Preliminary Exam 2018
Solutions to Morning Exam

Part I.

Solve four of the following five problems.

Problem 1. Consider the series $\sum_{n\geqslant 2}(n\log n)^{-1}$ and $\sum_{n\geqslant 2}(n(\log n)^2)^{-1}$. Show that one converges and one diverges by applying a standard convergence test.

Solution: Use the integral test. An antiderivative for $1/(x\log x)$ is $\log\log x$ (the substitution $u=\log x$ replaces the integrand by du/u) and consequently

$$\int_2^\infty \frac{dx}{x\log x} = \lim_{T\to infty}(\log\log T - \log\log 2) = \infty.$$

So $\sum_{n\geqslant 2}(n\log n)^{-1}$ diverges. On the other hand, an antiderivative for $1/(x(\log x)^2)$ is $-1/\log x$ (again, use the substitution $u=\log x$), so

$$\int_2^\infty \frac{dx}{x(\log x)^2} = \lim_{T\to infty}(-(\log T)^{-1} + (\log 2)^{-1}) = (\log 2)^{-1}.$$

Hence $\sum_{n\geqslant 2}(n(\log n)^2)^{-1}$ converges.

Problem 2. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\sqrt{x^2+y^2}}\,dx\,dy = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

by computing both sides.

Solution: Do the integrals in polar coordinates. On the left-hand side we obtain

$$\int_0^{2\pi}(\int_0^\infty e^{-r}r\,dr)\,d\theta = \int_0^{2\pi}(-e^{-r}r - e^{-r}|_0^\infty)\,d\theta,$$

which is $(2\pi)(1)=2\pi$. On the right-hand side we obtain

$$\int_0^{2\pi}(\int_0^\infty e^{-r^2}r\,dr)\,d\theta = \int_0^{2\pi}(-e^{-r^2}/2|_0^\infty)\,d\theta,$$

which is $(2\pi)(1/2)=\pi$.

Problem 3. Prove that if $f(x)$ is $\sin x$ or $\arctan x$ then $|f(b)-f(a)|\leqslant|b-a|$ for all $a,b\in\mathbb{R}$ and that this inequality also holds for $f(x)=\log x$ and $a,b\geqslant 1$.

Solution: By the Mean Value Theorem, $f(b)-f(a)=f'(c)(b-a)$ for some c strictly between a and b, and consequently

$$|f(b)-f(a)|\leqslant|f'(c)||b-a|.$$

Since the derivatives of $\sin x$ and $\arctan x$ are $\cos x$ and $1/(1+x^2)$ respectively, both of which are bounded by 1 in absolute value on $\mathbb{R}$, we obtain the stated inequality. Also, if $f(x)=\log x$ then $f'(x)=1/x$, which is bounded by 1 for $x\geqslant 1$.

Problem 4. Let y be a differentiable function and p a continuous function on $(0,\infty)$, and suppose that $y'(t)+p(t)y(t)=p(t)$ for all $t>0$. If $p(t)>c/t$ for some constant $c>0$ prove that $\lim_{t\to\infty}y(t)=1$.

Solution: Let $P(t)$ be an antiderivative of $p(t)$ on $(0,\infty)$, say

$$P(t)=\int_1^t p(t)\,dt,$$

and put $\mu(t) = e^{P(t)}$. Multiplying both sides of the differential equation by $\mu(t)$, we obtain $(y(t)\mu(t))' = \mu'(t)$, whence

$$y(t) = e^{-P(t)}(e^{P(t)} + \kappa) = 1 + \kappa e^{-P(t)}$$

for some constant κ. Now for $t > 1$ the fact that $p(t) > c/t$ implies that

$$P(t) = \int_1^t p(t)\, dt > c \log t,$$

whence $e^{-P(t)} < t^{-c}$. Returning to the equation $y(t) = 1 + \kappa e^{-P(t)}$, we conclude that $\lim_{t\to\infty} y(t) = 1$.

Problem 5. Let $f_n(x) = x^n$ on the interval $I = [0,1]$ in $\mathbb{R}$. Show that the sequence $\{f_n\}_{n\geqslant 1}$ does not converge uniformly on I. You may quote general theorems about uniform convergence.

Solution: Each function f_n is continuous on I, but the function f to which $\{f_n\}_{n\geqslant 1}$ is pointwise convergent is not continuous: Indeed f is 0 on $[0,1)$ and 1 at 1, so f is not continuous at 1. It follows that the convergence is not uniform.

Part II.

Solve three of the following six problems.

Problem 6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that $\partial f/\partial x$ and $\partial f/\partial y$ exist at $(0,0)$ but f is not differentiable at $(0,0)$. You may quote general facts about differentiability.

Solution: To show that $\partial f/\partial x$ exists at $(0,0)$ we set $x = 0$ and attempt to differentiate with respect to y at 0. Since $f(0,y)$ is identically 0, we see that $\partial f/\partial x(0,0)$ exists and equals 0. Similarly $\partial f/\partial y(0,0)$ exists and equals 0. However, if a function is differentiable at a point then it is continuous at that point, but our f is not continuous at $(0,0)$: Indeed f is identically $1/2$ on the line $x = y$ except at the point $(0,0)$, where the value is 0. Thus f is not differentiable at $(0,0)$.

Problem 7. Let I be any interval in $\mathbb{R}$. Show that if $f : I \to \mathbb{R}$ is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in I then $\{f(x_n)\}$ is also Cauchy. Is the assertion still true if we assume merely that f is continuous? Justify your answer.

Solution: Let $\varepsilon > 0$ be given. Since f is uniformly continuous there exists $\delta > 0$ such that if $x, x' \in I$ and $|x - x'| < \delta$ then $|f(x) - f(x')| < \varepsilon$. And since $\{x_n\}$ is Cauchy there exists N such that if $m, n > N$ then $|x_m - x_n| < \delta$. So if $m, n > N$ then $|f(x_m) - f(x_n)| < \varepsilon$, and we conclude that $\{f(x_n)\}$ is also Cauchy.

Unless I is closed and bounded and therefore compact, the assertion is false without the assumption that f is *uniformly* continuous. For example, take $I = (0,\infty)$, $f(x) = \log x$, and $x_n = 1/n$. Then $\{x_n\}$ is a Cauchy is a sequence in I but $\{f(x_n)\}$ is not even bounded, because $f(x_n) = -\log n$.

Problem 8. Show that

$$\frac{1}{(x-1)(x-2)(x-3)} = \sum_{n\geqslant 0} \left(-\frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{2\cdot 3^{n+1}}\right) x^n$$

for $|x| < 1$.

Solution: By the method of partial fractions,

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{1/2}{x-1} - \frac{1}{x-2} + \frac{1/2}{x-3}.$$

So

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{-1/2}{1-x} + \frac{1/2}{1-x/2} - \frac{1/6}{1-x/3}.$$

Using the geometric series $(1-r)^{-1} = \sum_{n\geqslant 0} r^n$, which converges for $|r| < 1$, we obtain

$$\frac{1}{(x-1)(x-2)(x-3)} = (-1/2)\sum_{n\geqslant 0} x^n + (1/2)\sum_{n\geqslant 0} x^n/2^n - (1/6)\sum_{n\geqslant 0} x^n/3^n.$$

So the coefficient of x^n is as asserted.

Problem 9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the functions

$$f(x,y) = (e^{2x-y} - e^x, e^{-3x+y} - e^{2y})$$

and

$$h(x,y) = (x^3 + x + y, y^2 + 2x + 3y).$$

There is an open neighborhood $\mathcal{U}$ of $(0,0) \in \mathbb{R}^2$ and a differentiable function $g : \mathcal{U} \to \mathbb{R}^2$ such that $g(0,0) = (0,0)$ and $f \circ g = h$. Compute $[g'(0,0)]$, the Jacobian matrix of g at $(0,0)$.

Solution: The Jacobian matrix of f is

$$[f'(x,y)] = \begin{pmatrix} 2e^{2x-y} - e^x & -e^{2x-y} \\ -3e^{-3x+y} & e^{-3x+y} - 2e^{2y} \end{pmatrix},$$

so

$$f'(0,0) = \begin{pmatrix} 1 & -1 \\ -3 & -1 \end{pmatrix},$$

which is invertible. Thus by the Inverse Function Theorem, f has a C^∞ inverse on an open neighborhood of $(0,0)$, whence $g = f^{-1} \circ h$ on an open neighborhood of $(0,0)$. The Chain Rule gives

$$[g'(0,0)] = [f'(0,0)]^{-1}[h'(0,0)].$$

Now

$$h'(x,y) = \begin{pmatrix} 3x^2+1 & 1 \\ 2 & 2y+3 \end{pmatrix},$$

so

$$h'(0,0) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

Therefore

$$[g'(0,0)] = \begin{pmatrix} 1 & -1 \\ -3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \frac{-1}{4}\begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \frac{-1}{4}\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}.$$

Problem 10. Let $P(x,y) = -y/(x^2+y^2)$ and $Q(x,y) = x/(x^2+y^2)$.
(a) Compute $\partial Q/\partial x - \partial P/\partial y$.
(b) Compute the line integral of $P(x,y)\,dx + Q(x,y)\,dy$ around the unit circle (oriented counterclockwise) $x^2+y^2 = 1$.

(c) Explain why (a) and (b) do not contradict Green's Theorem.

Solution: An easy calculation using the quotient rule shows that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

and another easy calculation using the parametrization $\mathbf{r}(t) = (\cos t, \sin t)$ shows that the line integral in (b) is 2π. But there is no contradiction to Green's Theorem

$$\int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dx\, dy = \int_C P(x,y)\, dx + Q(x,y)\, dy$$

(where R is the unit disk $x^2 + y^2 \leqslant 1$) because P and Q are not C^1 functions on R: Indeed neither is continuous at the origin.

Problem 11. Let C and C' be the circles in $\mathbb{R}^3$ parametrized by $(\cos t, \sin t, 0)$ and $(\cos t, \sin t, 2)$ respectively $(0 \leqslant t \leqslant 2\pi)$. Let $\mathbf{F}(x,y,z)$ be a C^∞ vector field in $\mathbb{R}^3$ such that $\nabla \times \mathbf{F} = \mathbf{0}$. Show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r},$$

where the integrals on the left and right are the line integrals of $\mathbf{F}$ along the oriented circles C and C' respectively.

Solution: Let S be the cylinder $x^2 + y^2 = 1$ for $0 \leqslant z \leqslant 2$, oriented so that a normal vector point outward. Then

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0,$$

where n is the unit outward normal vector and $d\sigma$ is the element of surface area on S. By Stokes' Theorem, we deduce that

$$0 = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} - \int_{C'} \mathbf{F} \cdot d\mathbf{r},$$

and the stated equality follows.

Actually, an even more direct use of Stokes' Theorem shows that the line integrals over C and C' are equal because both are 0. Indeed, instead of taking S to be the cylinder, take it to be disk having C as boundary. Since $\nabla \times \mathbf{F} = \mathbf{0}$, we deduce that the integral of $\nabla \times \mathbf{F}$ over D is 0, whence the line integral of $\mathbf{F}$ along C is 0 by Stokes' Theorem, and similarly for C'.

Part III.

Solve one of the following three problems.

Problem 12. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, put

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and let S denote the unit sphere $||x|| = 1$ in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be any linear transformation. Give a reason why the two sides of the equation

$$\max\{x \in S : ||T(x)||\} = \inf\{C \geqslant 0 : ||T(x)|| \leqslant C||x|| \text{ for all } x \in \mathbb{R}^n\}$$

both exist, and then prove the equation.

Solution: The left-hand side exists because a continuous real-valued function (such as $x \mapsto ||T(x)||$) on a compact set (like S) attains a maximum value, and the right-hand side exists because a nonempty set of real numbers which is bounded below has a greatest lower bound. Let us denote the left-hand side by M and the right-hand side by I. Then $||T(x/||x||)|| \leqslant M$ for all nonzero $x \in \mathbb{R}^n$, and since T is linear it follows that $||T(x)|| \leqslant M||x||$ for all $x \in \mathbb{R}^n$, including 0. So M is an element C of the set on the right-hand side, whence

$$M \geqslant I.$$

On the other hand, any element C of the set on the right-hand side satisfies $||T(x)|| \leqslant C$ if $x \in S$, and therefore the maximum of the values $||T(x)||$, namely M, satisfies $M \leqslant C$ also. In other words, M is a lower bound for the set on the right-hand side. Therefore M is less than or equal to the greatest lower bound, namely I, in other words

$$M \leqslant I.$$

We conclude that $M = I$.

Problem 13. Let X be a metric space with the following property: For every infinite subset S of X,

$$\inf\{d(x, y) : x \neq y,\ x, y \in S\} = 0.$$

Prove that X is *totally bounded*: In other words, show that for every $\varepsilon > 0$, the space X can be covered by *finitely many* open balls of radius ε.

Solution: Suppose the statement is false for a particular $\varepsilon > 0$, and choose $x_1 \in X$. Then there exists $x_2 \in X$ such that $d(x_2, x_1) \geqslant \varepsilon$. Also, there exists $x_3 \in X$ such that $d(x_3, x_1) \geqslant \varepsilon$ and $d(x_3, x_2) \geqslant \varepsilon$. Continuing in this way, we obtain an infinite sequence $\{x_n\}$ with the property that $d(x_n, x_m) \geqslant \varepsilon$ for $n \neq m$. Let $S = \{x_n : n \geqslant 1\}$. Then

$$\inf\{d(x, y) : x \neq y,\ x, y \in S\} \geqslant \varepsilon,$$

a contradiction.

Problem 14. Let S be the surface area of the sphere $x^2 + y^2 + z^2 = 1$ and V the volume of the ball $x^2 + y^2 + z^2 \leqslant 1$. Let S' be the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 1$ lying above the plane $z = 1/2$, and let V' be the volume of the portion of the ball $x^2 + y^2 + z^2 \leqslant 1$ lying above the plane $z = 1/2$. Show that $S' = S/4$ and $V' = 5V/32$.

Solution: Parametrize the sphere by $\Gamma(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$. Then

$$(\partial\Gamma/\partial\varphi) \times (\partial\Gamma/\partial\theta) = (-\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \cos\varphi\sin\varphi)$$

and consequently the area element is

$$||(\partial\Gamma/\partial\varphi) \times (\partial\Gamma/\partial\theta)|| = \sin\varphi\, d\varphi\, d\theta.$$

Consequently the surface area of S' is

$$\int_0^{2\pi}\int_0^{\pi/3} \sin\varphi\, d\varphi\, d\theta = 2\pi(1 - 1/2) = \pi.$$

(Here recall that φ is the angle between the radial vector $\Gamma(\varphi, \theta)$ and the positive z-axis.) Since the surface area of the whole sphere is 4π (as follows on replacing the integral from 0 to $\pi/3$ by an integral from 0 to π), we do indeed have $S' = S/4$.

For the volume it is easier to use cylindrical coordinates. Thus

$$V' = \int_0^{2\pi} \int_0^{\sqrt{3}/2} \int_{1/2}^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta.$$

The innermost integral is $r\sqrt{1-r^2} - r/2$, and then the integral with respect to r is

$$-((1-r^2)^{3/2}/3 + r^2/4)|_0^{\sqrt{3}/2} = -(\frac{1}{24} + \frac{3}{16}) + \frac{1}{3} = \frac{5}{48}.$$

After multiplying by 2π we get

$$V' = \frac{5}{24}\pi = \frac{5}{32} \cdot \frac{4}{3}\pi = \frac{5}{32}V$$

as claimed.