Preliminary Exam 2010
Morning Session (3 hours)

Part I. Solve four of the following five problems.

1. Using power series, derive Euler's formula. (Do not worry about issues of convergence, etc. In other words, do the derivation just as Euler would have done it.)

2. Consider the differential equation

$$\frac{dy}{dt} = 2y + 3\cos 4t.$$

For what initial values $y(0) = y_0$ are the solutions bounded for all t? Include a one-sentence justification of your answer.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the function

$$f(x, y, z) = \left(e^{x^2} + y + z, x + e^{y^2} + z, x + y + e^{z^2}\right).$$

Show that f is a one-to-one function on some neighborhood of the origin in $\mathbb{R}^3$.

4. Calculate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \arctan\left(\frac{k}{n}\right)$.

5. Give a proof or counterexample of the following statement: Let f be a real-valued function that is defined and continuous on all of $\mathbb{R}^2$ except at the origin. It has a removable discontinuity at the origin provided that the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exists along all parabolas that contain the origin.

Part II. Solve three of the following six problems.

6. Compute the flux of the vector field $\mathbf{F}(x, y, z) = (2x - y^2)\mathbf{i} + (2x - 2yz)\mathbf{j} + z^2\mathbf{k}$ through the surface consisting of the the side and bottom of the cylinder of radius two and height two, i.e., $\{(x, y, z) \mid x^2 + y^2 = 4 \ , \ \ 0 \le z \le 2\}$. (Note that this surface does not include the top of the cylinder.)

   That is, compute the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

   where $\mathbf{F}$ is the vector field above, S is the bottom and side (but not the top) of the cylinder above, and $\mathbf{n}$ is the outward pointing unit normal vector to the surface.

7. Canada has a total of \$10 billion in \$20 bills in circulation, and each day \$40 million of these \$20 bills passes through one bank or another. A new harder-to-forge version of the \$20 bill is developed, and the banks replace the old bills with new ones whenever they can. How long does it take for the new bills to reach 90% of the total number of \$20 bills in circulation?

8. For $x > 0$, let $f(x) = \displaystyle\int_0^\infty e^{-t-x^2/t}\, t^{-1/2}\, dt$.

   (a) Using a substitution, show that

$$f(x) = x \int_0^\infty e^{-t-x^2/t}\, t^{-3/2}\, dt.$$

   (b) Show that $f(x) = Ce^{-2x}$ for some positive constant C.

9. Suppose that $\mathbf{A}$ is an $n \times n$ matrix with $||\mathbf{A}|| \le a < 1$. Prove that the matrix $(\mathbf{I} - \mathbf{A})$ is invertible with

$$||(\mathbf{I} - \mathbf{A})^{-1}|| \le \frac{1}{1-a}.$$

   (The choice of norm does not matter.)

10. Two metrics d_1 and d_2 on a space X are said to be *numerically equivalent* if there are positive constants a and b such that

$$a\, d_1(x, y) \le d_2(x, y) \quad \text{and} \quad b\, d_2(x, y) \le d_1(x, y)$$

for all pairs (x, y) in $X \times X$.

Let $X = \mathbb{R}^n$ and d be the standard Euclidean metric. Also, for x and y in $\mathbb{R}^n$, let

$$d_1(x, y) = \sum_{1 \le i \le n} |x_i - y_i|$$

$$d_2(x, y) = \min\{1, d(x, y)\}$$

Both of these distance functions are metrics on $\mathbb{R}^n$.

(a) Show that d_1 is a metric.

(b) For both d_1 and d_2 in $\mathbb{R}^2$, sketch the open balls centered at the origin.

(c) Is d_2 numerically equivalent to d_1? Justify your answer.

11. For $\arctan x$ with $x > 1$, derive an infinite series representation (not a power series—negative powers of x are allowed) as follows:

(a) Derive $R_n(t)$ such that

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 \pm \ldots + (-1)^n t^{2n} + R_n(t).$$

(b) Derive the power series representation for $\arctan x$ for $|x| \le 1$, and verify convergence without using the term-by-term integration theorem for power series.

(c) Derive an infinite series for $\arctan x$ for $x > 1$.

Part III. Solve one of the remaining three problems.

12. Let $f(x, y)$, $g(x, y)$, $\varphi(u, v)$, and $\psi(u, v)$ be real-valued functions on $\mathbb{R}^2$ with continuous partial derivatives. Put

$$\begin{cases} J_{f,g}(x, y) = (\partial f/\partial x)(\partial g/\partial y) - (\partial f/\partial y)(\partial g/\partial x) \\ J_{\varphi,\psi}(u, v) = (\partial \varphi/\partial u)(\partial \psi/\partial v) - (\partial \varphi/\partial v)(\partial \psi/\partial u), \end{cases}$$

and assume that $J_{f,g}(0, 0) \neq 0$ and $J_{\varphi,\psi}(0, 0) \neq 0$. Assume also that $f(0, 0) = \varphi(0, 0)$ and $g(0, 0) = \psi(0, 0)$.

(a) Show that there is a C^1 function $(x, y) \mapsto (u(x, y), v(x, y))$ defined on some open neighborhood of $(0, 0)$ such that $(u(0, 0), v(0, 0)) = (0, 0)$ and

$$\begin{cases} f(x, y) = \varphi(u(x, y), v(x, y)) \\ g(x, y) = \psi(u(x, y), v(x, y)). \end{cases}$$

(b) With notation as in part (a), show that $J_{u,v}(0, 0) \neq 0$, where

$$J_{u,v} = (\partial u/\partial x)(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x).$$

(c) With notation as in parts (a) and (b), prove that if $r > 0$ is sufficiently small then

$$\int\int_{D_r} |J_{f,g}(x, y)| \, dx \, dy = \int\int_{W_r} |J_{\varphi,\psi}(u, v)| \, du \, dv,$$

where $D_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r\}$ and W_r is the image of D_r under the map $(x, y) \mapsto (u(x, y), v(x, y))$.

13. Consider the function $f(x) = \dfrac{x}{1 - x - x^2}$.

(a) Determine a recursive formula for the coefficients c_n of the Maclaurin series of f.

(b) Using the partial fractions decomposition of $f(x)$, determine the Maclaurin series of f in a second way, thereby finding an explicit formula for the coefficients c_n.

14. Consider the one-parameter family of differential equations

$$\frac{d\theta}{dt} = \frac{s^2 - \cos\theta}{s}$$

$$\frac{ds}{dt} = -\sin\theta - Ds^2$$

defined on the half-plane $s > 0$.

(a) Determine the equilibrium points assuming that the parameter $D \geq 0$.

(b) Classify the equilibria for all values of D in the interval $0 \leq D \leq 4$, and determine the bifurcation values of D.