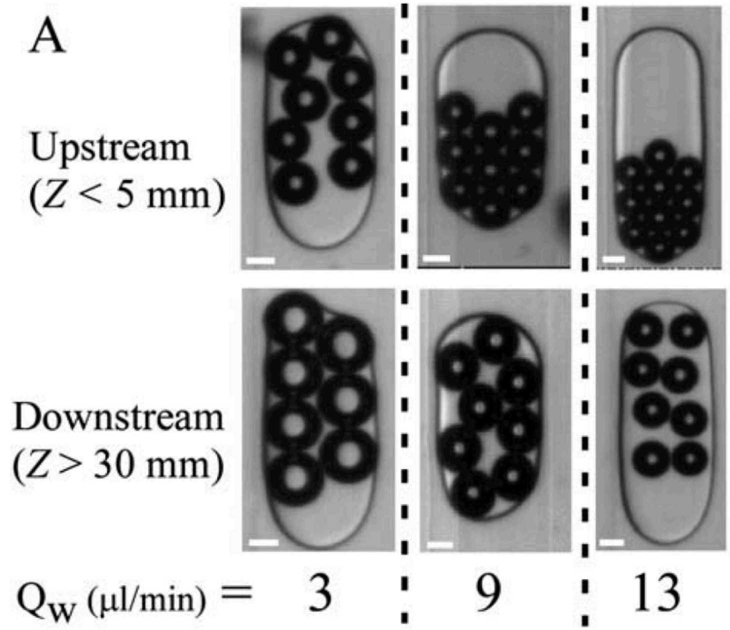
Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial





Jiandi Wan and Howard A. Stone. "Microfluidic generation of a high volume fraction of bubbles in droplets." Soft Matter 6, no. 19 (2010): 4677-4680.
Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
Droplet: http://www.rycobel.be/en/technical-info/articles/1337/measuring-dynamic-absorption-and-wetting

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Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial

Large **surface-to-volume** ratios in microfluidic devices

- Makes surface effects increasingly important.
- Important when free fluid surfaces are present.

Surface tensions can exert significant stress

- Result in free surface deformations.
- Can drive fluid motion.

Capillary forces tend to draw fluid into wetting microchannels

- Occurs when **solid-liquid** interfacial **energy** is **lower** than the **solid-gas** interfacial **energy**.

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
Droplet: http://www.rycobel.be/en/technical-info/articles/1337/measuring-dynamic-absorption-and-wetting

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial

Capillary forces tend to draw fluid into wetting microchannels

Dynamics of surface tension-driven intrusion into a pipe of radius w occurs from **balance** of **capillary** and **viscous forces**.

Curved meniscus at fluid-gas interface – Laplace pressure

$$\Delta p \sim \Delta\gamma / w$$

Difference in surface energies (solid-liquid to solid-gas)

$$\Delta\gamma = \gamma_{sl} - \gamma_{sg}$$

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial

Capillary forces tend to draw fluid into wetting microchannels

Dynamics of surface tension-driven intrusion into a pipe of radius w occurs from **balance** of **capillary** and **viscous forces**.

Stoke's flow:

$$\mu \nabla^2 \mathbf{u} = \nabla p$$

Scaling of the flow rate:

$$u \sim \frac{\Delta P w^2}{\mu L}$$

From the pressure – surface tension relationship:

$$u \sim \frac{\Delta \gamma w}{\mu L}$$

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial

Capillary forces tend to draw fluid into wetting microchannels

From the pressure – surface tension relationship:

$$u \sim \frac{\Delta\gamma w}{\mu L}$$

Capillary number determines the dynamics:

$$\mathscr{C} \sim w/z$$

The column length changes as the fluid front moves: ($u = \partial z / \partial t$)

- Fluid invades the channel at an ever slowing rate:

$$z \sim \left(\frac{\Delta\gamma w}{\mu} t\right)^{1/2}$$

Washburn equation

Washburn, 1921
Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial

Surface patterning can create "wall-less" microchannels which confine fluids.

Chemically treat a surface to make some areas hydrophilic and others hydrophobic.

Spreading along hydrophilic **stripe** is similar to the **Washburn** analysis, however the height of the fluid stripe:

$$h \sim w^2 / R_r$$

- R_r is the radius of curvature of the inlet reservoir drop.

$$z \sim \left(\frac{\Delta \gamma w^4}{\mu R_r^3} t \right)^{1/2}$$

- Spreading requires: $\mathscr{C} \ll 1$

Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.
Gau et al., 1999
Darhuber et al., 2001

Fluid Behavior

Viscous Forces

Interfacial Forces

Capillary Number: viscous/interfacial

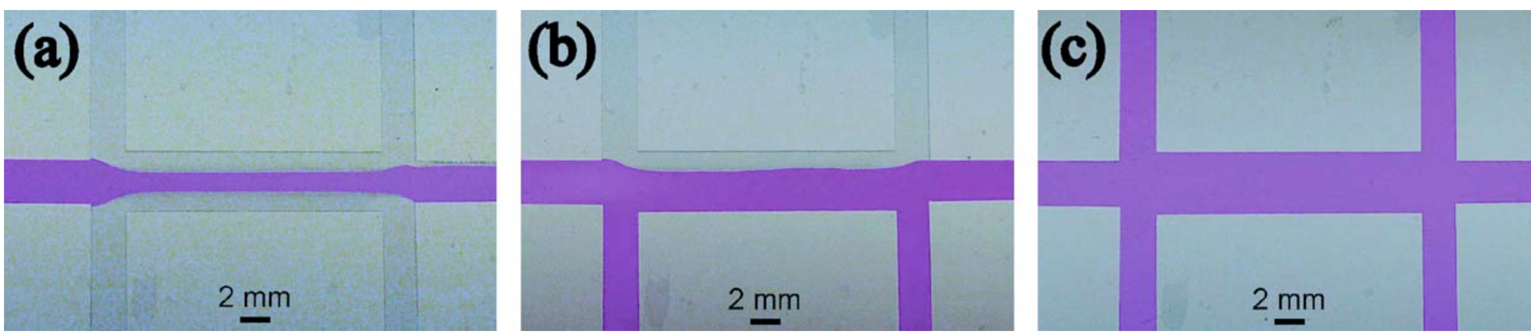
Chemically treat walls of **microfluidic channel** to make pressure-sensitive **valves**.

Pressure sensitive pumping

- From pressure drop across fluid channel (different from Quake valve)
- Fluid is confined to hydrophilic region if:

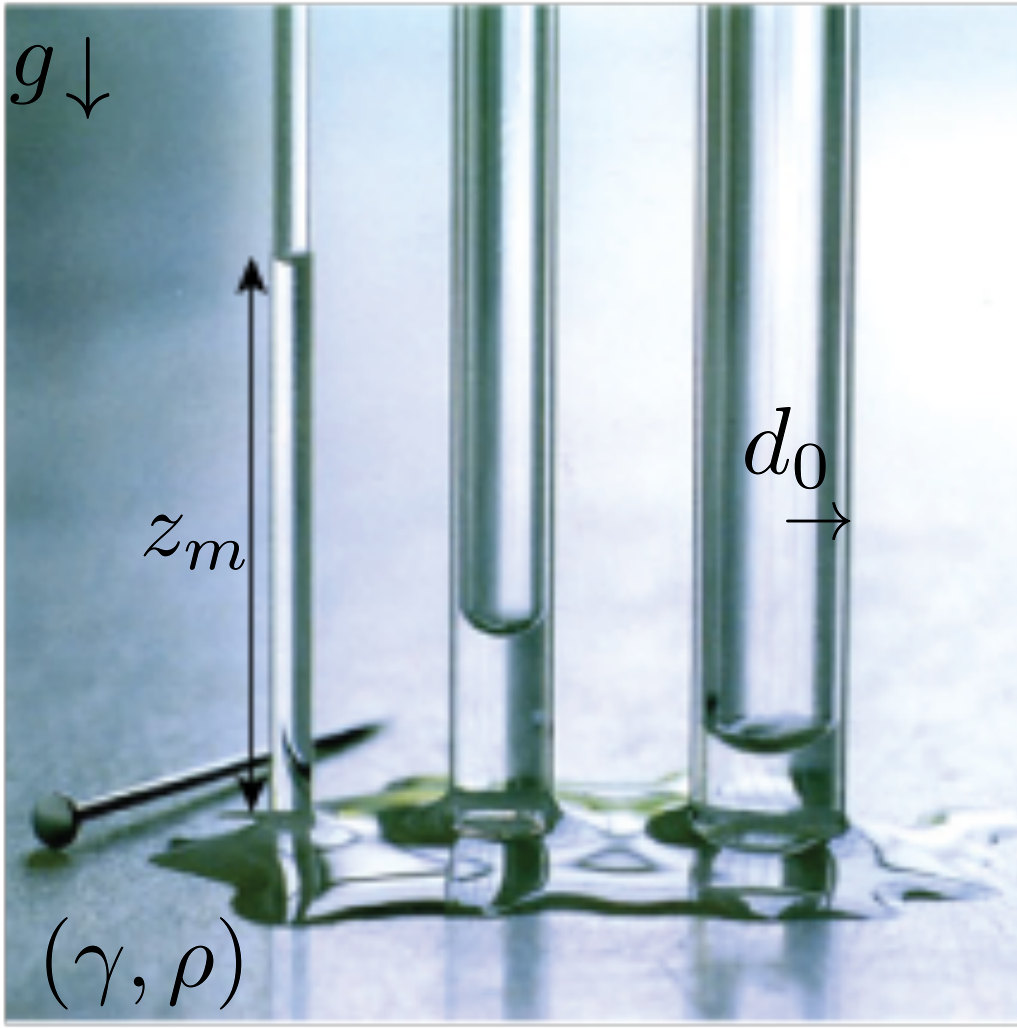
$$\Delta p \leq \gamma / w$$

Channel is patterned with hydrophilic stripe (center), weakly hydrophobic surface (bottom channels), and strongly hydrophobic (top channels).



Zhao et al., 2001
Squires, Todd M., and Stephen R. Quake. "Microfluidics: Fluid physics at the nanoliter scale." Reviews of modern physics 77.3 (2005): 977.

Capillary Rise



Classical Problem:

- Noted as early as 15th by Leonardo da Vinci.
- Attributed to circulation in plants in 17th century by Montanari.

$$U = \underbrace{2\pi\gamma\cos\theta_e r z_m}_{surface\ energy} + \underbrace{\frac{1}{2}\rho g \pi r^2 z_m^2}_{gravitational\ P.E.}$$

$$\frac{dU}{dz_m} = 0$$

$$\ell_{cg} \sim \frac{\gamma\cos\theta_e}{\rho g d_0}$$

Balance: Surface Tension & Gravity

J.M. Bell and F.K. Cameron, "The flow of liquids through capillary spaces," J. Phys. Chem. **10**, 658-674, (1906).

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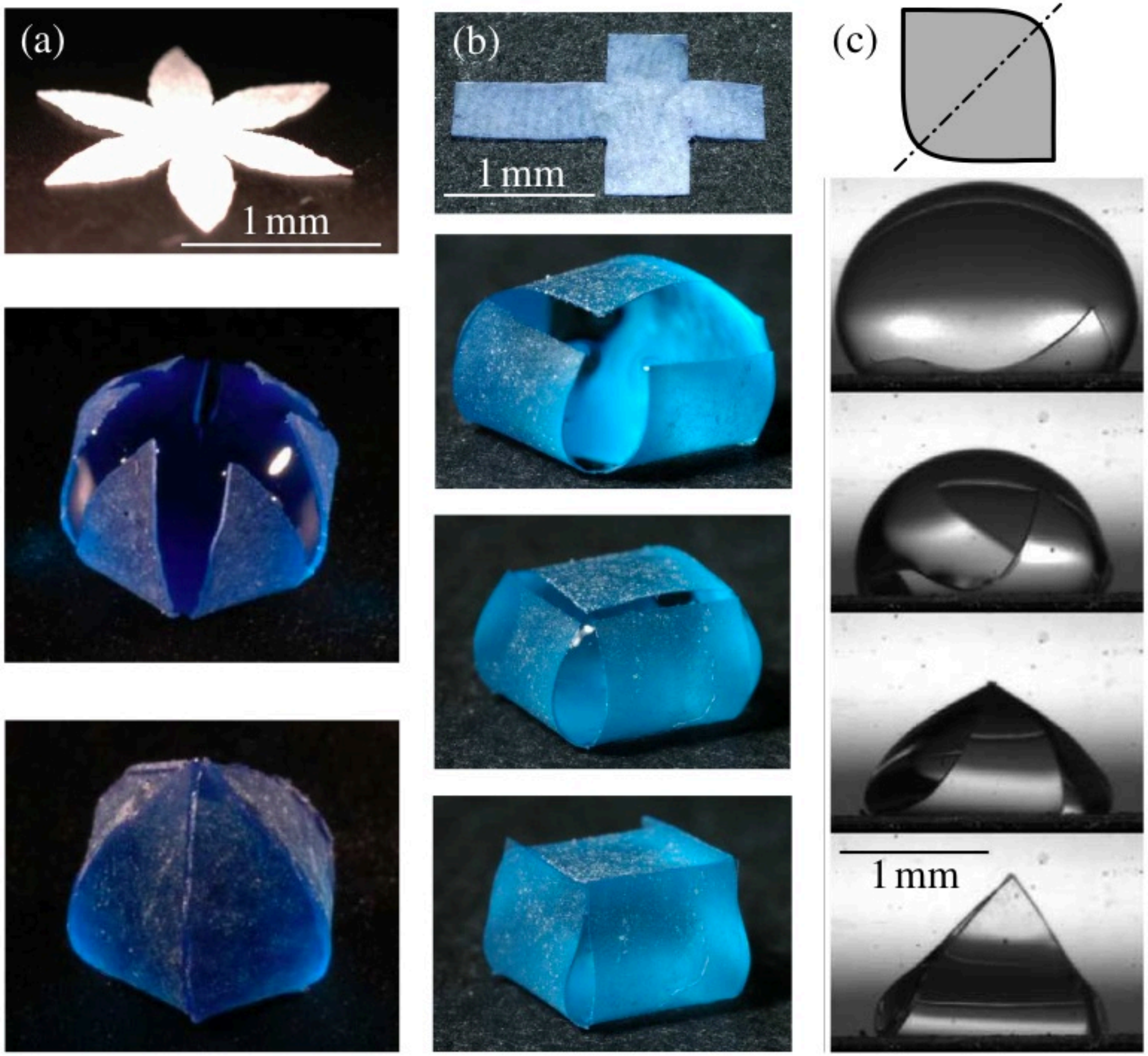
Elastocapillarity

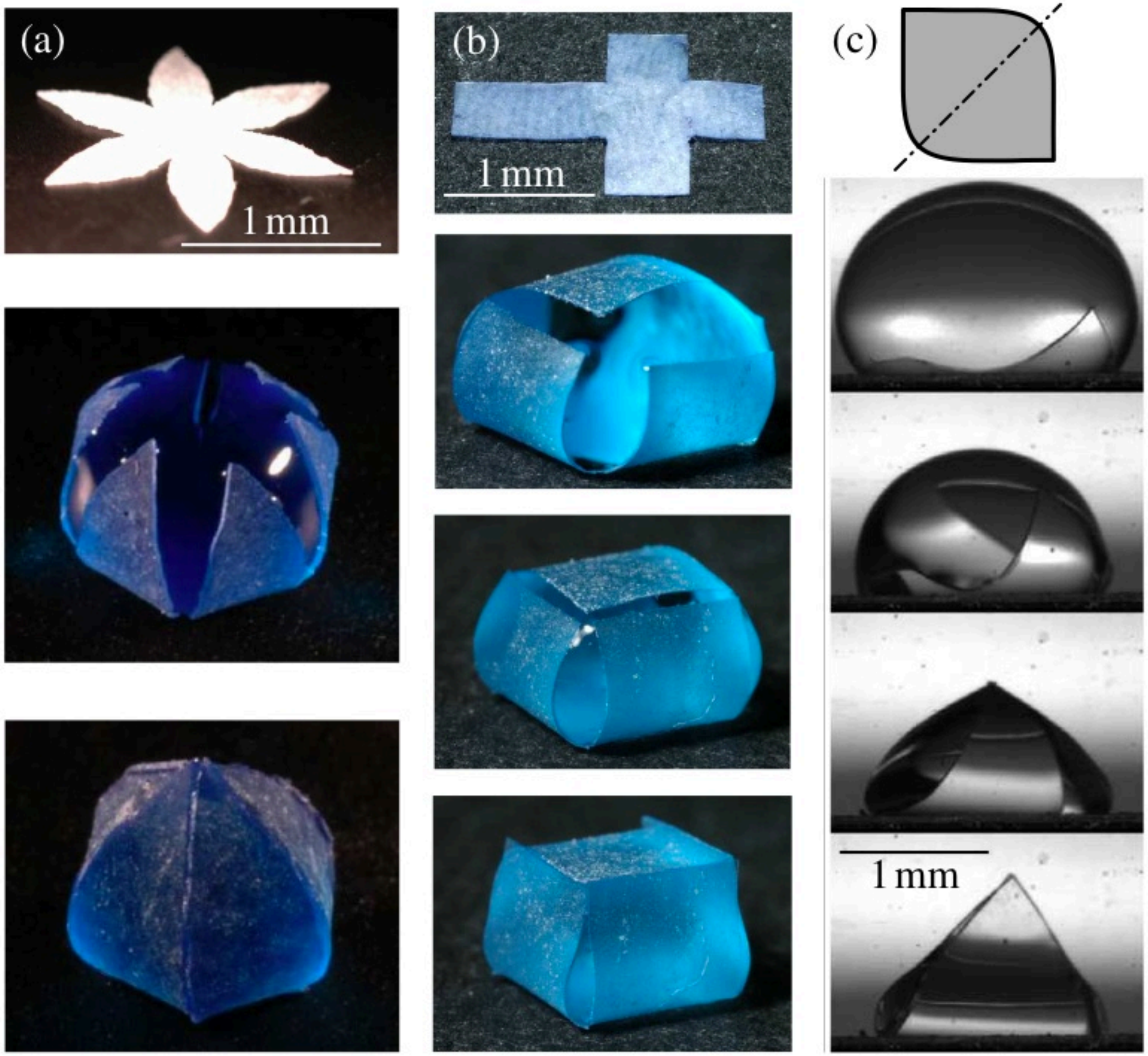


Fluid-structure interaction:

- Droplet bends and folds the sheet.
- Droplet is minimizing the amount of its surface in contact with air.
- Liquid-air surface area is minimized at the expense of bending the sheet.

Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

Elastocapillarity



Fluid-structure interaction:

Elastic energy of a plate – bending:

$$\mathcal{U}_e = \frac{1}{2}\iint_P \mathrm{d}x\mathrm{d}y \int_{-h/2}^{h/2} \mathrm{d}z\,(\sigma_{\alpha\beta}\varepsilon_{\alpha\beta})$$

Relation between in-plane strain to out-of-plane bending:

$$\varepsilon_{\alpha\beta}(x) = z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{z}{R}$$

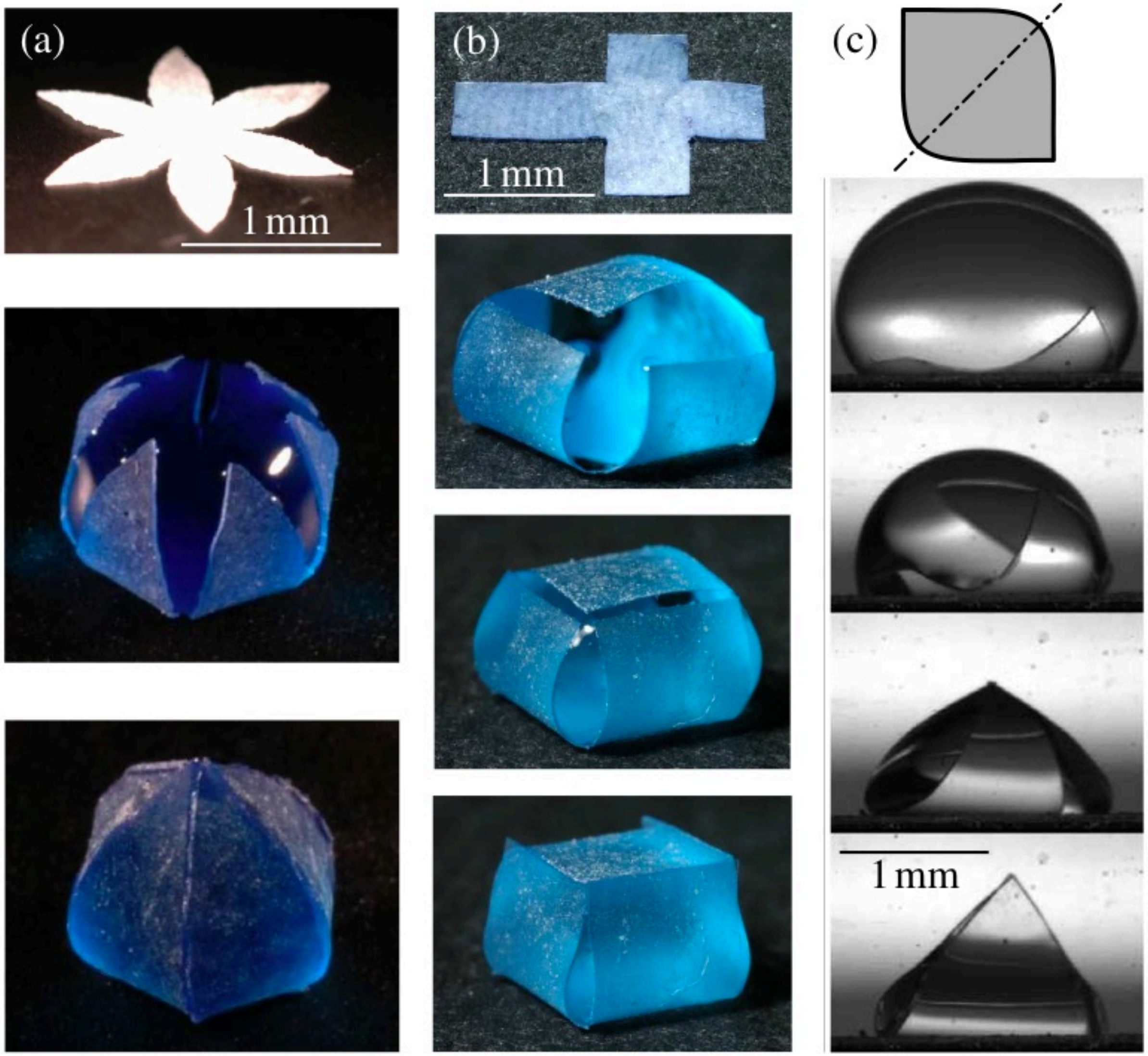
Bending energy:

$$\mathcal{U}_b = \frac{1}{2}\iint_P \mathrm{d}x\mathrm{d}y\,\frac{Eh^3}{12}\left(\frac{1}{R}\right)^2$$

$$\mathcal{U}_b \sim Eh^3$$

Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

Elastocapillarity



Fluid-structure interaction:

Bending energy:

$$\mathcal{U}_b \sim Eh^3$$

Surface energy:

$$\mathcal{U}_\gamma \sim \gamma L^2$$

Elastocapillary length:

$$\ell_{ec} \sim \sqrt{\frac{Eh^3}{\gamma}} \sim \sqrt{\frac{B}{\gamma}}$$

Elastocapillary bending of sheet:

$$7\ell_{ec} \leq L \leq 12\ell_{ec}$$

Py, Charlotte, Paul Reverdy, Lionel Doppler, José Bico, Benoit Roman, and Charles N. Baroud. "Capillary origami: spontaneous wrapping of a droplet with an elastic sheet." Physical Review Letters 98, no. 15 (2007): 156103.

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Elastocapillarity

Capillary rise between flexible fibers.

$$\underbrace{B\frac{\partial^4 d}{\partial x^4}}_{\text{Beam bending}} = \underbrace{\gamma\kappa_f}_{\text{Laplace pressure}}$$

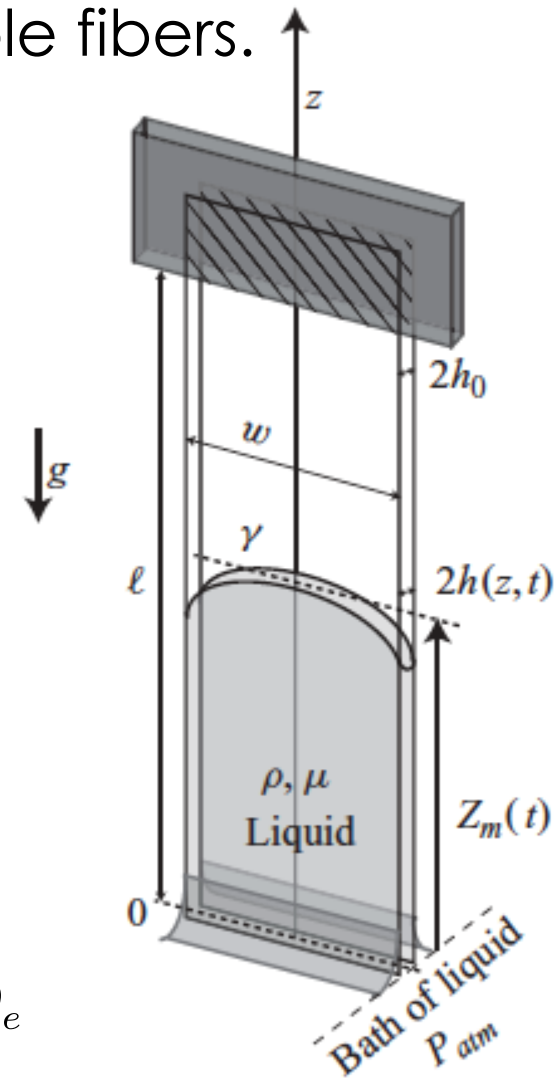
Curvature of meniscus

$$\kappa_f = \frac{1}{d(z_m)}\cos\theta_e + \frac{2}{b}\cos\theta_e$$

Assume gap is much smaller than width: $d_0 \ll b$

Initial gap: $d(z_m, t) = d_0$

Approximation of meniscus curvature: $\kappa_f \approx \frac{1}{d_0}\cos\theta_e$



H.-Y. Kim and L. Mahadevan, "Capillary rise between elastic sheets, " J. Fluid Mech. **548**, 141-150, (2006).
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Elastocapillarity

Capillary rise between flexible fibers.

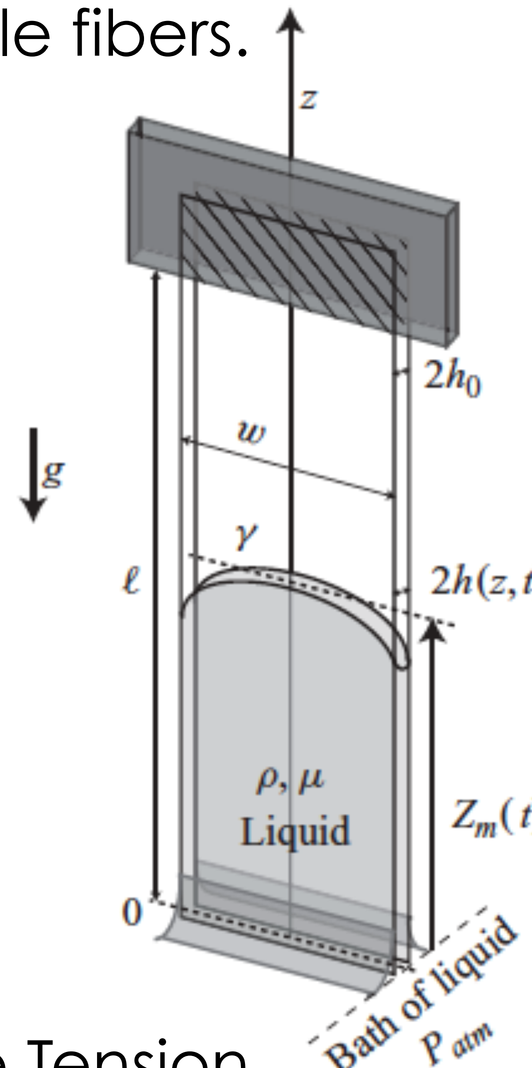
$$\underbrace{B\frac{\partial^4 d}{\partial x^4}}_{\text{Beam bending}} = \underbrace{\gamma \kappa_f}_{\text{Laplace pressure}}$$

Approximation: $\kappa_f \approx \dfrac{1}{d_0}\cos\theta_e$

Scaling: $B\dfrac{d_0}{\ell^4} \sim \dfrac{\gamma}{d_0}\cos\theta_e$

$$\ell_{ec} \equiv \left(\frac{B d_0^2}{\gamma \cos\theta_e}\right)^{1/4}$$

Balance: Bending & Surface Tension



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Elastocapillarity

Capillary rise between flexible fibers.

Potential energies:

Elastic energy

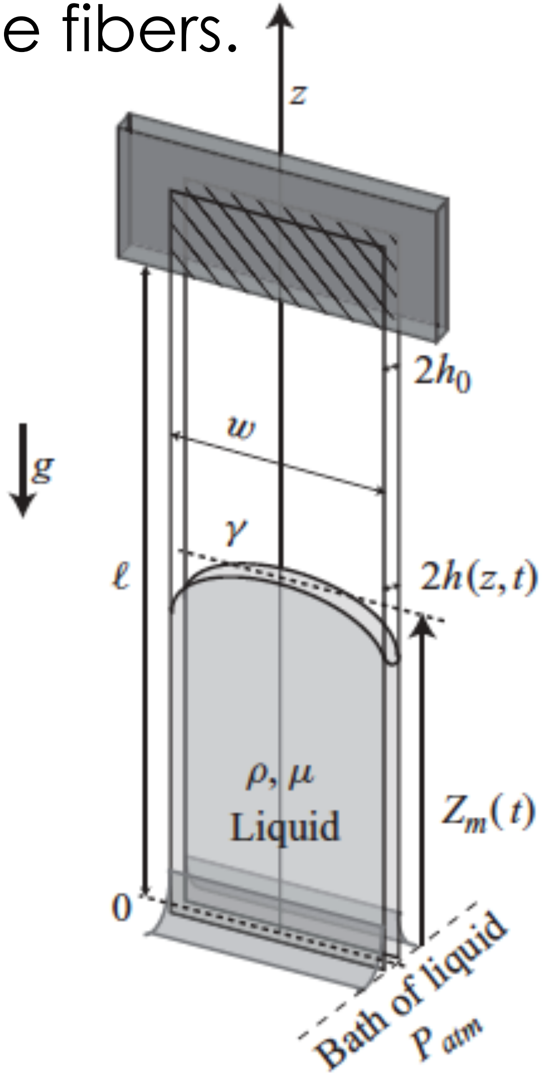
$$\mathcal{U}_e \sim \frac{Bwh_0^2}{\ell^3}$$

Gravitational potential energy

$$\mathcal{U}_g \sim \rho g w h_0 \ell^2$$

Surface energy

$$\mathcal{U}_c \sim w\ell\gamma$$



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Elastocapillarity

Capillary rise between flexible fibers.

Characteristic length scales:

Elastocapillary length

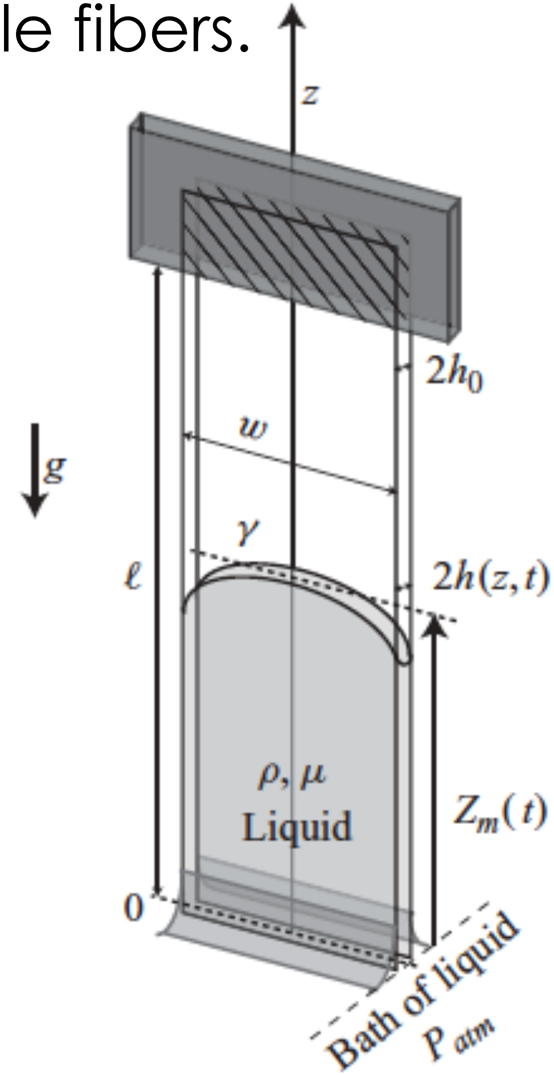
$$\ell_{ec} = \frac{\mathcal{U}_e}{\mathcal{U}_c} = \left(\frac{Bh_0^2}{\gamma}\right)^{1/4}$$

Capillary gravity length

$$\ell_{cg} = \frac{\mathcal{U}_c}{\mathcal{U}_g} = \frac{\gamma}{\rho g h_0}$$

Elastogravity length

$$\ell_{eg} = \frac{\mathcal{U}_e}{\mathcal{U}_g} = \left(\frac{Bh_0}{\rho g}\right)^{1/5}$$



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Elastocapillarity

Capillary rise between flexible fibers.

Dimensionless parameters

Bond number:

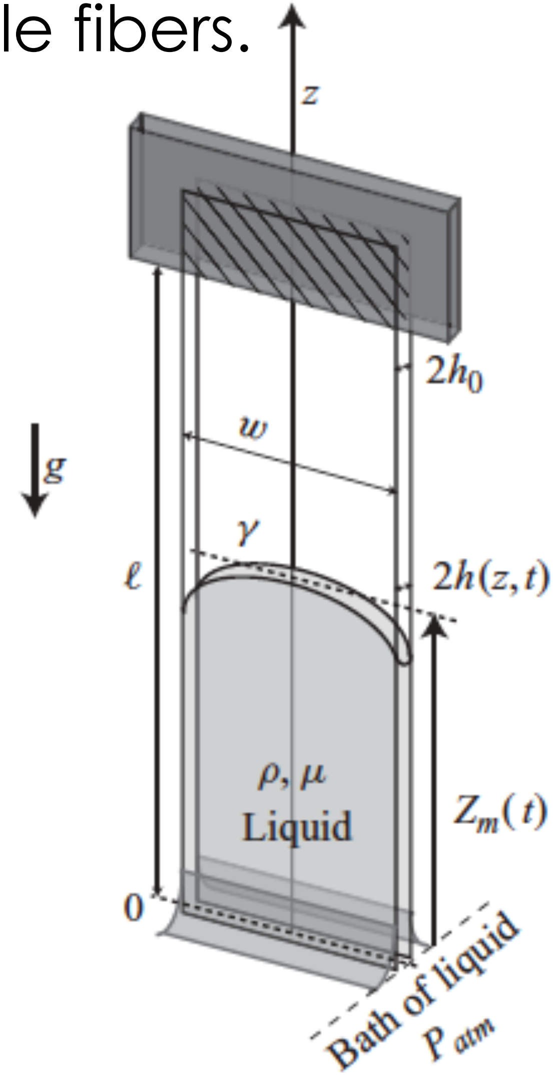
$$\mathscr{B} = \frac{\ell}{\ell_{cg}}$$

Elastocapillary number

$$\mathscr{E} = \left(\frac{\ell}{\ell_{ec}}\right)^4$$

Elastogravity number

$$\mathscr{G} = \mathscr{B}\mathscr{E} = \left(\frac{\ell}{\ell_{eg}}\right)^5$$



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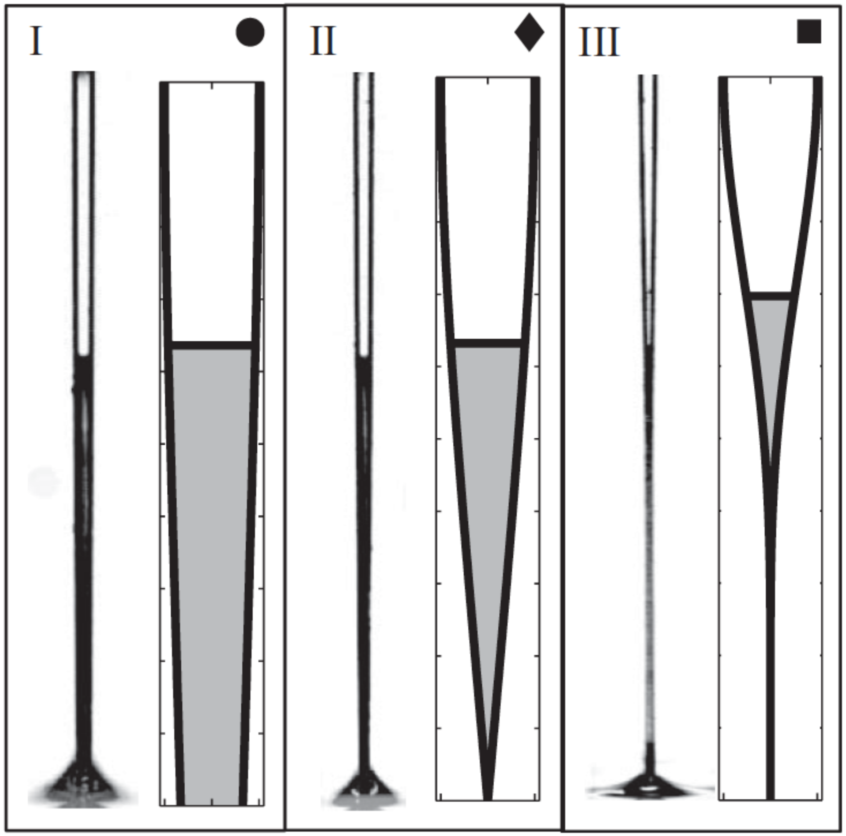
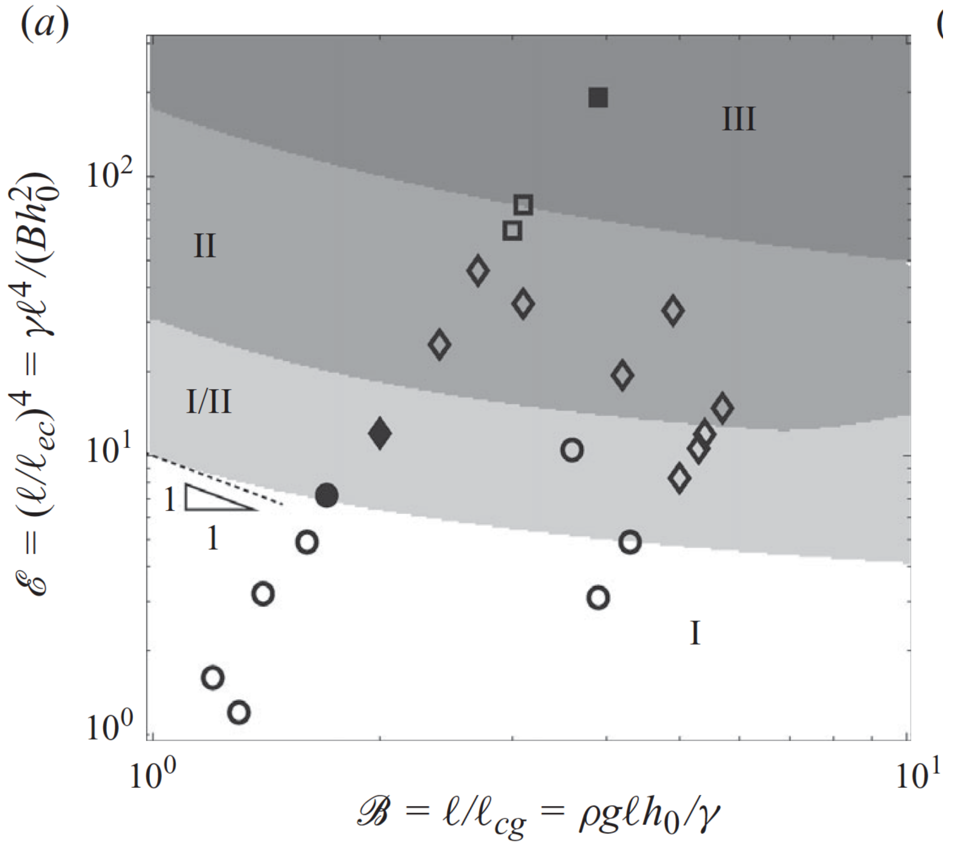
Elastocapillarity





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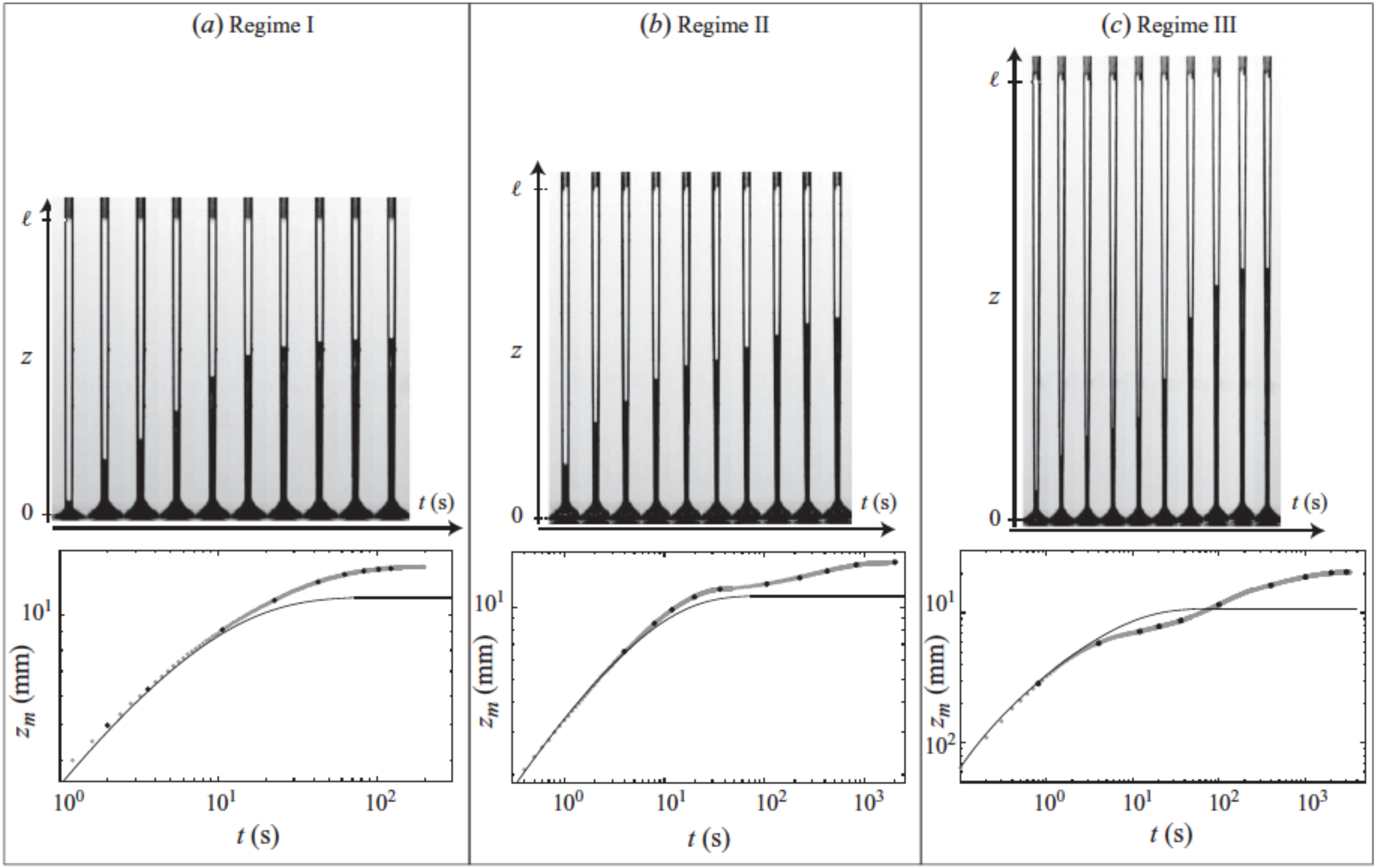
Elastocapillarity



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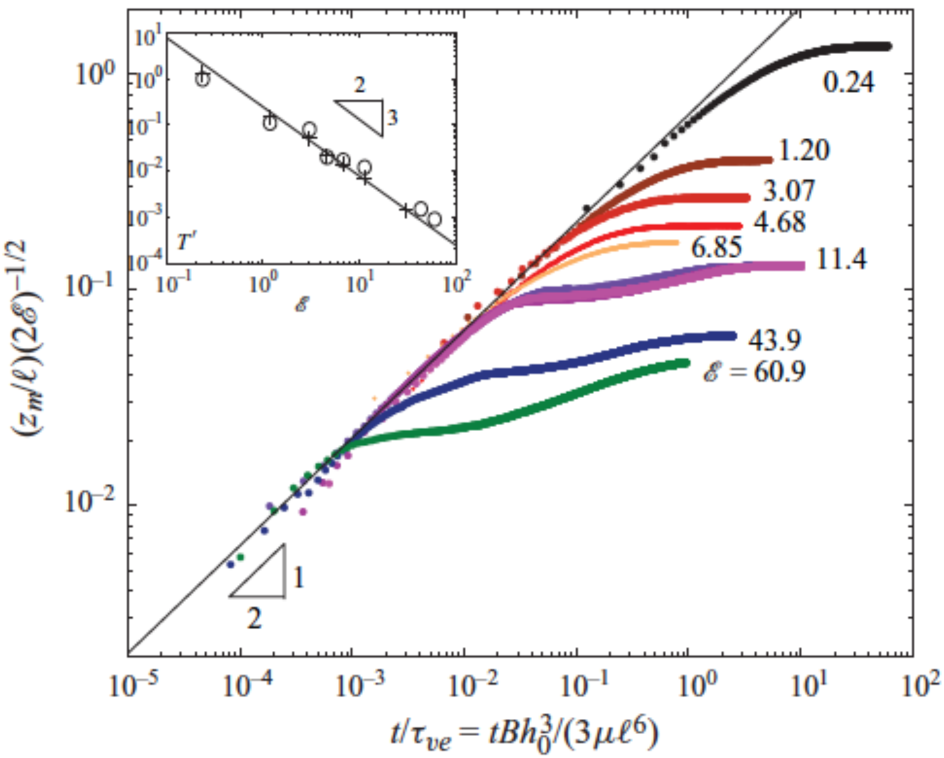
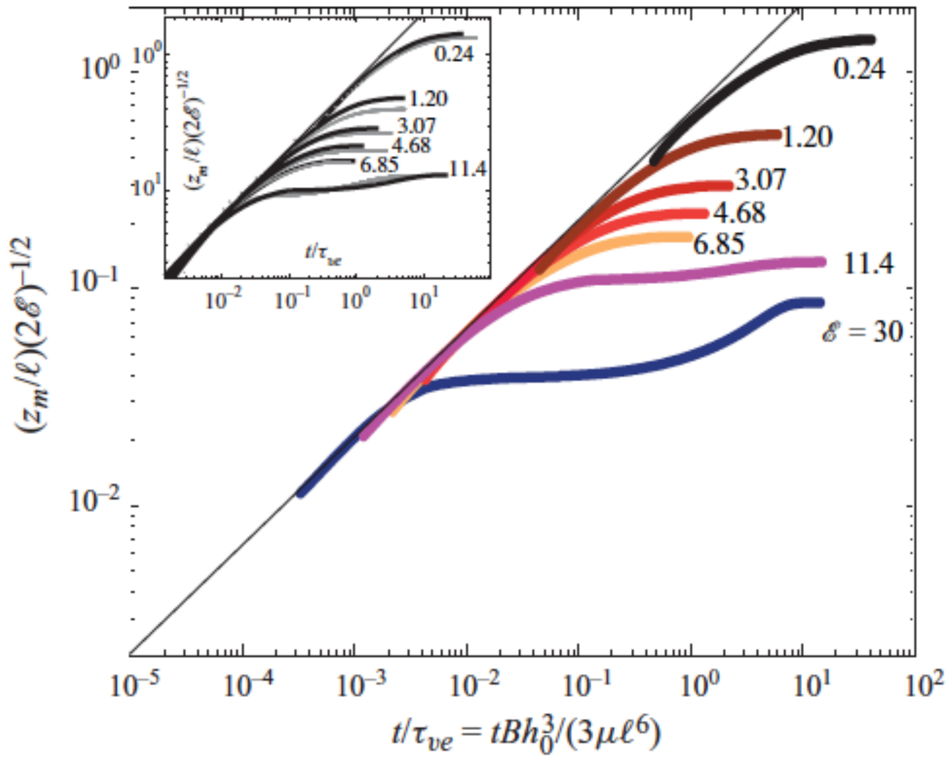
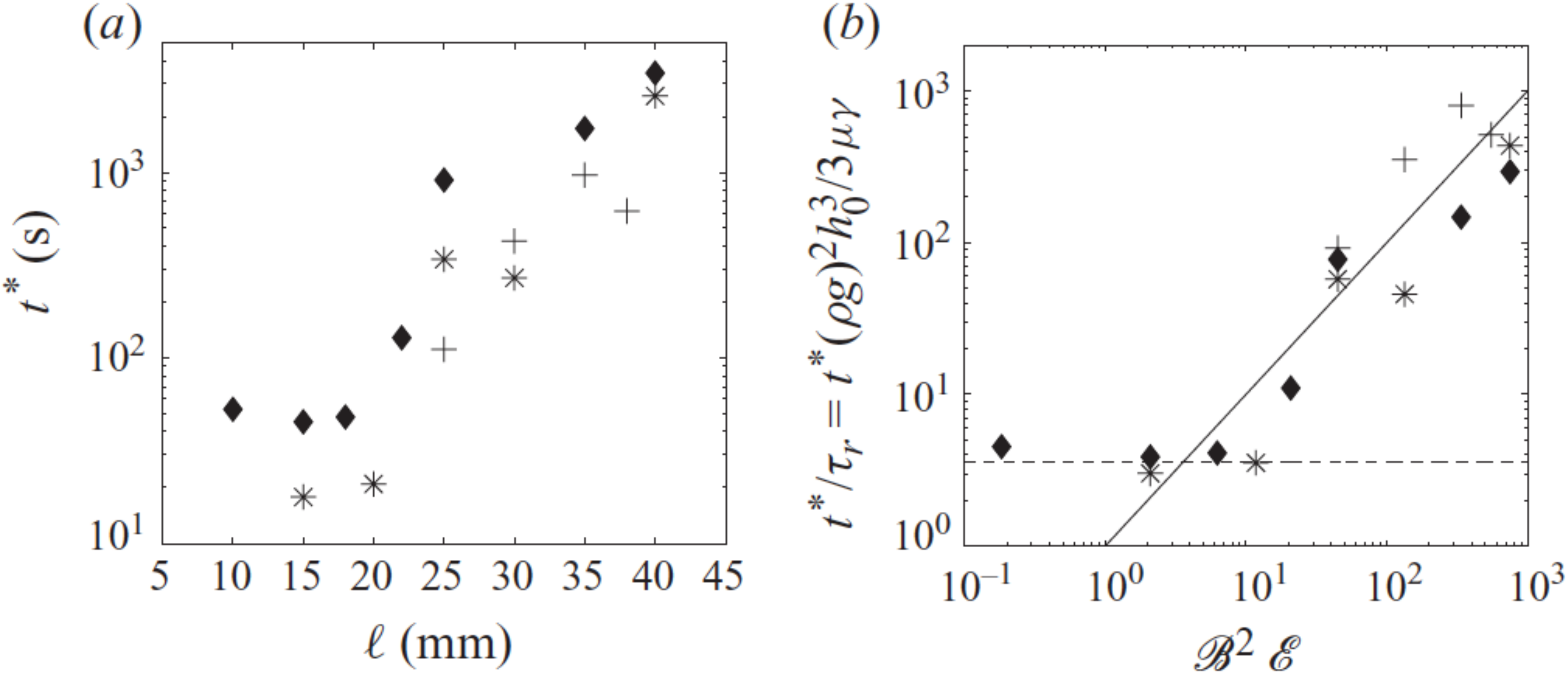
Elastocapillarity





Elastocapillarity



Capillarity & Swelling

Solid: Polyvinylsiloxane

Fluid: Silicone Oil (5 cSt)

20x faster than real time

$E \approx 1\, MPa\, (PVS)$

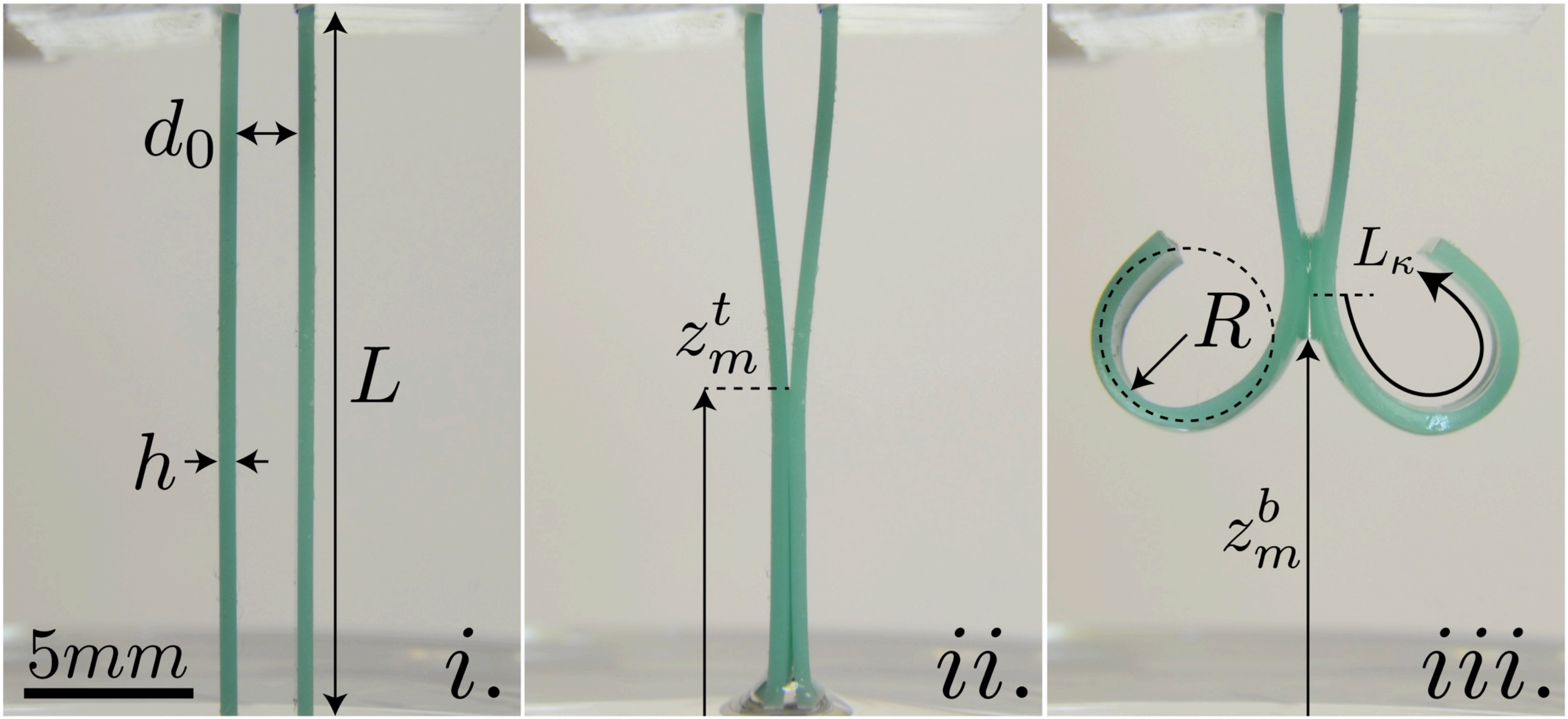
$L = 20\, mm$

$d \approx 2\, mm$

$h \approx 0.5\, mm$

D.P. Holmes, A. Pandey, P.-T. Brun, and S. Protière, *In Preparation*, (2015).

Capillarity & Swelling



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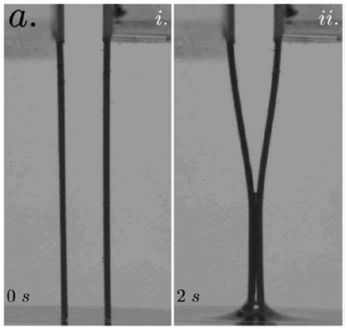
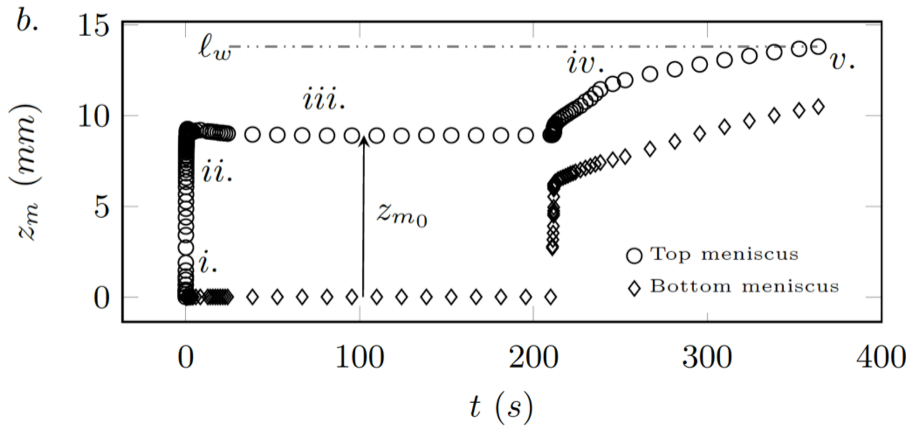
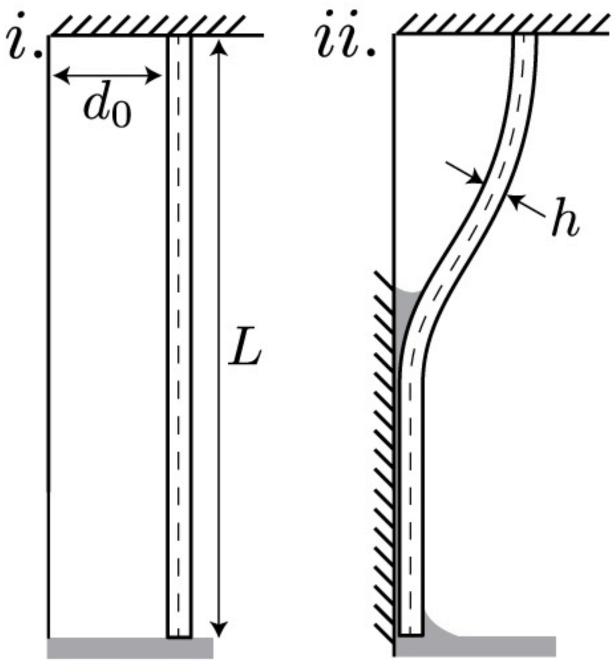
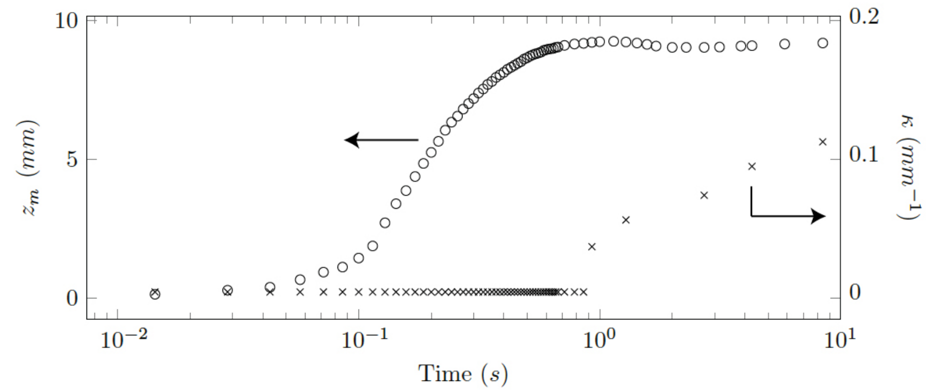
Capillarity & Swelling





1. Elastocapillary rise between flexible fibers.





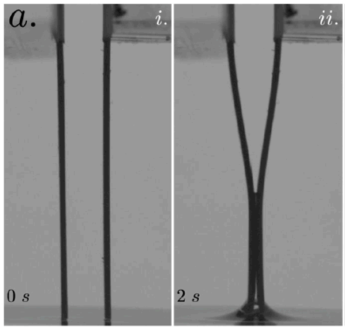
$$B\frac{\partial^4 d}{\partial x^4} = \gamma \kappa_f$$

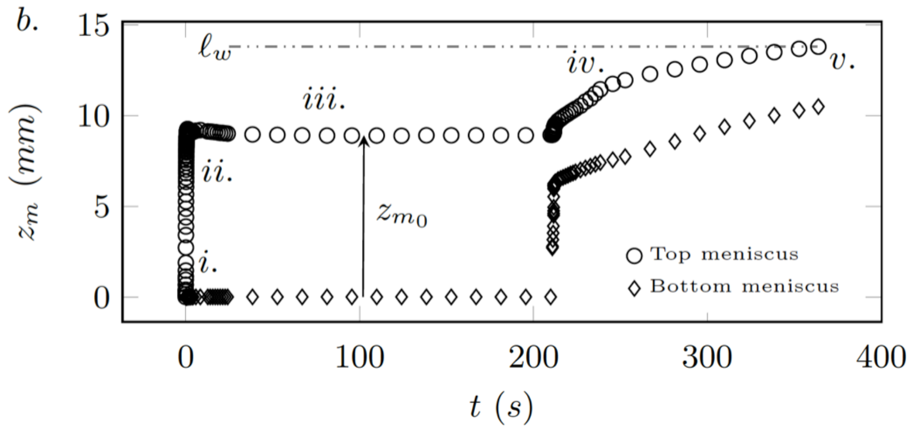
$$\ell_{ec} \equiv \left(\frac{B d_0^2}{\gamma \cos\theta_e}\right)^{1/4}$$

$$B\frac{d_0}{\ell^4} \sim \frac{\gamma}{d_0}\cos\theta_e$$

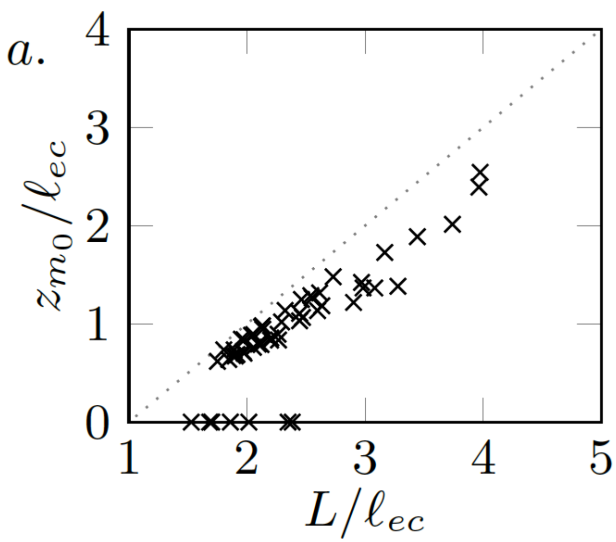
D.P. Holmes, A. Pandey, P.-T. Brun, and S. Protière, *In Preparation*, (2015).

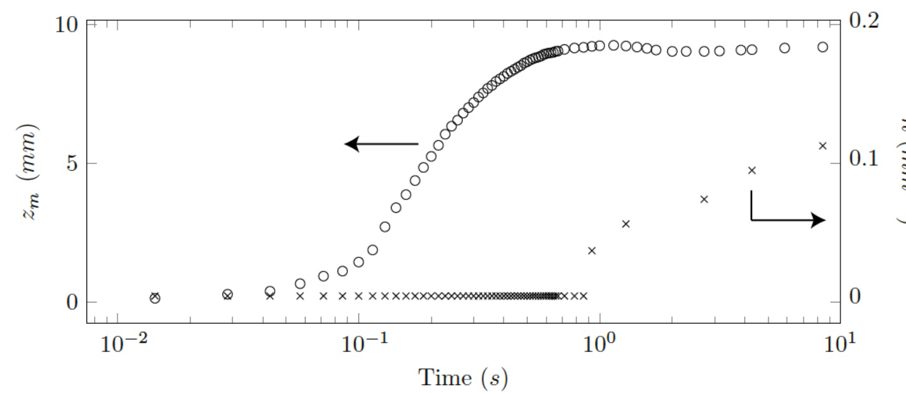
Capillarity & Swelling





1. Elastocapillary rise between flexible fibers.



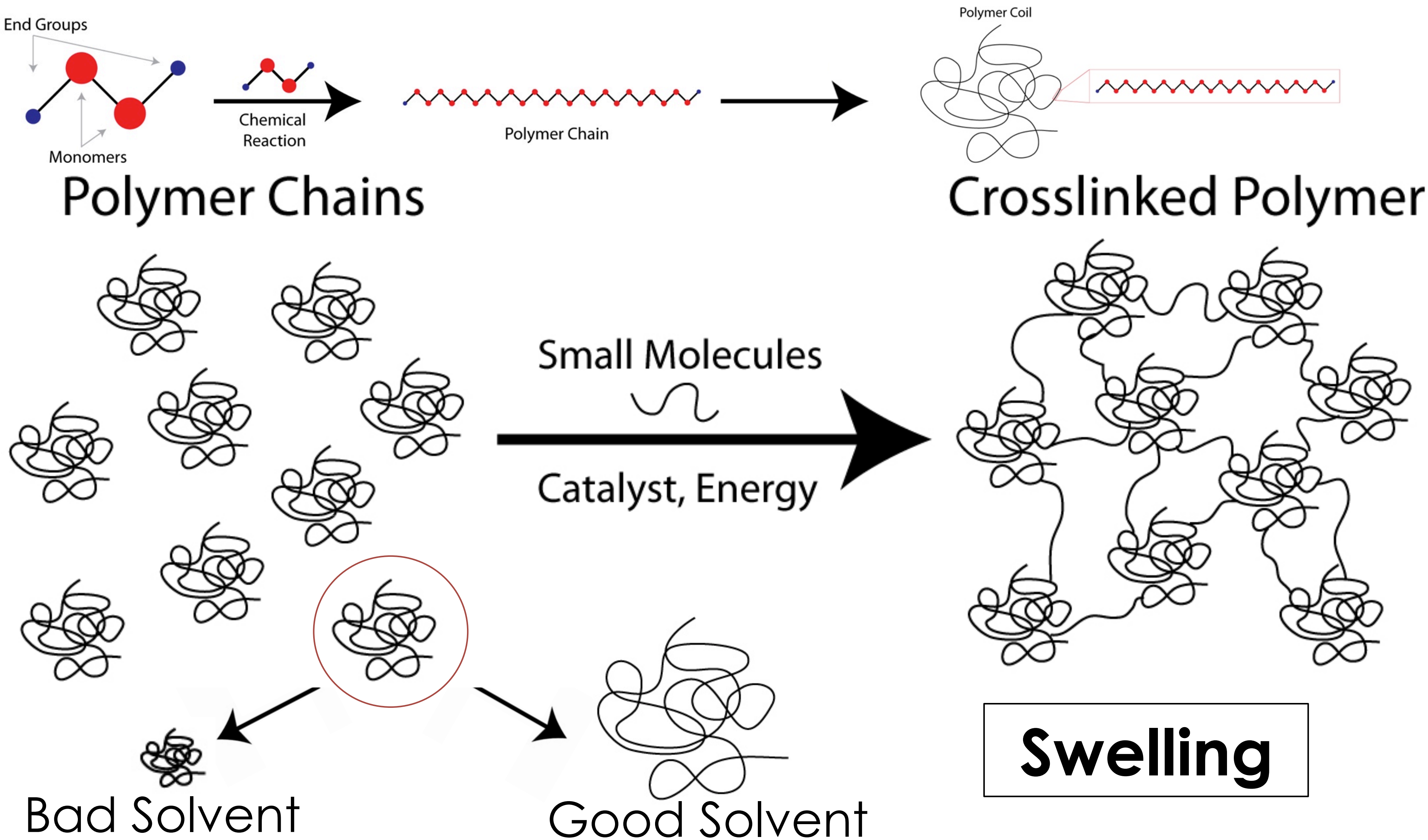


Stationary meniscus height rises linearly with elastocapillary length.

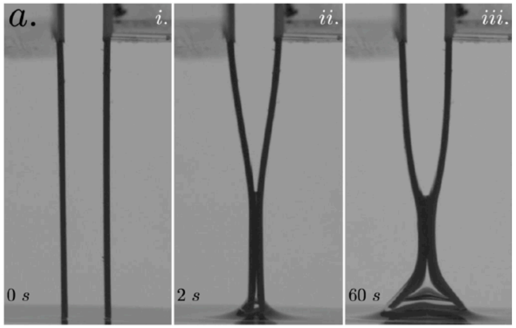
$$\ell_{ec} \sim \left(\frac{Bd_0}{\gamma}\right)^{1/4}$$

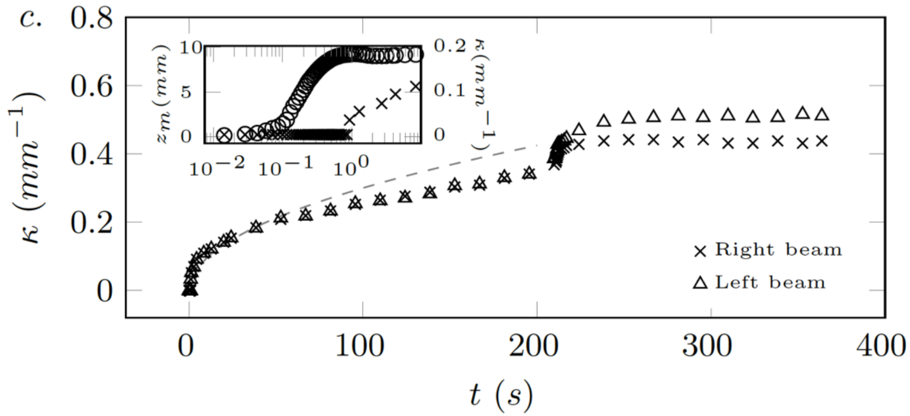
D.P. Holmes, A. Pandey, P.-T. Brun, and S. Protière, *In Preparation*, (2015).

Polymers & Swelling

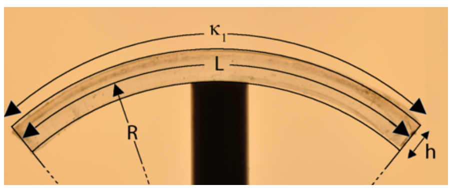


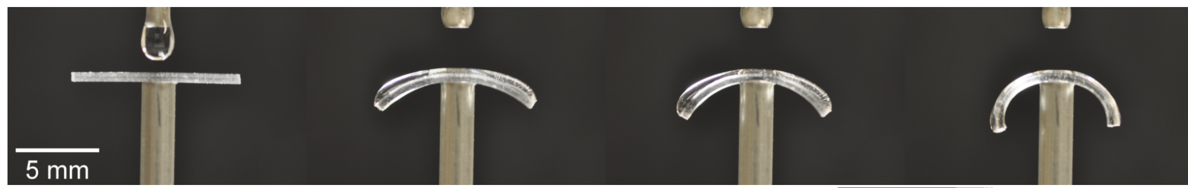
Capillarity & Swelling

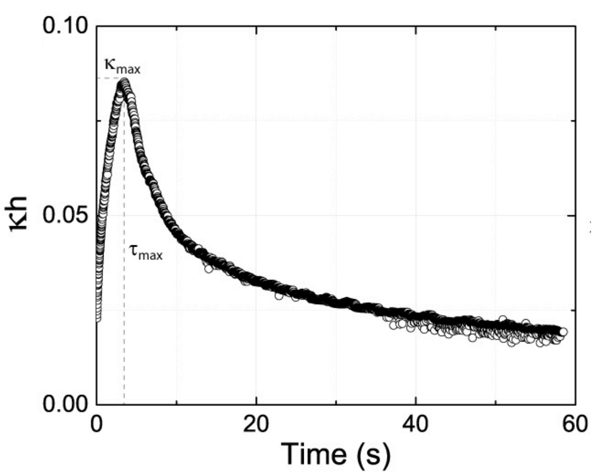


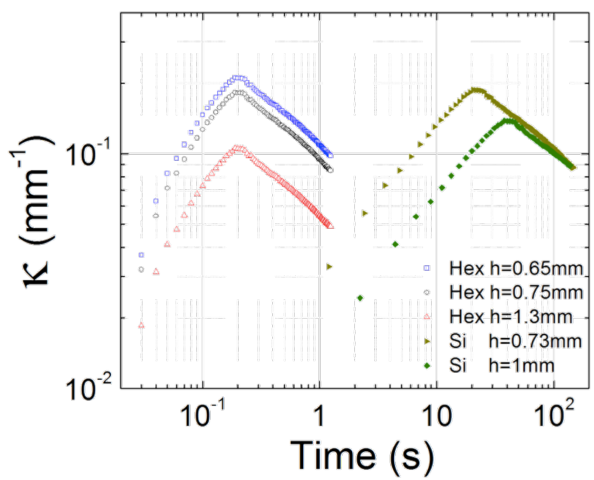


2. Swelling-induced bending.



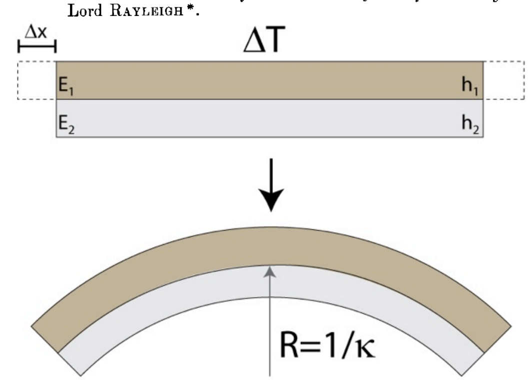






FEBRUARY 1901.

XV. *On the Stresses in Solid Bodies due to unequal Heating, and on the Double Refraction resulting therefrom.* By Lord RAYLEIGH*.



Dynamics of bending ~ Diffusion of temperature

D.P. Holmes, A. Pandey, and S. Protière, *In Preparation*, (2015).

D.P. Holmes, M. Roché, T. Sinha, and H.A. Stone. "Bending and Twisting of Soft Materials by Non-homogenous Swelling" *Soft Matter*, **7**, 5188, 2011.

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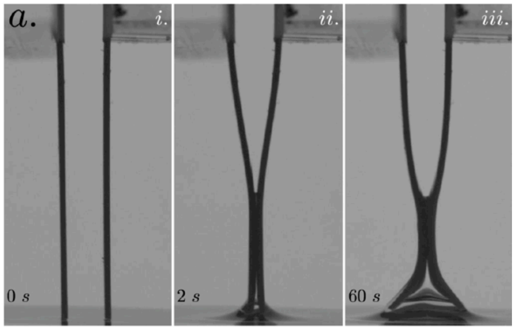
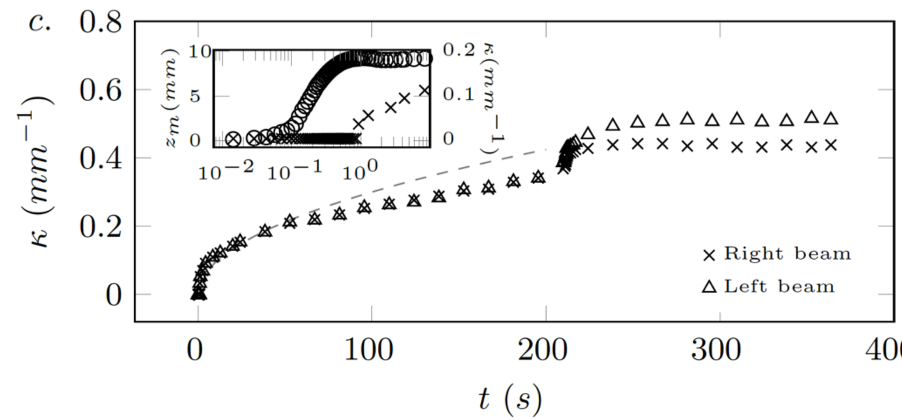
Capillarity & Swelling





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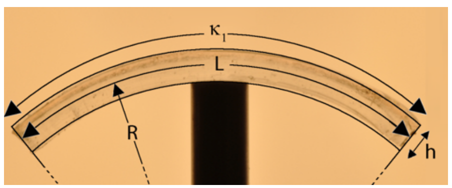
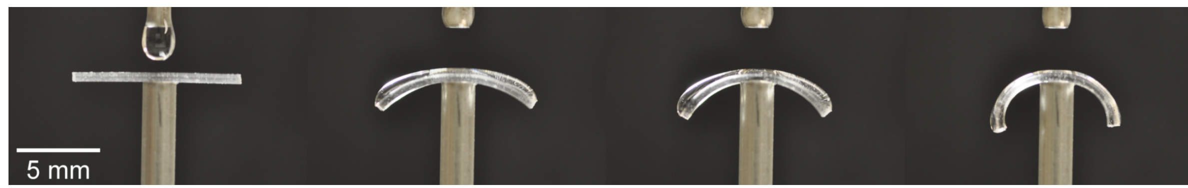




- Thermal diffusion through the beam thickness.
- Shape obtained by minimizing the bending moment in the beam.
- Beam curvature as temperature diffuses.

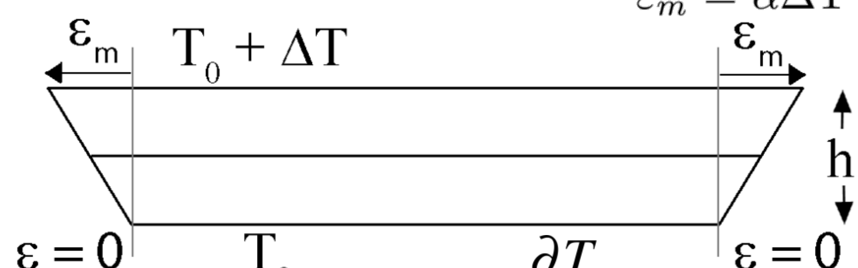
$$\frac{\kappa_1 h}{\varepsilon_m(1+\nu)} = 1.33e^{-\frac{\pi^2 t/\tau}{4}} - 0.77e^{-\frac{9\pi^2 t/\tau}{4}} + \ldots$$

- Poroelastic time scale

$$\tau_p \approx \frac{\mu h^2}{kE}$$

μ = Solvent viscosity
h = Thickness
k = Permeability (k ≈ 10^{-18} m^2/s)
E = Elastic modulus (E = 10^6 Pa)

$\varepsilon_m = \alpha \Delta T$



$$\frac{\partial T}{\partial x_3} = 0$$

κ_1 = Principal curvature
h = Thickness
ε_m = Maximum strain
ν = Poisson's ratio

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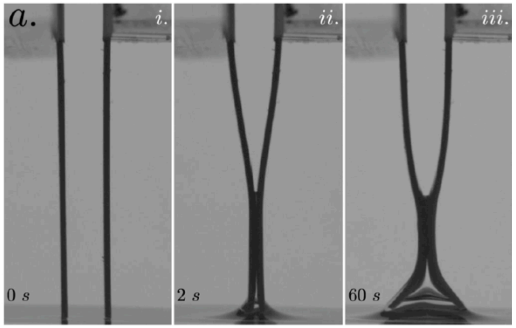

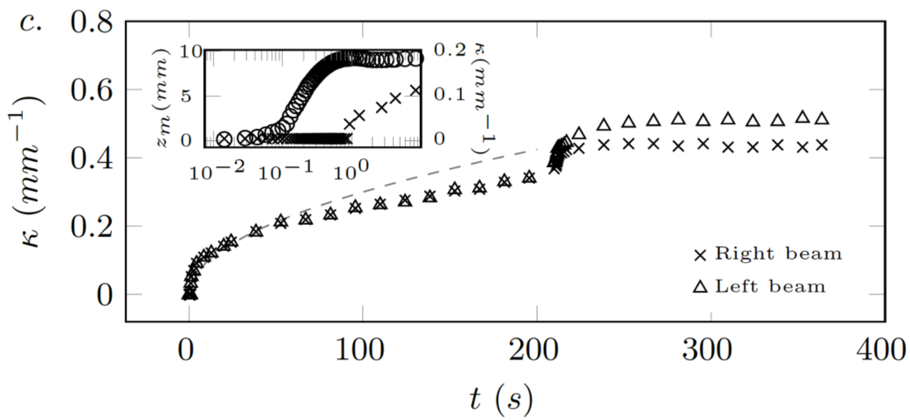

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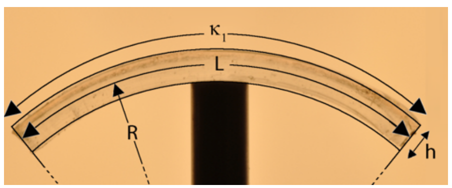

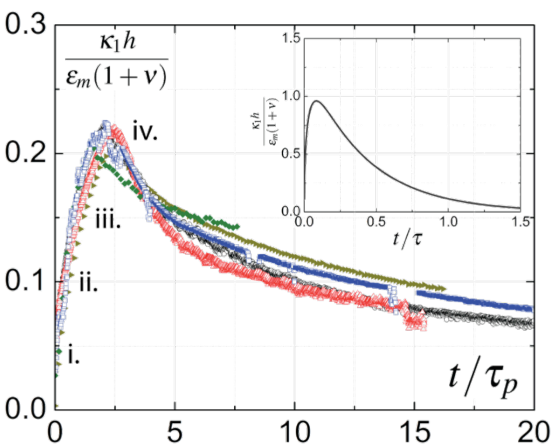

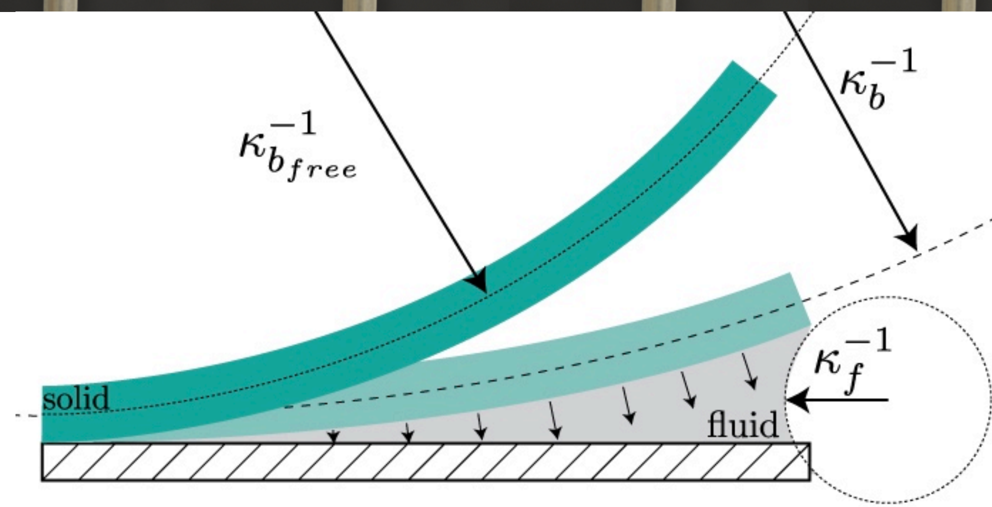

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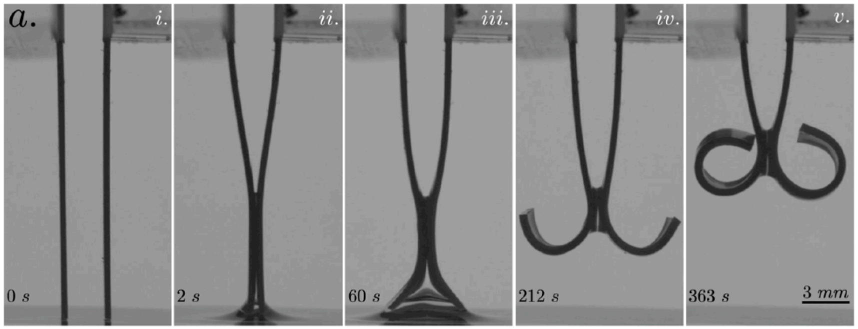
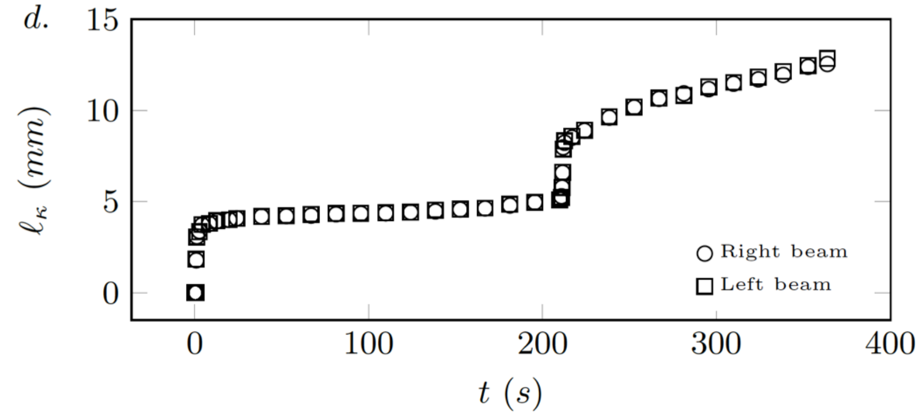
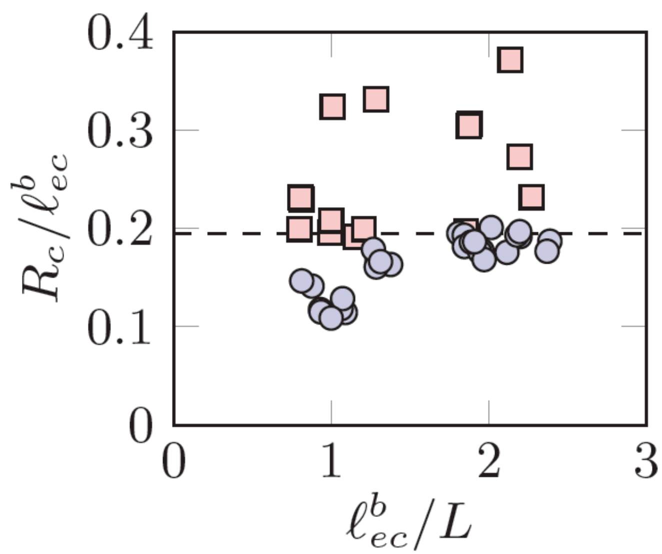
Capillarity & Swelling

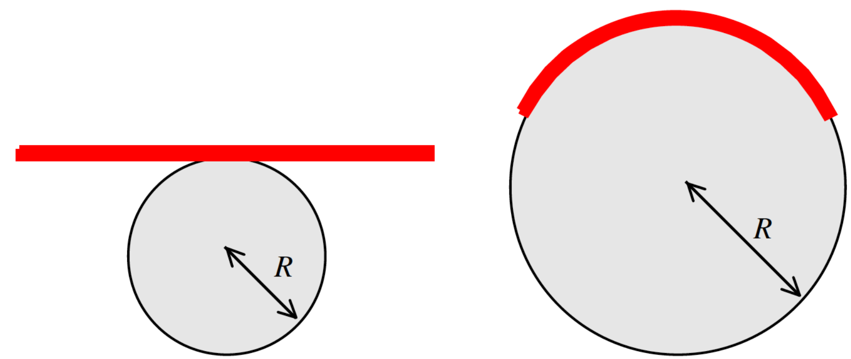




3. Bending dominates surface tension.



Peeling occurs if the curvature exceeds the bending capillary length.



Bending energy

$$\mathcal{U}_b \sim \frac{1}{2} B L b \kappa_b^2$$

Surface energy

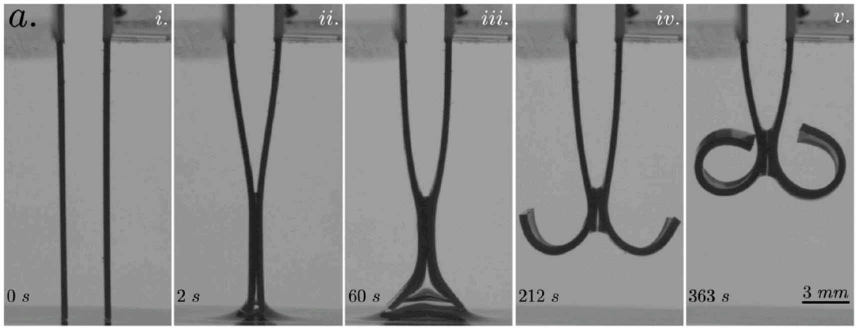
$$\mathcal{U}_\gamma \sim 2\gamma L b$$

$$\ell_{ec}^b \sim \left(\frac{B}{\gamma}\right)^{1/2}$$

D.P. Holmes, A. Pandey, and S. Protière, *In Preparation*, (2015).

Roman, Benoit, and José Bico. "Elasto-capillarity: deforming an elastic structure with a liquid droplet." Journal of Physics: Condensed Matter 22.49 (2010): 493101.

Capillarity & Swelling



1. Elastocapillary rise between flexible fibers.

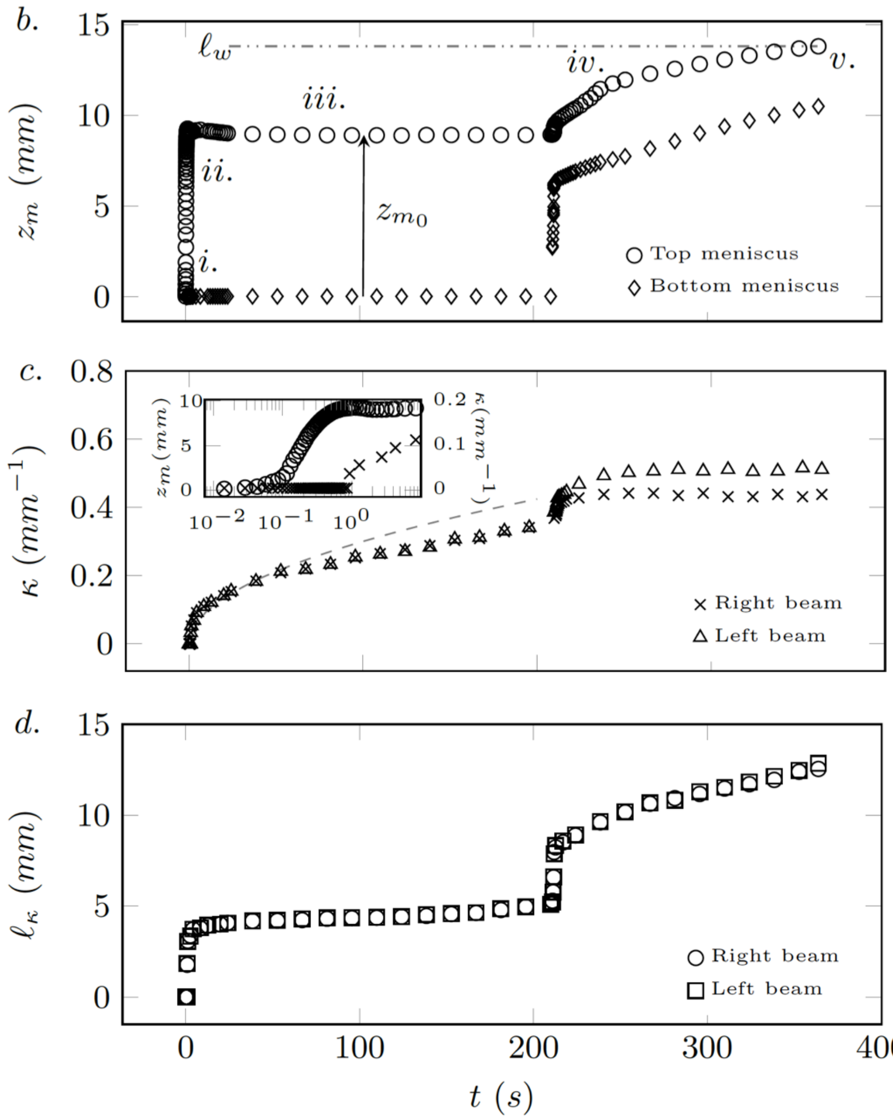
   At short times, elastocapillary rise dominates the deformation.

2. Swelling-induced bending.

   Bending is constrained by surface tension, as the beam bends with a lower curvature than a free swelling beam.

3. Bending dominates surface tension.

   Separation occurs as the "natural" curvature of the beam exceeds the fluids ability to confine it.



D.P. Holmes, A. Pandey, and S. Protière, *In Preparation*, (2015).

Capillarity & Swelling

$$\mathcal{U} \sim \underbrace{\mathcal{U}_m}_{\text{stretching}} + \underbrace{\mathcal{U}_b}_{\text{bending}} - \underbrace{\mathcal{U}_\gamma}_{\text{surface}}$$

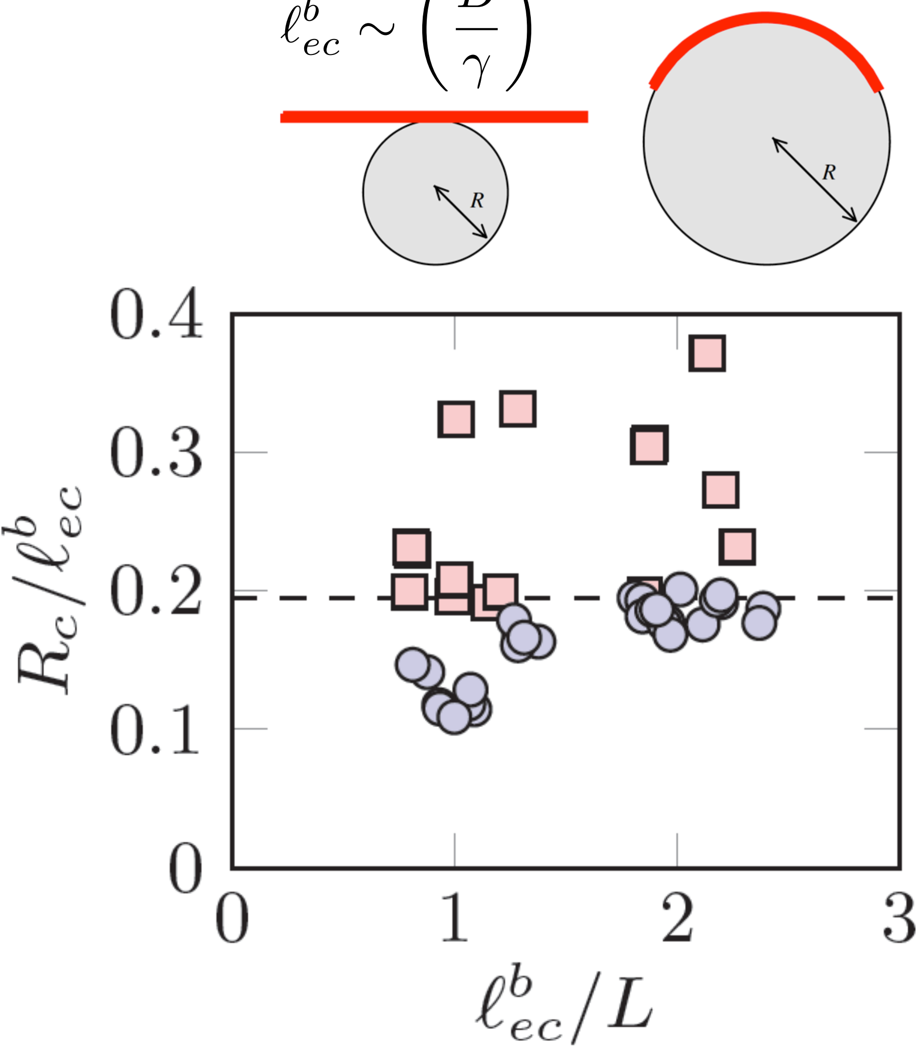
Stretching energy	Bending energy	Surface energy
$\mathcal{U}_m = Eh\int_A \varepsilon^2 \, \mathrm{d}A$	$\mathcal{U}_b = EI\int_A \kappa^2 \, \mathrm{d}A$	$\mathcal{U}_\gamma = \ell_w \int_A \gamma \mathrm{d}A$
$\downarrow$	$\downarrow$	$\downarrow$
$Ehlb\varepsilon^2$	$\frac{Eh^3lb}{12}\kappa^2$	$\kappa_f \ell_{ec}^b lb\gamma_{lv}$

$$Eh\varepsilon^2 + \frac{Eh^3}{12}\kappa_b^2 = \gamma\kappa_f\ell_{ec}^b \xrightarrow[\kappa_b \sim L \qquad \kappa_f \sim d_0^{-1}]{\mathcal{S} = B/EhL^2b = h^2/12L^2} \mathcal{S} \sim \frac{\varepsilon^2}{(L/\ell_{ec})^2 - 1}$$

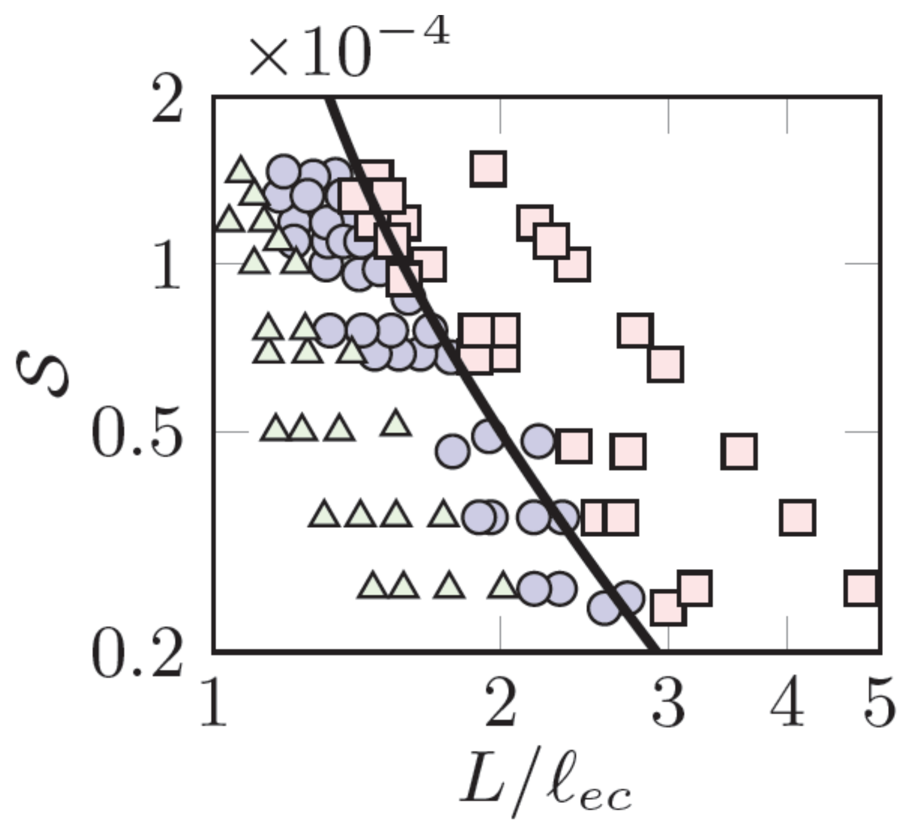
D.P. Holmes, A. Pandey, and S. Protière, *In Preparation*, (2014).

Capillarity & Swelling

$$\ell_{ec}^{b} \sim \left(\frac{B}{\gamma}\right)^{1/2}$$



$$\mathcal{S} \sim \frac{\varepsilon^2}{(L/\ell_{ec})^2 - 1}$$



D.P. Holmes, A. Pandey, and S. Protière, *In Preparation*, (2014).

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Baobab Flowering

BBC - Planet Earth: Seasonal Forests

Summary

Confined Fluid Flow: Microfluidics and Capillarity

Reynolds Number: Inertia vs. Viscous effects
- Review of characteristic flows…

Péclet Number: Transport phenomena in a continuum
- Diffusion, separation, and mixing…

Geometric confinement: Controlling and manipulating fluid flow
- Microfluidic fabrication, valving, pumping…

Capillary Number: Viscosity vs. Surface tension
- Droplet formation, capillary rise, elasticity…