Statistics and Biostatistics Ph.D. Theory Qualifier 2009

Total time: 4 hours.

Instructions: This is a closed book closed notes exam; however, you can use copies of the pages of your textbook containing the definitions of various pdfs and their moments. You need to answer any FIVE, but no more than FIVE, problems. If you answer more than FIVE problems, only the first FIVE will be graded. If you don't want me to grade a part of your answer, please cross out that part of your answer book completely. Also note that the problems are not arranged in any particular order, so I suggest that you read all eight problems before deciding on the FIVE that you wish to answer.

Ph.D. Theory Qualifying Exam 2009

1. Let $X_1, X_2, ..., X_n$ be i.i.d. $Beta(\theta.1)$, that is the densities are

$$f(x) = \theta x^{\theta-1}, \quad 0 < x < 1, \ \theta > 0,$$

and suppose the prior distribution for θ is $Exp(\lambda)$, where λ is a hyperparameter (a known constant).

a. Find the posterior distribution of θ.

b. Define the term "conjugate prior". Does exponential distribution constitute a family of conjugate priors for $f(x)$?

b. Find the posterior mode of θ (note: *mode*, not the mean or median).

2. Let $X_1, ..., X_n$ be i.i.d. discrete random variables having a distribution in the parametric family with probability functions

$$f(x|\lambda) = (1 - e^{-\lambda})e^{-\lambda x}, \quad x = 0, 1, 2, ...$$

where $\lambda > 0$ is the parameter.

a. Find the MLE of λ.

b. Find the expected Fisher Information of λ.

c. Give an asymptotic 95% confidence interval for λ.

3. Let Y be a random variable with pdf

$$f(y) = \frac{by^{b-1}e^{-(y/\theta)^b}}{\theta^b}, \quad y > 0.$$

a. Let $X = Y^b$, find the pdf of X.

b. Suppose b is a known fixed constant. Find the UMP test for testing

$$H_0 : \theta = 1 \quad \text{vs} \quad H_a : \theta > 1.$$

Give the exact critical value of the test.

4. Suppose that $X|n, \theta$ has a binomial distribution $B(n, \theta)$, where both n and θ are parameters of the model. Suppose we assign independent priors on n and θ, with n having a $Poisson(\lambda)$ prior, and θ having a $Beta(\alpha, \beta)$ prior, where α and β are known hyperparameters.

a. Prove that the posterior density of θ given $X = x$ and n is $Beta(x + \alpha, n - x + \beta)$.

b. Prove that the posterior probability distribution of $n - X$ given $X = x$ and θ is $Poisson[(1 - \theta)\lambda]$.

c. Suppose $\alpha = \beta = 1, \lambda = 20$, and $X = 10$, explain in detail how you can obtain 100 samples of $n's$ from the Posterior distribution of n given $X = 10$.

5. Suppose $X_1, ..., X_n$ are i.i.d. sample of size n from a population with density

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0, \theta > 0.$$

Consider the problem of estimating $\pi = P(X > 1) = e^{-1/\theta}$.

a. Compute the Cramér-Rao lower bound for the variance of an unbiased estimator of π.

a. Find the MLE of π.

b. Find the UMVUE of π. For large n, would you expect the MLE to be approximately the same as UMVUE? Why?

6. Let $X_1, .., X_n$ be a random sample from the probability mass function $f(x)$ that is positive only on the non-negative integers. We wish to test the simple hypothesis

$$H_0 : f(x) = \begin{cases} \frac{e^{-1}}{x!}, & x = 0, 1, 2, .. \\ 0, & \text{otherwise} \end{cases}$$

against the alternative simple hypothesis

$$H_a : f(x) = \begin{cases} \left(\frac{1}{2}\right)^{x+1}, & x = 0, 1, 2, .. \\ 0, & \text{otherwise} \end{cases}$$

a. Derive the rejection region of the UMP test of the form $R = \left\{ x | \frac{f(x|H_a)}{f(x|H_0)} > k \right\}$.

b. For the case $n = 1$ and $k = 1$, explicitly identify the rejection region R.

c. Compute the level of the test for the case $n = 1$ and $k = 1$.

d. Compute the power of the test for the case $n = 1$ and $k = 1$.

7. Let $X_1, X_2, .., X_n$ are i.i.d. $Exponential(\theta)$. Suppose X_i represent the lifetime of independent pieces of equipment in hundreds of hours, and warranty replacement period is (say) 200 hours, then we may be interested in warranty failure probability $\beta(\theta) = P_\theta(X \leq 2)$.

a. Obtain MLE of $\beta(\theta)$..

b. Compute the **approximate** bias and variance of your estimator in part a.

c. Derive the asymptotic distribution of your estimator in part a, and hence compute a $(1 - \alpha)$ confidence interval for $\beta(\theta)$.

8. a. If $Y_1, ..., Y_n$ is a random sample from density $\theta^{-1} e^{-\frac{x}{\theta}}$, show that the maximum likelihood estimator $\widehat{\theta}$ of θ has an asymptotic normal distribution with mean θ and variance $\frac{\theta^2}{n}$.

b. Hence deduce that an approximate $(1 - 2\alpha)$ confidence interval for θ is

$$\frac{\widehat{\theta}}{1 + z_{1-\alpha} n^{-1/2}} \leq \theta \leq \frac{\widehat{\theta}}{1 + z_\alpha n^{-1/2}}.$$

c. Show that $\frac{\widehat{\theta}}{\theta}$ is a pivot. Hence find an exact $(1 - 2\alpha)$ confidence interval for θ, and compare it with the approximate one when $n = 10$, $\widehat{\theta} = 100$ and $\alpha = 0.025$.