Preliminary Exam 2016
Afternoon Exam (3 hours)

Part I.

Solve four of the following five problems.

Problem 1. Let W be the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$. Find a basis for W which is orthonomal relative to the usual dot product.

Problem 2. Consider the matrix (with real coefficients)

$$A = \begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix}.$$

Find an invertible matrix C such that $C^{-1}AC$ is diagonal.

Problem 3. Let A be a 2×2 matrix with real coefficients, and suppose that $A^4 = I$ but $A^2 \neq I$, where I is the 2×2 identity matrix.

(a) Find $\mathrm{tr}\,(A)$.

(b) Give an example of a 2×2 matrix B with *complex* coefficients such that $B^4 = I$ and $B^2 \neq I$ but $\mathrm{tr}\,(B) \neq \mathrm{tr}\,(A)$.

Problem 4. Let $\varphi : \mathbb{Q} \to \mathbb{Z}$ be a group homomorphism. Show that $\varphi(x) = 0$ for all $x \in \mathbb{Q}$.

Problem 5. A sequence of row operations transforms the matrix

$$A = \begin{pmatrix} 1 & 4 & * & * \\ 3 & 2 & * & * \\ 5 & 3 & * & * \end{pmatrix} \quad \text{into the matrix} \quad B = \begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation which has A as its matrix relative to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^3$. Find a basis for the kernel and image of T, and determine the third and fourth columns of A.

Part II.

Solve three of the following six problems.

Problem 6. Let A be a 3×3 matrix such that $(A^2 - I)(A - I) = 0$ but $A^2 - I \neq 0$. What are the possibilites for the Jordan normal form of A? (You may take A to have complex coefficients.)

Problem 7. Make a list of abelian groups of order 32 so that every abelian group of order 32 is isomorphic to exactly one group on your list.

Problem 8. Let R be a commutative ring, and let I be the set of all $r \in R$ such that $r^k = 0$ for some positive integer k (which may depend on r). Show that I is an ideal of R.

Problem 9. Write $\mathbb{Z}^3/L$ as a direct sum (or a direct product) of cyclic groups, where L is the subgroup of $\mathbb{Z}^3$ generated by $(1, 1, -2)$, $(7, 9, -14)$, and $(5, 9, -6)$.

Problem 10. Find an integer $n \geqslant 1$ such that the Galois group over $\mathbb{Q}$ of the polynomial $x^n - n$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Problem 11. Let A be an $n \times n$ matrix, let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A, and let a_{ij} be the entry in the ith row and jth column of A. For $1 \leqslant h, i \leqslant n$, what is $\sum_{j=1}^{n} a_{hj}(-1)^{i+j} \det(A_{ij})$? Why? Your answer should depend on whether $h = i$ or $h \neq i$.

Part III.

Solve one of the following three problems.

Problem 12. Factor the polynomial $x^3 - 2$ into irreducible polynomials in $\mathbb{F}_p[x]$ for $p = 3, 5$, and 7, where $\mathbb{F}_p$ is the field with p elements.

Problem 13. Let G be a finite group and H a subgroup, and let $\mathcal{S}$ be the group of permutations of G/H, the set of left cosets of H in G. Define a homomorphism $\varphi : G \to \mathcal{S}$ by setting $\varphi(a)(bH) = abH$ for $a, b \in G$. Show that the kernel of φ is the largest normal subgroup of G contained in H.

Problem 14. Let A be a square matrix with coefficients in $\mathbb{C}$, and suppose that A is not diagonalizable but that A^n is diagonalizable for some $n \geqslant 2$. Show that $\det(A) = 0$.