THESIS SUMMARY
SEMISTABLE REDUCTION IN CHARACTERISTIC ZERO

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Abstract. Let $f : X \to B$ be a family of varieties; i.e., a morphism of algebraic varieties, where we consider the fibers $f^{-1}(b)$ for $b \in B$ as elements in the family. The general theme of the thesis is to make simultaneously all fibers of f as nice as possible. The formal definition of "as nice as possible" is to say that f is semistable. Semistable reduction then asks to replace the base B parametrizing the family by a generically finite cover $B' \to B$, and the family X pulled back to B' by a birational modification X' such that the induced morphism $f' : X' \to B'$ is semistable. We solve the semistable reduction problem for the case of families of surfaces and threefolds. A slightly weaker version is proved for general families. In practice, the weak semistable reduction is often sufficient, as we show by an application in the moduli space theory.

0.1. **Semistable reduction.** Suppose we are given a family of varieties $f : X \to B$ over an algebraically closed field of characteristic zero. In general, the fibers of the morphism may have arbitrarily bad singularities. The question we ask is: can one replace the family $f : X \to B$ by a new family $f' : X' \to B'$ such that all fibers of f' are as nice as possible. Consider two well-known special cases of the problem:

(1) If B is a point, then by Hironaka's theorem, there exists a proper birational morphism $X' \to X$ such that X' is nonsingular. Here X' is the only closed fiber of $f' : X' \to B$, and being nonsingular, it is indeed as good as one may wish.
(2) If B is a curve, then the semistable reduction theorem of Kempf, Knudsen, Mumford and Saint-Donat [KKMS] states that there exists a finite base change $f' : B' \to B$ and a proper birational morphism $X' \to X \times_B B'$ such that the induced morphism $f' : X' \to B'$ is semistable; i.e., both X' and B' are nonsingular, the generic fiber of f' is nonsingular, and the special fibers ${f'}^{-1}(b)$ for $b \in B'$ are reduced divisors of normal crossings in X'.

The main goal of the thesis is to extend the two examples above to the case of a family over a base B of arbitrary dimension. The operations we are allowed to perform on the varieties are (i) a generically finite proper base change (an alteration) $B' \to B$; and (ii) a proper birational morphism (a modification) $X' \to X \times_B B'$. Recall that a semistable morphism $f : X \to B$ over a curve as in example (2) above is locally given by $t = x_1 \cdots x_n$ for some local parameters $x_1, \ldots, x_n$ of X and t of

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B. We define a morphism to be semistable if it is locally a product of semistable morphisms as above:

Definition 0.1. A surjective flat morphism $f : X \to B$ of projective varieties is **semistable** if

(i) Both X and B are nonsingular.
(ii) For every point $x \in X$ there exist formal coordinates $x_1, \ldots, x_n$ at x and $t_1, \ldots, t_m$ at $f(x) \in B$ such that f is given by

$$t_i = \prod_{j=l_{i-1}+1}^{l_i} x_j$$

for some $0 = l_0 < l_1 < \ldots < l_m \leq n$.

We prove in Chapter 3 of the thesis:

Theorem 0.2. *Let $f : X \to B$ be a surjective morphism of projective varieties defined over an algebraically closed field k of characteristic zero. Assume that the relative dimension of f is at most 3. Then there exist an alteration $B' \to B$ and a modification $X' \to X \times_B B'$ such that the induced morphism $f' : X' \to B'$ is semistable.*

The similar statement for a morphism of relative dimension > 3 remains open. For a general morphism we prove in Chapter 2 a slightly weaker version of the theorem.

0.2. **Weak semistable reduction.** Theorem 0.2 is proved by modifying the morphism $f : X \to B$ to a toroidal (locally toric) morphism, writing the condition of semistability and the operations allowed on X and B in terms of the associated polyhedral complexes, and solving the resulting combinatorial problem. For a toroidal morphism $f : X \to B$, the conditions of Definition 0.1 are equivalent to the following: (i) both X and B are nonsingular; and (ii) f has reduced and equidimensional fibers. In defining weak semistability we give up nonsingularity of X:

Definition 0.3. A flat surjective toroidal morphism $f : X \to B$ without horizontal divisors is called **weakly semistable** if

1. B is nonsingular.
2. f is equidimensional with reduced fibers.

The main theorem of Chapter 2 is:

Theorem 0.4. *Let $f : X \to B$ be a surjective morphism of projective varieties defined over an algebraically closed field k of characteristic zero. Then there exist an alteration $B' \to B$ and a modification $X' \to X \times_B B'$ such that the induced morphism $f' : X' \to B'$ is weakly semistable.*

We also show that a weakly semistable morphism $f : X \to B$ behaves well under certain base changes. As a consequence, we get a universal Gorenstein property of weakly semistable morphisms:

Proposition 0.5. *Let $f : X \to B$ be weakly semistable, and let $B_1 \to B$ be any dominant morphism. If B_1 has at most rational Gorenstein singularities, then $X \times_B B_1$ has at most rational Gorenstein singularities.*

0.3. **Boundedness for stable smoothable n-folds.** The last chapter of the thesis contains an application of weak semistable reduction in the theory of moduli spaces. Consider nonsingular projective n-folds X with ample canonical divisor K_X and a fixed Hilbert polynomial $H(l) = h^0(X, lK_X)$ for large l. It is known that there exists a quasi-projective coarse moduli space for such varieties [Vie]. We are interested in compactifying this moduli space; i.e, adding points on the boundary corresponding to limits of smooth n-folds.

The general procedure for constructing singular varieties corresponding to points on the boundary of the moduli space goes as follows. One takes a 1-parameter semistable family of n-folds, with generic fiber as above, and finds the relative canonical model of this family by applying the minimal model program in dimension $n+1$ (MMP(n+1)). The special fibers of the relative canonical model are exactly the (singular) varieties we need to add in order to compactify the moduli space. We let the moduli functor $\mathcal{M}_H^{sm}$ assign to a scheme S the set of isomorphism classes of certain "good" families of stable smoothable n-folds over S with a given Hilbert polynomial H. A theorem of Kollár [Kol] states that the moduli functor $\mathcal{M}_H^{sm}$ is coarsely represented by a projective scheme, provided that the functor is bounded, locally closed, separated, complete, with tame automorphisms, and if the canonical polarization is semi-positive. All these properties, except boundedness, have either been proved in the literature, or follow from known results. We prove that the minimal model program assumption in dimension $n+1$ implies boundedness of the moduli functor for stable smoothable n-folds:

Theorem 0.6. *Assuming MMP(n+1), there exists a family $\tilde{X} \to B$ over a projective scheme B in $\mathcal{M}_H^{sm}(B)$ whose geometric fibers include all stable smoothable n-folds with Hilbert polynomial H.*

As a corollary we get:

Corollary 0.7. *Assuming MMP(n+1), the moduli functor $\mathcal{M}_H^{sm}$ is coarsely represented by a projective scheme M_H^{sm}.*

References

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