PH.D. QUALIFYING EXAMINATION 1994

MA 782

1. Let $X_1, X_2, \ldots, X_n$ be iid Poisson(λ).

a) Find a UMP test for $H_0 : \lambda \le \lambda_0$ versus $H_a : \lambda > \lambda_0$.

b) Consider a specific case $H_0 : \lambda \le 1$ versus $H_a : \lambda > 1$. Use the central limit theorem to determine the sample size n so that a UMP test of size 0.05 will have the power 0.9 at $\lambda = 2$.

(For part b, you don't need to consider randomized tests)

2. Let $X_1, X_2, \ldots, X_n$ be iid observations from Pareto density given by:

$$f(x,q) = c^{\theta}\, \theta\, x^{-(1+\theta)}, \quad x > c,$$

where $\theta > 1$ and $c > 0$ known.

a) Express mean μ in terms of θ.

b) Find the UMP test for testing $H_0 : \mu \le \mu_0$ versus $H_a : \mu > \mu_0$.

c) In a certain state the mean income is \$9000 per year. In a sample of 100 persons of a certain occupation in that state, we obtain the incomes $X_1, X_2, \ldots, X_{100}$ in thousands of dollars and compute $\Sigma \log X_i = 230$. Using Pareto model with $c = 5$, test the hypothesis $H_0 : \mu \le 9$ versus $H_a : \mu > 9$. at level $\alpha = 0.01$.

(For part c, use normal approximation with $E(\log X) = \log c + (1/\theta)$ and $Var(\log X) = 1/\theta^2$).

MA 782
Ph.D. Qualifying Exam

1. Let $Y_1, \ldots, Y_n$ be a random sample from a beta distribution with parameters $(\theta, 1)$. Derive a test procedure for the hypotheses:

$$H_0 : \theta = 1$$
$$H_a : \theta \neq 1.$$

(a) Show that the UMPU test rejects H_0 if

$$Q = -\sum_{i=1}^{n} \log Y_i$$

is sufficiently large or sufficiently small.

(b) Find the exact distribution of $2Q$, and use it to find critical values of the test in (a).

(c) Determine the power function of the test derived in parts (a) and (b).

2. Let $\theta = (p_1, p_2, p_3, p_4, p_5)$ where $\sum_{i=1}^{5} p_i = 1$ and $p_i \geq 0$, $i = 1, 2, 3, 4, 5$. Suppose $X_1, \ldots, X_n$ i.i.d. discrete random variables and $P_\theta(X_i = j) = p_j$, $j = 1, 2, 3, 4, 5$.

(a) Find the likelihood ratio test statistic to test:

$$H_0 : \ p_1 = p_2 = p_3 \text{ and } p_4 = p_5 \quad \text{vs.} \qquad H_a : \ H_0 \text{ is not true.}$$

(b) What is the approximate critical value of the test derived in part (a)?

MA 782

Ph.D. Qualifying Exam

1. Suppose $X_1 \ldots X_n$ are i.i.d. uniform $(0, \theta)$ distribution.
Let $M_n = \max(X_1 \ldots X_n)$
We want to test

$$\text{Ho} : \theta \leq \theta_0 \qquad \text{Ha} : \theta > \theta_0$$

for some $\theta_0 > 0$.
Consider the following test statistics

$$d_c(\underline{x}) = \begin{cases} 1 & \text{if } M_n \geq c \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the power function of $d_c(\underline{x})$ and show that it is a monotone increasing function of θ.

(b) In testing Ho: $\theta \leq \frac{1}{2}$ vs. Ha: $\theta > \frac{1}{2}$, what choice of c would make $d_c(\underline{x})$ have size exactly $0 \cdot 05$?

(c) Draw a rough graph of the power function of $d_c(\underline{x})$ specified in (b) when $n = 20$.

(d) How large n be so that $d_c(\underline{x})$ specified in (b) has power $0 \cdot 98$ for $\theta = \frac{3}{4}$?

(e) If in a sample of size $n = 20$, $M_n = 0 \cdot 48$; what is the p-value of the test?

2. Let $X_1 \ldots X_n$ denote the number of hours a random sample of internet subscribers spent per week on the internet. Suppose the data is modeled as a sample from an exponential (θ) distribution.

(a) Construct a two-sided $(1 - \alpha)$ CI for θ.

(b) What kind of optimality property your procedure in part (a) has? Justify your answer.

MA 782
Ph.D. Qualifying Exam

1.

Let $X_i = (\frac{\theta}{2})\, t_i^2 + \epsilon_i,\ i = 1, 2, \ldots, n$, where ϵ_i are normal random variables with mean 0 and *known* variance σ^2.

(a) Obtain the UMA confidence interval for θ with confidence coefficient $(1-\alpha)$.

(b) Show that $T = 2\sum_{i=1}^n X_i / \sum_{i=1}^n t_i^2$ is an unbiased estimator of θ. Use T to construct another $(1-\alpha)$ confidence interval for θ.

(c) Show that the confidence interval in (a) is more accurate than the one in (b) by showing that the length of the confidence interval in (a) is uniformly smaller than (b).

2.

Let $X_1, \ldots, X_n$ be the times in months until failure of n similar pieces of equipment. If the equipment is subject to wear, a model often used is the one where $X_1, \ldots, X_n$ is a sample from the density

$$f(x, \lambda) = \lambda c \ \ x^{c-1} \ \ e^{-\lambda x^c}, \quad x > 0,$$

c is a known positive constant and $\lambda > 0$ is the scale parameter of interest.

(a) Find the UMP test for testing

$$Ho\colon \frac{1}{\lambda} \le \frac{1}{\lambda_0} \quad \text{vs.} \quad Ha\colon \frac{1}{\lambda} > \frac{1}{\lambda_0}.$$

(b) Find explicitly the critical value for the size α test.

(c) Suppose $\frac{1}{\lambda_0} = 12$. Find the sample size needed for a level 0.01 test to have power at least 0.95 at the alternative value $\frac{1}{\lambda_1} = 15$.

for part (c):

Hint: [For this problem, do not use the critical value obtained in part (b) using the exact distribution of the test statistic. Instead, use normal approximation to find the critical value and the probability of rejection.]

Ph.D. Qualifying Exam 2004 (MA782)

1.

a. If $Y_1, ..., Y_n$ is a random sample from density $\theta^{-1}e^{-\frac{y}{\theta}}$, show that the maximum likelihood estimator $\widehat{\theta}$ of θ has an asymptotic normal distribution with mean θ and variance $\frac{\theta^2}{n}$.

b. Hence deduce that an approximate $(1-2\alpha)$ confidence interval for θ is

$$\frac{\widehat{\theta}}{1+z_{1-\alpha}n^{-1/2}} \leq \theta \leq \frac{\widehat{\theta}}{1+z_{\alpha}n^{-1/2}}.$$

c. Show that $\frac{\widehat{\theta}}{\theta}$ is a pivot. Hence find an exact $(1-2\alpha)$ confidence interval for θ, and compare it with the approximate one when $n = 10$, $\widehat{\theta} = 100$ and $\alpha = 0.025$.

2. Let $X_1, ..., X_n$ be independent observations from distributions $X_i \tilde{} N(i\theta, \sigma^2)$, both θ and σ^2 are unknown. Note $E(X_i) = i\theta$, for $i = 1, 2, .., n$. Consider the problem of developing LRT for testing

$$H_o : \theta = \theta_o \text{ against } H_a : \theta \neq \theta_0$$

a. Show that the unrestricted MLE of θ and σ^2 are given by

$\widehat{\theta} = \frac{\sum_{i=1}^n iX_i}{b_n}$ and $\widehat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - i\widehat{\theta})^2}{n}$, where $b_n = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

b. Derive the MLE $\widehat{\theta}_0$ and $\widehat{\sigma}_0^2$ under H_0.

c. Show that the LRT statistic is $\Lambda = (1 + \frac{t^2}{n-1})^{n/2}$, where $t = b_n^{1/2}(\widehat{\theta} - \theta_0)/s$, and $s^2 = \frac{n}{n-1}\widehat{\sigma}^2$. Hence show that LRT rejects H_0 if $|t| > t_{n-1,\alpha/2}$.

(Hint: show that $\widehat{\sigma}_0^2 = \widehat{\sigma}^2 + b_n(\widehat{\theta} - \theta_o)^2/n$).

3. (Biostatistics only)

Let $X_1, ..., X_n$ are independent observations such that X_i is distributed as $Poisson(i\theta)$ (as in problem 2 in MA781).

a. Does there exits an UMP test for testing

$$H_o : \theta = 2 \text{ against } H_a : \theta > 2?$$

If yes (justify), derive the *exact* test for $\alpha = 0.05$ and $n = 3$.

b. Now consider testing the hypothesis

$$H_o : \theta = 2 \text{ against } H_a : \theta \neq 2.$$

Does there exist an UMP test for this problem? If yes, derive the test for $\alpha = 0.05$ and $n = 3$; otherwise, justify your answer.

MA782 Qualifying Exam 2003

1. Consider the pdf $f(x,\theta) = \frac{1}{\beta}\exp(-\frac{x-\theta}{\beta}), \beta$ is known, $\theta \in R, x \geq \theta$. Suppose $X_1, .., X_n$ be i.i.d. $f(x,\theta)$,let $Y = \min(X_1, ..., X_n)$.

a. Show that $T = \frac{Y-\theta}{\beta}$ is a pivot.

b. Determine a and b, such that $P(a < T < b) = 1 - \alpha$,and hence conclude that $[Y - \frac{b\beta}{n}, Y - \frac{a\beta}{n}]$ confidence interval for θ.

c. Note that the length of any $(1-\alpha)$ interval of this form is $l = \frac{\beta}{n}(b-a)$. Set $b = b(a)$ and show that the shortest confidence interval is given by $[Y + \frac{b\log\alpha}{n}, Y]$.

{Hint for c. Set $b = b(a)$;and from $e^{-a} - e^{-b} = 1 - \alpha$, obtain $\frac{db}{da}$ by differentiating both sides w.r.t. a.Then replace $\frac{db}{da}$ in $\frac{dl}{da}$ and observe that it is >0. Hence argue that l attains its minimum at $a = 0$.}

2. Let X be a discrete random variable whose pmf under H_0 and H_a is given by

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_a)$	.06	.05	.04	.03	.02	.01	.79

Use Neyman-Pearson Lemma to find the most powerful test for H_o vs. Ha with size $\alpha = 0.04$. Compute the probability of Type II error for this test.

3. (Biostatistics only) Let $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ be independent $\exp(\theta)$ and $\exp(\lambda)$ samples respectively. Let $\Delta = \frac{\theta}{\lambda}$.

a. If $f(\alpha)$ denotes the αth quantile of the $F_{2n_1,2n_2}$ distribution, show that $[\bar{Y}f(\frac{\alpha}{2})/\bar{X}, \bar{Y}f(1-\frac{\alpha}{2})/\bar{X}]$ is a confidence interval of Δ with confidence coefficient $1 - \alpha$.

b. Show that the test with acceptance region $[f(\frac{\alpha}{2}) \leq \frac{\bar{X}}{\bar{Y}} \leq f(1 - \frac{\alpha}{2})]$ has size α for testing $H_0 : \Delta = 1$ vs. $H_a : \Delta \neq 1$.

c. The following are times until breakdown in days of air monitors operated under two different maintenance policies at a nuclear power plant. Experience has shown that the exponential assumption is warranted. Give a 90% confidence interval for the ratio Δ of mean life times.

x 3, 150, 40, 34, 32, 37, 34, 2, 31, 6, 6, 14, 150, 27, 4, 27, 10, 30, 37

y 8, 26, 10, 8, 29, 20, 10