Preliminary Exam 2018
Solutions to Afternoon Exam

Part I.

Solve four of the following five problems.

Problem 1. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix}.$$

Solution: Any sequence of row operations which puts the matrix $(A|I)$ into row-reduced upper-echelon form terminates in the matrix $(I|B)$, where

$$B = A^{-1} = \begin{pmatrix} 3/8 & 0 & 1/8 \\ 0 & 1/2 & 0 \\ -5/8 & 0 & 1/8 \end{pmatrix}.$$

Alternatively, one can use the formula $A^{-1} = (b_{ij})$, where

$$b_{ij} = (-1)^{i+j} (\det A)^{-1} (\det A_{ji}),$$

where A_{ji} is the matrix obtained from j by removing the jth row and ith column.

Problem 2. The 2×2 matrix A has trace 1 and determinant -2. Find the trace of A^{100}, indicating your reasoning.

Solution: The characteristic polynomial of A is $x^2 - x - 2 = (x+1)(x-2)$. Thus the eigenvalues of A are -1 and 2, and consequently the eigenvalues of A^{100} are 1 and 2^{100}. So $\operatorname{tr}(A) = 1 + 2^{100}$.

Problem 3. Find a basis for the space of solutions to the simultaneous equations

$$\begin{cases} x_1 + 2x_3 + 3x_4 + 5x_5 = 0 \\ x_2 + 5x_3 + 4x_5 = 0. \end{cases}$$

Solution: The associated matrix is already in row-reduced upper-echelon form, so one can solve for the pivotal variables x_1 and x_2 in terms of the nonpivotal variables x_3, x_4, and x_5. Thus the general solution to the system is

$$(-2x_3 - 3x_4 - 5x_5, -5x_3 - 4x_5, x_3, x_4, x_5) = x_3 v_1 + x_4 v_2 + x_5 v_3,$$

where $v_1 = (-2, -5, 1, 0, 0)$, $v_2 = (-3, 0, 0, 1, 0)$, $v_3 = (-5, -4, 0, 0, 1)$, and x_3, x_4, and x_5 are arbitrary scalars. Hence a basis for the solution space is $\{v_1, v_2, v_3\}$.

Problem 4. Let A be a 3×3 matrix with coefficients in $\mathbb{R}$. Show that if $A^4 = 0$ then $A^3 = 0$, and give an example where $A^3 = 0$ but $A^2 \neq 0$.

Solution: If $\lambda \in \mathbb{C}$ is an eigenvalue of A then for some nonzero column vector $X \in \mathbb{C}^3$ we have $AX = \lambda X$. By iteration, $A^4 X = \lambda^4 X$ and consequently $\lambda^4 X = 0$ and hence $\lambda = 0$. Thus 0 is the only eigenvalue of A. It follows that the characteristic polynomial of A is X^3, whence $A^3 = 0$ by the Cayley-Hamilton theorem. If

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then $A^3 = 0$ but

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 5. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$w_1 = (1/\sqrt{3}, -1/\sqrt{3}, 0, 1/\sqrt{3})$$

and

$$w_2 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0).$$

Let $v = (\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3})$. Write $v = w + w^{\perp}$, where $w \in W$ and $w^{\perp}$ is orthogonal to W relative to the dot product.

Solution: Since $\{w_1, w_2\}$ is already an orthonormal basis for W, we have

$$w = (v \cdot w_1)w_1 + (v \cdot w_2)w_2 = 1w_1 + 3w_2$$

and $w^{\perp} = v - w$. So

$$w = (4/\sqrt{3}, 2/\sqrt{3}, \sqrt{3}, 1/\sqrt{3})$$

and

$$w^{\perp} = (-1/\sqrt{3}, 1/\sqrt{3}, 0, 2/\sqrt{3}).$$

Part II.

Solve three of the following six problems.

Problem 6. In a certain group there are elements g and h satisfying $ghg^{-1} = h^{-1}$. Show that $(gh)^2 = g^2$.

Solution: Multiplying both sides of the equation $ghg^{-1} = h^{-1}$ by g^{-1} on the left and g on the right, we obtain $h = gh^{-1}g^{-1}$. Returning to $ghg^{-1} = h^{-1}$ and conjugating by g a second time, we deduce that $g^2hg^{-2} = h$. Thus

$$(gh)^2 = (ghg^{-1})(g^2h) = h^{-1}(g^2hg^{-2})g^2 = g^2.$$

Problem 7. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix (relative to the standard basis)

$$A = \begin{pmatrix} 5/8 & 3/4 & \sqrt{3}/8 \\ -3/4 & 1/2 & \sqrt{3}/4 \\ \sqrt{3}/8 & -\sqrt{3}/4 & 7/8 \end{pmatrix}$$

is a rotation about an axis through the origin. Find the axis.

Solution: The axis is the line spanned by an eigenvector of eigenvalue 1. So we need to find a basis for the null space of

$$A - I = \begin{pmatrix} -3/8 & 3/4 & \sqrt{3}/8 \\ -3/4 & -1/2 & \sqrt{3}/4 \\ \sqrt{3}/8 & -\sqrt{3}/4 & -1/8 \end{pmatrix}$$

Subtract 2 times the first row from the second and add $1/\sqrt{3}$ times the first row to the third. Then divide the first row by $-3/8$ and subtract the second row from the first row. Finally, divide the second row by -2. The matrix is now in row-reduced upper-echelon form, and we see that a basis for the kernel in $(1, 0, \sqrt{3})$. So the axis is the line spanned by this vector.

Problem 8. In this problem you may quote the Rank-Nullity Theorem and general facts from set theory without proof.

(a) Let V be a finite-dimensional vector space, let id_V be the identity map on V (so $\mathrm{id}_V(v) = v$ for all $v \in V$), and let $f, g : V \to V$ be linear maps satisfying $f \circ g = \mathrm{id}_V$. Prove that $g \circ f = \mathrm{id}_V$.

(b) Is (a) still true if we remove the assumption that V has finite dimension? Either prove or give a counterexample.

Solution: (a) It is given that g is a right inverse of f, and we must show that g is actually a genuine two-sided inverse of f. In other words, we are given that g is injective, and we must show that g is also surjective. But by the Rank-Nullity Theorem, $\dim(V)$ is the sum of the rank of g and the nullity of g. Since the nullity is 0 (g is injective), the rank is $\dim(V)$; in other words, the image of g is all of V. So g is surjective.

(b) The statement no longer holds if the dimension of V is infinite. For example, let V be the space of infinite sequences $(a_1, a_2, a_3, \dots)$ of real numbers, and let $f(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$. Let $g(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. Then $f \circ g = \mathrm{id}_V$, but $g \circ f \neq \mathrm{id}_V$: Indeed if $v = (1, 0, 0, \dots)$ then $g(f(v)) = 0$, not v.

Problem 9. Let V be the real vector space consisting of polynomials with real coefficients and degree at most 2, and consider the linear transformation $T : V \to V$ given by $T(f(x)) = f(x) + f'(x) + f''(x)$. What is the Jordan normal form of T? Your answer should be a matrix together with justification.

Solution: As ordered basis for V we take 1, x, x^2. The matrix of T relative to this basis is

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

A computation shows that $(A - I)^3 = 0$ but $(A - I)^2 \neq 0$, whence the Jordan normal form of T is

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 10. Let L be the lattice in $\mathbb{Z}^3$ spanned by the vectors $(1, 1, 60)$, $(2, 0, 60)$, and $(1, 1, 0)$. Write $\mathbb{Z}^3/L$ as a direct sum of cyclic factors,

$$\mathbb{Z}^3/L \cong (\mathbb{Z}/a_1\mathbb{Z}) \oplus (\mathbb{Z}/a_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/a_k\mathbb{Z}),$$

where $a_j \geqslant 2$ for $1 \leqslant j \leqslant k$ and k, the number of cyclic direct summands, is (i) minimal, (ii) maximal.

Solution: Using row and column operations over $\mathbb{Z}$, we can put the relevant matrix in elementary divisor form:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 60 & 60 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 60 \end{pmatrix}.$$

Thus

$$\mathbb{Z}^3/L \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/60\mathbb{Z})$$

and hence by the Chinese Remainder Theorem also

$$\mathbb{Z}^3/L \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z}).$$

The first decomposition, where $k = 2$, is the decomposition with minimal k, and the second expression, where $k = 4$, is the decomposition with maximal k. Indeed $k = 1$ is not possible because $\mathbb{Z}^3/L$ is not cyclic (indeed the subgroup $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$ is not cyclic) and $k = 5$ is also not possible because $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$ is not isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Problem 11. Let S_n denote the group of permutations of $\{1, 2, 3, \ldots, n\}$. In each case, give an example of the indicated type or explain why none exists:
(a) an element of order 40 in S_{13}.
(b) an element of order 34 in S_{16}

Solution: The order of an element σ of S_n is the least common multiple of the lengths of the cycles in a disjoint cycle decomposition of σ. In case (a) we can take

$$\sigma = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10, 11, 12, 13).$$

However in case (b) we would need either a cycle of length 34 (impossible in S_{16}) or else the product of disjoint cycles of length 2 and length 17. This is again impossible because a cycle of length 17 does not exist in S_{16}.

Part III.

Solve one of the following three problems.

Problem 12. Give an example, with supporting justification, of two commutative rings which are isomorphic as abelian groups under addition but not isomorphic as rings.

Solution: There are many possiblities. For example, let p be a prime, let $\mathbb{F}_p$ be the field with p elements, and take $R = \mathbb{F}_p \times \mathbb{F}_p$, and let $R' = \mathbb{F}_p[x]/(x^2)$. Then both R and R' are two-dimensional vector spaces over $\mathbb{F}_p$ (the cosets of 1 and x are a basis for R'). But R' has nilpotent elements, namely the multiples of the coset of x, whereas R does not have nilpotents. So R and R' are not isomorphic as rings.

Of course there are many other solutions. For example, we could let $R' = \mathbb{F}_{p^2}$ and leave R unchanged. Then R and R' are not isomorphic as rings because R' is a field and R is not. Or take $R = \mathbb{Q} \times \mathbb{Q}$ and $R' = \mathbb{Q}(i)$: Again, R' is a field and R is not.

Problem 13. Let $\mathbb{F}_p$ be the field with p elements. What are the degrees and multiplicities of the monic irreducible factors of the polynomial

$$f(x) = x^3 + x^2 + 3x + 2$$

viewed over (i) $\mathbb{F}_2$, (ii) $\mathbb{F}_3$, (iii) $\mathbb{R}$, and (iv) $\mathbb{Q}$? Give reasons for your answers.

Solution: (i) Over $\mathbb{F}_2$ we have

$$f(x) = x^3 + x^2 + x = x(x^2 + x + 1).$$

Since $x^2 + x + 1$ is irreducible over F_2 (for $f(0) = f(1) = 1 \neq 0$) there are two distinct irreducible factors, one of degree 1 and one of degree 2.

(ii) Over $\mathbb{F}_3$ we have $f(x) = x^3 + x^2 + 2$. Since $f(0) = 2$, $f(1) = 1$, and $f(2) = 2$, we see that f does not have any degree-one factors and is therefore irreducible of degree 3.

(iii) Since f is a cubic, it has at least one real zero. But for all $x \in R$,

$$f'(x) = 3x^2 + 2x + 3 > x^2 + 2x + 1 = (x+1)^2 \geqslant 0,$$

so f is strictly increasing on $\mathbb{R}$ and therefore has exactly one real zero. So f has one irreducible factor of degree 1 and one of degree 2.

(iv) Since f is irreducible over $\mathbb{F}_3$ it is irreducible over $\mathbb{Q}$.

Problem 14. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 7 \end{pmatrix}.$$

State the Jordan normal form, with justification, for A as a matrix with coefficients in (i) $\mathbb{Q}$, (ii) $\mathbb{F}_2$, (iii) $\mathbb{F}_3$, and (iv) $\mathbb{F}_5$.

Solution: (i) Over $\mathbb{Q}$, the matrix A has 3 distinct eigenvalues, namely 1, 3, and 7. Since the eigenvalues are distinct the matrix is diagonalizable, so the Jordan normal form is

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

(ii) Over $\mathbb{F}_2$, the matrix A is in Jordan normal form as it stands:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) Over $\mathbb{F}_3$, there are two eigenvalues, 0 with multiplicity one and 1 with multiplicity two. But A is not diagonalizable, for $A(A-I) \neq 0$ by direct calculation. Hence the Jordan normal form of A is

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iv) Finally, over $\mathbb{F}_5$ there are again 3 distinct eigenvalues, namely 1,2, and 3. So A is diagonalizable and the Jordan normal form is the matrix J in (a), although now $7 = 2$.