Preliminary Exam 2007
Afternoon Session (3 hours)

Part I. Solve four of the following five problems.

1. Let
$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$
   (a) Calculate the characteristic polynomial and eigenvalues of $\mathbf{A}$.
   (b) Diagonalize $\mathbf{A}$. In other words, find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. (**Do not calculate** $\mathbf{P}^{-1}$.)

2. Let
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -2 \\ 1 & 1 & 3 & 2 & -1 \end{bmatrix}.$$
   Calculate bases for the row space of $\mathbf{A}$, the column space of $\mathbf{A}$, and the null space of $\mathbf{A}$ (as subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$ for the appropriate m and n).

3. Give an example of a 6×6 matrix whose characteristic polynomial is
$$(x-1)(x+2)^3(x-3)^2$$
   and whose minimal polynomial is $(x-1)(x+2)(x-3)^2$. Give a brief explanation to justify your answer.

4. The matrix
$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$
   determines a rotation of $\mathbb{R}^3$ (relative to the standard basis). Find a vector that spans the axis of rotation.

5. Does the polynomial $p(x) = 2x^2 + x$ lie in the ideal generated by the polynomials $q_1(x) = x^3 + x^2 + x + 2$ and $q_2(x) = x^2 + 1$ in $\mathbb{Q}[x]$? Justify your answer.

Part II. Solve three of the following six problems.

6. (a) Let $\mathbf{A}$ be a 2×2 matrix with real entries such that $\mathbf{A}^3$ is the identity matrix. Assume that $\mathbf{A}$ is not the identity matrix and prove that the trace of $\mathbf{A}$ is -1 and the determinant of $\mathbf{A}$ is $+1$.
   (b) Is result in part (a) true if $\mathbf{A}$ has complex entries?

7. Let R be a commutative ring. Recall that an element x is said to be nilpotent if $x^n = 0$ for some n. Let N denote the set of all nilpotent elements of R.
   (a) Prove that N is an ideal.
   (b) Prove that N is contained in every prime ideal of R.
   (c) Prove that R/N has no nonzero nilpotent elements.

8. Let V be $\mathbb{C}[x]/(x^3 + 5x^2 + 6x + 2)$ and let $T : V \to V$ be defined by
$$T(p(x)) = (x + 1)p(x).$$
Find bases for the kernel (null space) and image (range) of T.

9. Let A be the additive group $\mathbb{Z} \oplus \mathbb{Z}$ and B be the subgroup
$$\{(5m + 7n, 2m + 4n) \,|\, m, n \in \mathbb{Z}\}.$$
Show that A/B is cyclic and determine its order.

10. Let n be a positive integer and let $\mathbb{Z}_n$ denote the cyclic group of order n, i.e., $\mathbb{Z}/n\mathbb{Z}$.
   (a) Suppose that a, b, c, and d are positive integers such that b is an integer multiple of a and d is an integer multiple of c. Prove that, if the direct sums
$$\mathbb{Z}_a \oplus \mathbb{Z}_b \quad \text{and} \quad \mathbb{Z}_c \oplus \mathbb{Z}_d$$
are isomorphic, then $a = c$ and $b = d$.
   (b) Prove that the groups $\mathbb{Z}_6 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$ are isomorphic.

11. Let m and n be odd integers. Show that the polynomial $x^3 + mx + n$ is irreducible over $\mathbb{Q}$.

Part III. Solve one of the remaining three problems.

12. Let $\mathbf{A}$ be an $n \times n$ matrix with entries in $\mathbb{R}$. Suppose that $\mathbf{A}^n$ is the zero matrix and the dimension of the null space of $\mathbf{A}$ is one. Show that the set of matrices

$$\{\mathbf{A}^j \mid j = 0,\ 1,\ \ldots\ n-1\}$$

is linearly independent over $\mathbb{R}$.

13. Find Galois extensions K of $\mathbb{Q}$ such that the Galois group of $K/\mathbb{Q}$ is

    (a) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

    (b) S_3, the symmetric group of degree 3.

    Provide brief justifications for your answers.

14. Suppose that $\mathbf{A}$ is an $n \times n$ matrix with real entries. Let s denote the trace of the matrix $\mathbf{A}^2$.

    (a) If $\mathbf{A}^t = \mathbf{A}$, show that $s \geq 0$.

    (b) If $\mathbf{A}^t = -\mathbf{A}$, show that $s \leq 0$.