Ph.D. Qualifying Exam 2004 (MA781)

1. Eggs are thought to be infected with a bacterium *salmonela enteriditis* so that the number of the organisms, Y, in each has a Poisson distribution with mean μ. The value of Y cannot be observed directly, but after a period it becomes certain whether the egg is infected $(Y > 0)$ or not $(Y = 0)$. Out of m such eggs, r are found to be infected. Find the maximum likelihood estimator $\widehat{\mu}$ of μ and derive it's asymptotic variance.

2. Let $X_1, ..., X_n$ are independent observations such that X_i is distributed as $Poisson(i\theta)$;i.e., $E(X_1) = \theta, E(X_2) = 2\theta, E(X_3) = 3\theta$....etc. Define:

$$a_n = \sum_{i=1}^n i = \frac{n(n+1)}{2} \qquad \text{and} \qquad b_n = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

a. Consider the following two estimators of θ.

$$P = \sum_{i=1}^n \frac{X_i}{a_n} \qquad \text{and} \qquad Q = \sum_{i=1}^n \frac{iX_i}{b_n}.$$

Show that both estimators are unbiased for θ.

b. Compute the variances of the estimators, and hence identify the better of the two estimators. Denote it by $\widehat{\theta}$. (Hint: $(\sum_{i=1}^n a_i b_i)^2 < (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$, unless $a_i = kb_i$)

d. Show that $\widehat{\theta}$ attains the Cramer-Rao lower bound, and hence it is efficient.

e. Show that $\widehat{\theta}$ is, in fact, the MLE of θ; and hence derive the asymptotic distribution of $\widehat{\theta}$.

3. (Biostatistics only)

Suppose a random variable X is distributed as

$$f(x|\theta) = \exp[-(x-\theta)],\ x > \theta > 0.$$

Consider the problem of deriving the Bayes estimator θ based on a single observation x_0.

a. Compute the posterior distribution $g(\theta|x_0)$ with respect to the prior

$$\pi(\theta) = \exp[-\theta],\ \theta > 0.$$

b. Identify the Bayes estimator of θ under squared error loss.

c. A $(1-\alpha)$ posterior confidence set (*credible set*) is defined to be a set C of values of θ, whose posterior probability content is at least $(1-\alpha)$. When θ is continuous, this is

$$(1-\alpha) = P(\theta \in C|x) = \int_C g(\theta|x)d\theta.$$

Compute a $(1-\alpha)$ *credible interval* of θ based on the posterior distribution computed in part (a).

Ph.D. Qualifying Exam 2004 (MA782)

1.

a. If $Y_1, ..., Y_n$ is a random sample from density $\theta^{-1}e^{-\frac{x}{\theta}}$, show that the maximum likelihood estimator $\widehat{\theta}$ of θ has an asymptotic normal distribution with mean θ and variance $\frac{\theta^2}{n}$.

b. Hence deduce that an approximate $(1-2\alpha)$ confidence interval for θ is

$$\frac{\widehat{\theta}}{1+z_{1-\alpha}n^{-1/2}} \leq \theta \leq \frac{\widehat{\theta}}{1+z_{\alpha}n^{-1/2}}.$$

c. Show that $\frac{\widehat{\theta}}{\theta}$ is a pivot. Hence find an exact $(1-2\alpha)$ confidence interval for θ, and compare it with the approximate one when $n = 10$, $\widehat{\theta} = 100$ and $\alpha = 0.025$.

2. Let $X_1, ..., X_n$ be independent observations from distributions $X_i \tilde{} N(i\theta, \sigma^2)$, both θ and σ^2 are unknown. Note $E(X_i) = i\theta$, for $i = 1, 2, .., n$. Consider the problem of developing LRT for testing

$$H_o : \theta = \theta_o \text{ against } H_a : \theta \neq \theta_0$$

a. Show that the unrestricted MLE of θ and σ^2 are given by

$\widehat{\theta} = \frac{\sum_{i=1}^n iX_i}{b_n}$ and $\widehat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - i\widehat{\theta})^2}{n}$, where $b_n = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

b. Derive the MLE $\widehat{\theta}_0$ and $\widehat{\sigma}_0^2$ under H_0.

c. Show that the LRT statistic is $\Lambda = (1+\frac{t^2}{n-1})^{n/2}$, where $t = b_n^{1/2}(\widehat{\theta}-\theta_0)/s$, and $s^2 = \frac{n}{n-1}\widehat{\sigma}^2$. Hence show that LRT rejects H_0 if $|t| > t_{n-1,\alpha/2}$.

(Hint: show that $\widehat{\sigma}_0^2 = \widehat{\sigma}^2 + b_n(\widehat{\theta} - \theta_o)^2/n$).

3. (Biostatistics only)

Let $X_1, ..., X_n$ are independent observations such that X_i is distributed as $Poisson(i\theta)$ (as in problem 2 in MA781).

a. Does there exits an UMP test for testing

$$H_o : \theta = 2 \text{ against } H_a : \theta > 2?$$

If yes (justify), derive the *exact* test for $\alpha = 0.05$ and $n = 3$.

b. Now consider testing the hypothesis

$$H_o : \theta = 2 \text{ against } H_a : \theta \neq 2.$$

Does there exist an UMP test for this problem? If yes, derive the test for $\alpha = 0.05$ and $n = 3$; otherwise, justify your answer.