MA 781 Problems
Ph.D. Qualifying

1.

Consider an experiment where, for given θ, the outcome X has density $f(x,\theta) = 2x/\theta^2$, $0 < x < \theta$.

Let π denote a prior distribution for θ.

(a) Find the posterior density of θ when

$$\pi(\theta) = 1, \quad 0 \le \theta \le 1.$$

(b) Find the posterior density of θ when

$$\pi(\theta) = 3\theta^2, \quad 0 \le \theta \le 1.$$

(c) Compute the predictor of θ under squared error loss in (a) and (b) above.

(d) Suppose $X_1, \ldots, X_n$ are independent with the same distribution as X. Find the posterior density of θ given $X_1 = x_1, \ldots, X_n = x_n$ when $\pi(\theta) = 1, \ 0 \le \theta \le 1$.

2.

Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \sigma^2)$. Our objective is to estimate θ^2.

(a) Compute UMVU estimator (δ_{1n}) of θ^2 when σ^2 is known.

(b) Compute UMVU estimator (δ_{2n}) of θ^2 when σ^2 is unknown.

(c) Write down the MLE (δ_{3n}) of θ^2.

(d) Find the asymptotic distributions of

$$\Delta_i = \sqrt{n}(\delta_{in} - \theta^2), \quad \text{for} \quad i = 1, 2, 3.$$

(e) Comparing the asymptotic distributions in part (d), what can you say about the relative performance of these three estimators for large n?

1. a. Suppose $X_1 \ldots X_n$ i.i.d Bernoulli (θ) distribution. Compute the Bayes estimator θ with respect to the Beta (α, β) prior, and loss function given by

$$L(\theta, a) = \frac{(\theta - a)^2}{\theta(1-\theta)}, \quad 0 < \theta < 1.$$

b. Specification of a prior distribution is an important issue in Bayesian estimation problems. In the problem stated in (a) above, one can use (parametric) empirical Bayes approach to estimate α and β from the data. To do this,

b1) Compute the unconditional mean and variance of X, where

$$X|\theta \sim \text{Bernoulli } (\theta)$$

and

$$\theta \sim \text{Beta } (\alpha, \beta)$$

b2) Hence compute a method of moment estimator of α and β, and also the empirical Bayes estimator $\widehat{\theta}^B$ under the same loss function as in (a).

2. Let $X_1 \ldots X_n$, $n \geq 2$ be i.i.d. with pdf

$$f(x; \theta) = \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}; \; x \geq \mu$$

where $\theta = (\mu, \sigma^2); -\infty < \mu < \infty, \sigma^2 > 0$.

(a) Find MLEs of μ and σ^2.

(b) Find the MLE of

$$P_\theta(\overline{x} \geq t) \quad \text{for} \quad t > \mu.$$

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(d) Find the asymptotic distributions of

$$\text{or} \quad i = 1, 2, 3.$$

(e) Comparing the asymptotic distributions in part (d), what can you say about the relative performance of these three estimators for large n?

MA780 : #4, #5
MA781, MA782: #1, #2, #3

1. Assume that X and Y are independent binomial random variables with distributions $B(m, p_1)$ and $B(n, p_2)$ respectively. Derive the UMP unbiased test of testing $H_0 : p_1 = p_2$ versus $H_a : p_1 \neq p_2$ at α level of significance. factorization: 5%
Conditional p.m.f.: 10%, test: 5%

2. Let $X_1, \ldots, X_n$ be i.i.d. random variables with probability mass function $P(X_1 = k) = \frac{1}{N}$, for $k = 1, 2, \ldots, N$, where N is an unknown parameter.

5%(a) Find the sufficient statistic for N

5%(b) Let M_n denote the sufficient statistic found in (a). Derive the probability mass function of M_n.

5%(c) Show that M_n is a complete sufficient statistic.

5%(d) Find the UMVU estimator of N.

3. Let $X_1, X_2, \ldots,$ be i.i.d. with distribution function F. For any positive integer k, the kth moment of F is defined as $\alpha_k = E(X_1^k)$. The kth central moment of F is defined as $\mu_k = E(X_1 - E(X_1))^k$.

Then α_k may be estimated by $a_k = \frac{1}{n}\sum_{i=1}^n X_i^k$. And μ_k may be estimated by $m_k = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^k$. Assume that all the moments referred to do exist.

2%(a) Show that $a_k \xrightarrow{a.e.} \alpha_k$ and $E(a_k) = \alpha_k$.

3%(b) Find $\mathrm{Var}(a_k)$.

5%(c) Find the asymptotic distribution of the random vector $(a_1, a_2, \ldots, a_k)$.

10%(d) Find the asymptotic distribution of M_k.

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4. Suppose $X, X_1, X_2, \ldots$ are pairwise independent, identically distributed random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Fix $0 < p < 2$, and then define $W_n = n^{-1/p}\sum_{j=1}^n X_j$. Put $L = \{\omega \in \Omega : \lim_{n\to\infty} W_n(\omega) \text{ exists and is finite}\}$.

(a) Show directly that $L \in \mathcal{F}$. (*Hint:* Only Cauchy sequences converge.)

(b) Assume $P(L) > 0$. Show that $E|X|^p < \infty$.

(c) Assuming that $P(L) > 0$; show that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n |X_j|^p$ exists almost surely and identify the limit.

5. (a) Compute (and justify your computation):

$$\lim_{n\to\infty} \sum_{j=0}^{n} \cos\left(\frac{j - np}{\sqrt{n}}\right) \binom{n}{j} p^j (1-p)^{n-j}. \qquad (0 < p < 1)$$

(b) Let X be a random variable such that $Ee^{itX} = \cancel{2}\, e^{it}/(2 - e^{it})$.

(i) Identify explicitly the distribution of X, and justify. (*Hint:* The form $\frac{1}{1-z}$ is convenient.)

(ii) Compute the variance of X.

MA 781 Ph.D. Qualifier 2003

1. Let $X_1, X_2, .., X_n$ are i.i.d. *Exponential*(θ). Suppose X_i represent the lifetime of independent pieces of equipment in hundreds of hours, and warranty replacement period is (say) 200 hours, then we may be interested in warranty failure probability $\beta(\theta) = P_\theta(X \leq 2)$.

a. Obtain MLE of $\beta(\theta)$.

b. Compute the **approximate** bias and variance of your estimator in part a.

c. Derive the asymptotic distribution of your estimator in part a, and hence compute a $(1-\alpha)$ confidence interval for $\beta(\theta)$.

2. Suppose $X_1, X_2, ..., X_n$ be i.i.d. $N(\mu_0, \frac{1}{\theta})$, μ_0 is known, and $\theta = \frac{1}{\sigma^2}$ is called the precision of the distribution of X_i.

a. Consider the prior distribution

$$\pi(\theta) \propto \theta^{\frac{1}{2}(\lambda-2)} \exp(-\tfrac{1}{2}\nu\theta), \nu > 0, \lambda > 0; \theta > 0.$$

Compute the posterior distribution of θ given $(X_1, .., X_n)$, $\pi(\theta|X_1, .., X_n)$.

b. Show that if λ is an integer, given $(X_1, ..., X_n)$, $\theta(t+v)$ has a $\chi^2_{n+\lambda}$, where $t = \sum_{i=1}^{n}(X_i - \mu_0)^2$.

c. Suggest a Bayes estimator of θ under squared error loss.

3. (Biostatistics Students Only) The following data show the number of occupants in passenger cars observed during one hour at a busy intersection in Los Angeles. Suppose it can be assumed that these data follow a geometric distribution, $p_X(k;p) = (1-p)^{k-1}p, k = 1, 2, ...$ Find the MLE of p. Do you think that the model adequately describe the data? Justify your answer.

Number of Occupants	Frequency
1	678
2	227
3	56
4	28
5	8
6+	14
Total	1011

MA782 Qualifying Exam 2003

1. Consider the pdf $f(x,\theta) = \frac{1}{\beta}\exp(-\frac{x-\theta}{\beta}), \beta$ is known, $\theta \in R, x \geq \theta$. Suppose $X_1, .., X_n$ be i.i.d. $f(x,\theta)$,let $Y = \min(X_1, ..., X_n)$.

a. Show that $T = \frac{Y-\theta}{\beta}$ is a pivot.

b. Determine a and b, such that $P(a < T < b) = 1-\alpha$,and hence conclude that $[Y - \frac{b\beta}{n}, Y - \frac{a\beta}{n}]$ confidence interval for θ.

c. Note that the length of any $(1-\alpha)$ interval of this form is $l = \frac{\beta}{n}(b-a)$. Set $b = b(a)$ and show that the shortest confidence interval is given by $[Y + \frac{b\log\alpha}{n}, Y]$.

{Hint for c. Set $b = b(a)$;and from $e^{-a} - e^{-b} = 1-\alpha$, obtain $\frac{db}{da}$ by differentiating both sides w.r.t. a.Then replace $\frac{db}{da}$ in $\frac{dl}{da}$ and observe that it is >0. Hence argue that l attains its minimum at $a = 0$.}

2. Let X be a discrete random variable whose pmf under H_0 and H_a is given by

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_a)$	.06	.05	.04	.03	.02	.01	.79

Use Neyman-Pearson Lemma to find the most powerful test for H_o vs. Ha with size $\alpha = 0.04$. Compute the probability of Type II error for this test.

3. (Biostatistics only) Let $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ be independent $\exp(\theta)$ and $\exp(\lambda)$ samples respectively. Let $\Delta = \frac{\theta}{\lambda}$.

a. If $f(\alpha)$ denotes the αth quantile of the $F_{2n_1,2n_2}$ distribution, show that $[\bar{Y}f(\frac{\alpha}{2})/\bar{X}, \bar{Y}f(1-\frac{\alpha}{2})/\bar{X}]$ is a confidence interval of Δ with confidence coefficient $1-\alpha$.

b. Show that the test with acceptance region $[f(\frac{\alpha}{2}) \leq \frac{\bar{X}}{\bar{Y}} \leq f(1-\frac{\alpha}{2})]$ has size α for testing $H_0 : \Delta = 1$ vs. $H_a : \Delta \neq 1$.

c. The following are times until breakdown in days of air monitors operated under two different maintenance policies at a nuclear power plant. Experience has shown that the exponential assumption is warranted. Give a 90% confidence interval for the ratio Δ of mean life times.

x	3, 150, 40, 34, 32, 37, 34, 2, 31, 6, 6, 14, 150, 27, 4, 27, 10, 30, 37
y	8, 26, 10, 8, 29, 20, 10

Statistics and Biostatistics Ph.D. Theory Qualifier 2010

Total time: 4 hours

October 16, 2010

Instructions:

- This is a closed book closed notes exam; however, you can use copies of the pages of your textbook containing the definitions of various pdfs and their moments.
- You need to answer any FOUR, but no more than FOUR, problems. If you dont want me to grade a part of your answer, please cross out that part of your answer book completely. Also note that the problems are not arranged in any particular order, so I suggest that you read all six problems before deciding on the FOUR that you wish to answer.
- Answer each question in separate BLUE BOOKS.

4. Let $f(x|\theta)$ be the logistic location pdf given by

$$f(x|\theta) = \frac{e^{(x-\theta)}}{1+e^{(x-\theta)}}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

   (a) Show that this family has an MLR.

   (b) Based on one observation, X, find the most powerful size α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. For $\alpha = .2$, find the size of the Type II Error.

   (c) Show that the test in part (b) is UMP size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 1$.

5. Let $(X_{j,1}, X_{j,2})$, $j = 1, 2, \ldots, n$ be independent pairs of exponentially distributed random variables with $E(X_{j,1}) = \lambda_1$ and $E(X_{j,2}) = \lambda_2$. for $j = 1, 2, \ldots, n$

   (a) Construct the likelihood ratio test to test $H_0 : \lambda_1 = \lambda_2$ vs $H_A : \lambda_1 = \lambda_2$. Provide a size α critical value for the above hypothesis based on the LRT.

   (b) Using appopriate reparameterization construct the Wald Test and the Score Test for the same hypothesis i.e. $H_0 : \lambda_1 = \lambda_2$ vs $H_A : \lambda_1 = \lambda_2$. Provide size α critical values for each of Wald Test and the Score Test.

   (c) Obtain the UMPU level α test of $H_0 : \lambda_1 = \lambda_2$ vs $H_A : \lambda_1 = \lambda_2$. Be sure to express this test as an unconditional test.

6. Suppose $X_1, X_2, ..., X_n$ are independent random samples from $N(\theta; \sigma^2)$, where both θ and σ^2 are unknown, but there is only interest on the inference of θ. Consider the prior pdf of θ and σ^2 given by

$$\pi(\theta, \sigma^2|\mu, \tau^2, a, b) = \frac{1}{\sqrt{2\pi\tau^2\sigma^2}} e^{-(\theta-\mu)^2/(2\tau^2\sigma^2)} \frac{1}{\Gamma(a)b^a} \left(\frac{1}{\sigma^2}\right)^{a+1} e^{-1/(b\,\sigma^2)},$$

which is the joint pdf of a $N(\mu; \tau^2\sigma^2)$ multiplied by an $IG(a, b)$.

   (a) Show that this is a conjugate prior for this problem.

   (b) Find the posterior distribution of θ and use it to construct a $(1-\alpha)$ credible set for θ.

   (c) The classical $(1-\alpha)$ credible set for θ can be expressed as

$$\left\{\theta : |\theta - \bar{x}|^2 \leq F_{1,n-1,\alpha}\frac{s^2}{n}\right\}.$$

   Is there any limiting sequence of τ^2, a and b that would allow this set to be approached by a Bayes set from part(b)?

10-781-Qualifying Problems.

1. Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from a discrete uniform distribution with support $S(\theta) = \{1, 2, \ldots, \theta\}$ with unknown θ ($\theta \geq 1$).
   1) Find the maximum likelihood estimator (MLE) $\hat{\theta}_{MLE}$ of θ.
   2) Find the sufficient statistic for θ, call $T(\underline{X})$.
   3) Find the distribution (pmf) of $T(\underline{X})$.
   4) Show that $T(\underline{X})$ is a complete sufficient statistic for θ.
   5) Find the UMVUE for θ.

2. The number of breakdowns X per day for a certain machine is a Poisson RV with unknown mean θ ($0 < \theta < \infty$). Let $X_1, X_2, \ldots, X_n$ be the observed number of breakdowns for n independently selected days.
   1) Find a minimal sufficient statistic for parameter θ.
   2) Find the maximum likelihood estimator (MLE) $\hat{\theta}_{MLE}$ of θ, and show that it is efficient estimator for θ.
   3) The daily cost of repairing the breakdowns is given by $Y = 3X^2$.
      Find MLE and UMVUE for expected value of Y.
   4) Find the asymptotic distribution of $e^{-\hat{\theta}_{MLE}}$ as $n \to \infty$?
   5) Find the limiting distribution (as $n \to \infty$) of the sequence of random variables $W_n = \dfrac{n}{\hat{\theta}_{MLE}}(\hat{\theta}_{MLE} - \theta)^2$.
   6) Use the pivotal method to construct a confidence interval for θ with coverage probability $1 - \alpha$.
   7) Find the variance stabilizing transformation for the underlying Poisson family.

3. Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from the normal distribution $N(\theta, 1)$ with unknown θ. Assume that the prior distribution $\pi(\theta)$ of θ is standard normal $N(0,1)$.
   1) Find the posterior distribution $\pi(\theta \mid \underline{x})$ of θ.
   2) Find the Bayes estimator $\delta^*(\underline{X})$ of θ.
   3) Find the Bayes risk of $\delta^*(\underline{X})$.
   4) Find the UMVUE $\hat{\theta} = \hat{\theta}(\underline{X})$ of θ.
   5) Show that the UMVUE $\hat{\theta} = \hat{\theta}(\underline{X})$ is a minimax and admissible estimator of θ under quadratic loss function $l(\theta, a) = (\theta - a)^2$.

2.

a. Show that if $X_1, \cdots X_n$ are independent Exponential λ (mean) random variables, then $2\sum_{i=1}^{n} \frac{X_i}{\lambda}$ has a χ^2_{2n} distribution.

b. If $X_1, \cdots X_m$; $Y_1, \cdots, Y_n$ are independent Exponential with mean λ random variables, then identify the distribution of $S = \left(\frac{n}{m}\right) \frac{\sum_{i=1}^{m} X_i}{\sum_{i=1}^{n} Y_i}$.

c. Let $X_1, \cdots X_m$ and $Y_1, \cdots, Y_n$ be independent Exponential θ and λ samples respectively, and let $\Delta = \theta/\lambda$. Identify an appropriate pivot for Δ and construct a $(1-\alpha)$ confidence interval for Δ.

d. The following are times of breakdown in days of air monitors separated under two different maintenance policies at a nuclear power plant. Experience has shown that the exponential assumption is warranted. Give 90% confidence interval for ratio Δ of mean life times.

X 3, 150, 40, 34, 32, 37, 34, 2, 31, 6, 5, 14, 150, 27, 4, 6, 27, 10, 30, 37

Y 8, 26, 10, 8, 29, 20, 10

Ph.D. Qualifying Exam 1997
MA 782

1. Let $N_1, N_2, \ldots, N_p$ be independent, N_i is distributed as $B(c_i, \theta_i)$, $i=1,2,\ldots,p$.

a. If $\theta_1 = \theta_2 = \cdots = \theta_p = \theta$, find the MLE $\hat{\theta}$ of θ.

b. If $\theta_1, \ldots, \theta_p$ vary freely, find the MLE $\hat{\theta}_i$ of θ_i.

c. Show that the likelihood ratio test statistic for $H_0: \theta_1 = \theta_2 = \cdots = \theta_p$ vs. $H_a: \theta_i \neq \theta_j$ *for some* i, j, is given by,

$$2\log\lambda = 2\sum_{i=1}^{p}\{N_i[\log\hat{\theta}_i - \log\hat{\theta}] + (c_i - N_i)[\log(1-\hat{\theta}_i) - \log(1-\hat{\theta})]\}$$

d. Show that the test statistic can be approximated by the usual goodness-of-fit Chi-Square statistic,

$$\chi^2 = \sum_{i=1}^{p}\frac{(N_i - c_i\hat{\theta})^2}{c_i\hat{\theta}(1-\hat{\theta})}.$$

e. The following table gives the data on mortality among mothers having their first child in four successive periods between 1894-1948 in New South Wales. Test the hypothesis that there is no difference in mortality rate between the four periods, and give an approximate p-value.

Period	1894-1910	1911-1920	1920-1937	1938-1948
No. of Mothers	1072	1133	2455	1995
No. of Deaths	22	23	49	33

[Hint: For part d, you may want to consider a Taylor expansion of logarithm]

Ph.D. Qualifying Exam 2004 (MA781)

1. Eggs are thought to be infected with a bacterium *salmonela enteriditis* so that the number of the organisms, Y, in each has a Poisson distribution with mean μ. The value of Y cannot be observed directly, but after a period it becomes certain whether the egg is infected $(Y > 0)$ or not $(Y = 0)$. Out of m such eggs, r are found to be infected. Find the maximum likelihood estimator $\widehat{\mu}$ of μ and derive it's asymptotic variance.

2. Let $X_1, ..., X_n$ are independent observations such that X_i is distributed as $Poisson(i\theta)$;i.e., $E(X_1) = \theta, E(X_2) = 2\theta, E(X_3) = 3\theta$....etc. Define:

$$a_n = \sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \text{and} \quad b_n = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

a. Consider the following two estimators of θ.

$$P = \sum_{i=1}^n \frac{X_i}{a_n} \quad \text{and} \quad Q = \sum_{i=1}^n \frac{iX_i}{b_n}.$$

Show that both estimators are unbiased for θ.

b. Compute the variances of the estimators, and hence identify the better of the two estimators. Denote it by $\widehat{\theta}$. (Hint: $(\sum_{i=1}^n a_i b_i)^2 < (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$, unless $a_i = kb_i$)

d. Show that $\widehat{\theta}$ attains the Cramer-Rao lower bound, and hence it is efficient.

e. Show that $\widehat{\theta}$ is, in fact, the MLE of θ; and hence derive the asymptotic distribution of $\widehat{\theta}$.

3. (Biostatistics only)

Suppose a random variable X is distributed as

$$f(x|\theta) = \exp[-(x-\theta)],\ x > \theta > 0.$$

Consider the problem of deriving the Bayes estimator θ based on a single observation x_0.

a. Compute the posterior distribution $g(\theta|x_0)$ with respect to the prior

$$\pi(\theta) = \exp[-\theta],\ \theta > 0.$$

b. Identify the Bayes estimator of θ under squared error loss.

c. A $(1-\alpha)$ posterior confidence set (*credible set*) is defined to be a set C of values of θ, whose posterior probability content is at least $(1-\alpha)$. When θ is continuous, this is

$$(1-\alpha) = P(\theta \in C|x) = \int_C g(\theta|x)d\theta.$$

Compute a $(1-\alpha)$ *credible interval* of θ based on the posterior distribution computed in part (a).

PH.D. QUALIFYING EXAMINATION 1994

MA 781

1. Suppose that the random variables $Y_1, Y_2, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i ,$$

where $x_1, x_2, \ldots, x_n$ are fixed constants and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are iid $N(0,\sigma^2)$, σ^2 unknown.

a) Find the MLE for β, and show that it is an unbiased estimator of β.

b) Find the distribution of the MLE of β.

c) Show that $\Sigma Y_i / \Sigma x_i$ is also an unbiased estimator of β.

d) Calculate the variance of the estimator in part c, and compare it to the variance of the MLE.

2. Suppose a random variable X is distributed as Uniform $(0, \theta)$. If θ has the prior density $\pi(\theta) = \theta \exp(-\theta)$, $\theta > 0$, then compute the Bayes estimate of θ under squared error loss based on a sample of size 1.

Ph.D. Qualifying Exam 1997
MA 781

1.
Suppose for given θ, X has geometric distribution given by $P(X=k)=\theta(1-\theta)^k, k=0,1,\cdots$.

a. Suppose that the parameter space (discrete) of θ is given by $\Theta=\{1/4, 1/2, 1/4\}$. Consider uniform prior $\pi(\theta)$ over Θ and compute the posterior distribution $\pi(\theta|x)$.

b. Suppose you have decided to estimate θ based on the highest posterior probability (i.e., the value of θ in Θ that has highest posterior probability). Given X=2, what will be your estimate? How your estimate would change given X = k, k = 0, 1,.....?

c. If $\Theta=[0,1]$, find the posterior probability of θ given X=k when the prior distribution is Beta(α,β). What is the Bayes estimate of θ under squared error loss?

2.
Suppose X have a Poisson(θ) distribution. We want to estimate $P(X=0) = e^{-\theta}$.

a. For any unbiased estimator T(X) of $e^{-\theta}$., compute the Cramer-Rao lower bound for T.

b. Define

$$T(X) = 1 \text{ if } X=0,$$
$$= 0 \text{ if otherwise.}$$

greater

Show that the variance of T(X) is strictly ~~less~~ than the Cramer-Rao lower bound.

c. Show that nevertheless T is an UMVUE of $e^{-\theta}$.

1998

MA 781
Ph.D. Qualifying Exam

1. Two laboratories each take c measurements on the same standard μ. Consider the model:

$$Y_{ij} = \mu + \epsilon_{ij}, \quad i = 1, 2$$

$$j = 1, 2, \ldots, c,$$

where ϵ_{ij}'s are independent random variables with mean 0. From past experience, assume that

$$\mathrm{Var}(Y_{1j}) = \sigma^2, \quad \mathrm{Var}(Y_{2j}) = 4\sigma^2,$$

$j = 1, 2, \ldots, c$; where σ^2 is unknown.

(a) Find the minimum variance linear unbiased estimate of μ.

(b) More generally, if $\dfrac{\mathrm{Var}(Y_{2j})}{\mathrm{Var}(Y_{ij})} = a$, where a is known, and $\mathrm{Var}(Y_{ij})$ is independent of j, find the minimum variance linear unbiased estimate.

(c) Solve the problem (b), if there are p-laboratories, the ith laboratory takes n_i independent measurements, the errors are all independent with mean 0, and the errors for i^{th} laboratory all have the same variance $\sigma^2 a_i$, $i = 1, 2, \ldots, p$, where a_i's are known.

2. The discrete random variable Y has probability distribution

$$P(Y = k) = \frac{\theta^k a_k}{f(\theta)}; \qquad \begin{array}{c} k = 0, 1, 2, \ldots, \\ \theta > 0 \end{array}$$

which is called power series distribution, with $a_k > 0$ for all k independent of θ.

(a) Is this an exponential family distribution?

(b) State whether or not Binomial Geometric and Poisson distribution are of this type.

(c) Show that f is infinitely differentiable.

(d) Obtain EY and $\mathrm{Var}(Y)$.

(e) For a random sample $(Y_1, \ldots, Y_n)$ from Y with $\bar{Y} = \dfrac{\sum_{i=1}^n Y_i}{n}$, show that the MLE of θ satisfies the equation

$$\bar{Y} = \frac{\hat{\theta} f'(\hat{\theta})}{f(\hat{\theta})}.$$

(f) Show that the Fisher Information is $\frac{n\sigma^2(\theta)}{\theta^2}$ where $\sigma^2(\theta) = \mathrm{Var}(Y)$.