Preliminary Exam 2018
Afternoon Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix}.$$

Problem 2. The 2×2 matrix A has trace 1 and determinant -2. Find the trace of A^{100}, indicating your reasoning.

Problem 3. Find a basis for the space of solutions to the simultaneous equations

$$\begin{cases} x_1 + 2x_3 + 3x_4 + 5x_5 = 0 \\ x_2 + 5x_3 + 4x_5 = 0. \end{cases}$$

Problem 4. Let A be a 3×3 matrix with coefficients in $\mathbb{R}$. Show that if $A^4 = 0$ then $A^3 = 0$, and give an example where $A^3 = 0$ but $A^2 \neq 0$.

Problem 5. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$w_1 = (1/\sqrt{3}, -1/\sqrt{3}, 0, 1/\sqrt{3})$$

and

$$w_2 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0).$$

Let $v = (\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3})$. Write $v = w + w^\perp$, where $w \in W$ and $w^\perp$ is orthogonal to W relative to the dot product.

Part II.

Solve three of the following six problems.

Problem 6. In a certain group there are elements g and h satisfying $ghg^{-1} = h^{-1}$. Show that $(gh)^2 = g^2$.

Problem 7. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix (relative to the standard basis)

$$A = \begin{pmatrix} 5/8 & 3/4 & \sqrt{3}/8 \\ -3/4 & 1/2 & \sqrt{3}/4 \\ \sqrt{3}/8 & -\sqrt{3}/4 & 7/8 \end{pmatrix}$$

is a rotation about an axis through the origin. Find the axis.

Problem 8. In this problem you may quote the Rank-Nullity Theorem and general facts from set theory without proof.

(a) Let V be a finite-dimensional vector space, let id_V be the identity map on V (so $\mathrm{id}_V(v) = v$ for all $v \in V$), and let $f, g : V \to V$ be linear maps satisfying $f \circ g = \mathrm{id}_V$. Prove that $g \circ f = \mathrm{id}_V$.

(b) Is (a) still true if we remove the assumption that V has finite dimension? Either prove or give a counterexample.

Problem 9. Let V be the real vector space consisting of polynomials with real coefficients and degree at most 2, and consider the linear transformation $T : V \to V$ given by $T(f(x)) = f(x) + f'(x) + f''(x)$. What is the Jordan normal form of T? Your answer should be a matrix together with justification.

Problem 10. Let L be the lattice in $\mathbb{Z}^3$ spanned by the vectors $(1, 1, 60)$, $(2, 0, 60)$, and $(1, 1, 0)$. Write $\mathbb{Z}^3/L$ as a direct sum of cyclic factors,

$$\mathbb{Z}^3/L \cong (\mathbb{Z}/a_1\mathbb{Z}) \oplus (\mathbb{Z}/a_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/a_k\mathbb{Z}),$$

where $a_j \geqslant 2$ for $1 \leqslant j \leqslant k$ and k, the number of cyclic direct summands, is (i) minimal, (ii) maximal.

Problem 11. Let S_n denote the group of permutations of $\{1, 2, 3, \ldots, n\}$. In each case, give an example of the indicated type or explain why none exists:

(a) an element of order 40 in S_{13}.

(b) an element of order 34 in S_{16}

Part III.

Solve one of the following three problems.

Problem 12. Give an example, with supporting justification, of two commutative rings which are isomorphic as abelian groups under addition but not isomorphic as rings.

Problem 13. Let $\mathbb{F}_p$ be the field with p elements. What are the degrees and multiplicities of the monic irreducible factors of the polynomial

$$f(x) = x^3 + x^2 + 3x + 2$$

viewed over (i) $\mathbb{F}_2$, (ii) $\mathbb{F}_3$, (iii) $\mathbb{R}$, and (iv) $\mathbb{Q}$? Give reasons for your answers.

Problem 14. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 7 \end{pmatrix}.$$

State the Jordan normal form, with justification, for A as a matrix with coefficients in (i) $\mathbb{Q}$, (ii) $\mathbb{F}_2$, (iii) $\mathbb{F}_3$, and (iv) $\mathbb{F}_5$.