Appearing in Stochastic Theory and Adaptive Control, T.E. Duncan and B. Pasik-Duncan, Eds., Springer-Verlag Lecture Notes in Control and Information Sciences, Vol. 184, 1992.

Self-organizing Behavior in a Simple Controlled Dynamical System

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Abstract: A standard paradigm in control theory involves the use of feedback to change the dynamics of a system in some significant way. In the language of Willems ([9]), this paradigm prescribes the use of feedback to create desirable "behavior" in the system. There is growing interest (see e.g. [8]) in exploring an alternative paradigm applied to the control of systems in which there is a set of various behaviors pre-existing within the natural (uncontrolled) dynamics of the system, and wherein control acts in a minimalistic way to entrain a mode of behavior chosen from this set. We shall explore the latter in the context of some mechanical systems in which the control is only allowed to act intermittently. The systems we look at involve the controlled one dimensional scattering of a certain number of particles. In the absence of control, the systems are similar to the Toda lattices that have been considered by Moser ([7]) and others. We introduce boundary controls and confine our analysis to two classes of open loop controls—roughly corresponding to constant and periodic forcing. For the constant controls, the set of possible behaviors is easily described using fixed point analysis. For periodic forcing, on the other hand, the behavior set is very rich, and is modeled as the dynamics of an iterated 2-d mapping. Results on the stability and bifurcations of periodic orbits are given.

1 Introduction

This paper describes the dynamics of a system of colliding particles whose motions will be controlled by varying the position, velocity, and inertia of a "racquet" at the boundary of a domain to which the particles are confined. While the system is extremely simple, the dynamics are nevertheless nonlinear, and there is a rich set of natural behaviors which can be produced by appropriate control actions.

The motivation for studying this system comes in part from a toy recently shown to the author by one of his children. The toy contains several hard elastic spheres which are constrained to move so

*The author gratefully acknowledges the support of the U. S. Air Force through grant AFOSR-90-0226. Thanks are also extended to Professor S.K. Mitter for hospitality shown during the author's recent sabbatical visit to MIT.

as to guarantee collisions. The object of play is to produce any of several immediately perceivable patterns of collisions. A broader motivation is to understand the general problem of controlling complex systems for which any controlling agent must learn how to influence the internal dynamics so as to elicit desired modes of behavior. The objective is non-classical in the sense that we wish to use feedback only to change the observed behavior of the system but not the qualitative nature of the system itself. For the system of colliding particles studied below, we find a large number of possible modes of behavior, but our understanding of how these depend on the control variables is not yet complete. Before a mature behavior-oriented approach to nonlinear control theory can be applied to such systems, it will be necessary to develop a more complete stability and bifurcation theory of the open-loop dynamics of prototypical systems. These objectives are currently being pursued in conjunction with controlling the gait of walking robots (c.f. [3]), controlling juggling ([5] and [6]), and controlling kinematic chains in which the number of degrees of freedom exceeds the number of actuators which are directly controlled. (See [1] and [2].)

While there is not presently a large body of research literature on the behavior-oriented control of nonlinear systems (c.f. the linear theory proposed in [9]), recent work by Ott *et al.* ([8]) has been directed somewhat along these lines. We believe that the existence or non-existence of chaos in the dynamics we are trying to control is perhaps a red herring, however, since we only need a rich set of natural dynamics, not necessarily fully developed chaos, to make the behavior oriented approach to control interesting. Also, we wish to avoid chaotic transients in switching from one stable mode of behavior to another.

The paper is organized as follows. In the next section, we review some elementary facts regarding the dynamics of elastic collisions. Section 3 describes the control problem to be studied, and discusses the system's response to steady state forcing. Section 4 presents preliminary results on the system's response to periodic forcing. Concluding remarks are given which summarize simulations showing the co-existence of distinct stable modes of behavior. While it must be pointed out that we only treat the response of the system to open loop forcing, simulations indicate that the set of possible responses is extremely rich, and further study of prescriptive control strategies seems warranted.

2 Preliminaries on the dynamics of elastic collisions

Before describing the system in detail, it is useful to recall some elementary facts about elastic collisions, and in so doing, it is important to distinguish between collisions involving two and collisions involving simultaneously more than two particles. Consider first two particles of mass m_1 and m_2 constrained to move without friction on a line. In an elastic collision, both total momentum and energy are conserved. Suppose the particles in our system have velocities v_1^i and v_2^i just prior to colliding and v_1^f and v_2^f just after colliding. Because momentum $\mathcal{M}$ and energy $\mathcal{E}$ are conserved, both the initial velocity pair, (v_1^i, v_2^i), and final velocity pair, (v_1^f, v_2^f), simultaneously satisfy the equations

$$m_1 v_1 + m_2 v_2 = \mathcal{M} \tag{1}$$

$$m_1 v_1^2 + m_2 v_2^2 = 2\mathcal{E} \tag{2}$$

It is easy to see that there are precisely two points in the (v_1, v_2)-plane at which the momentum locus intersects the energy locus. Thus if the initial velocity pair (v_1^i, v_2^i) is given, it defines values $\mathcal{M}$ and $\mathcal{E}$, and from (1)-(2) we find the post collision velocity pair (v_1^f, v_2^f). On the other hand, if the same pair of particles were to collide with initial velocities (v_1^f, v_2^f), the post collision velocity pair would be (v_1^i, v_2^i). The implication of this observation is that the pairs $\vec{v}^i = (v_1^i, v_2^i)$ and $\vec{v}^f = (v_1^f, v_2^f)$ are related by an idempotent matrix. I.e., there is a 2×2 matrix A such that

$$\vec{v}^f = A\vec{v}^i, \tag{3}$$

and $A^2 = I$. We have the following explicit characterization.

Lemma 2.1 *Suppose that two particles of mass m_1 and m_2 slide without friction along an infinitely long linear tract in the absence of exogenous forces. If the particles undergo an elastic collision and if the pre- and post- collision velocities are $\vec{v}^i = (v_1^i, v_2^i)$ and $\vec{v}^f = (v_1^f, v_2^f)$ respectively, then these are related by (3) where*

$$A = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}.$$

Proof: That such a matrix A exists follows from our above remarks. That it has this particular form follows from a simple calculation involving the conservation laws (1)-(2). □

Conservation of momentum and energy does not suffice to characterize elastic collisions simultaneously involving three or more particles. The post collision distribution of velocities in such a collision will depend not only on the pre-collision velocities but also on the relative amounts of time each particle spends in contact with the others. Such collisions are not easily analyzed, and the velocity transition must generally be determined by integrating the actual equations of motion.

3 The control of colliding particles in 1-dimension

The dynamics of the system to be studied are described in terms of Figure 1. It consists of a frictionless line or track along which n particles of unit mass may slide. The particles undergo elastic collisions, and they are confined to move between two barriers with which elastic collisions also occur. We assume that the left hand barrier is fixed, but that the right hand barrier moves and functions as a racquet which strikes the particles to influence their motions. The effect of the racquet striking a particle is described by a scattering law of the form (3). More specifically, suppose the racquet has mass M and velocity v_r^i just prior to striking the particle which has unit mass and

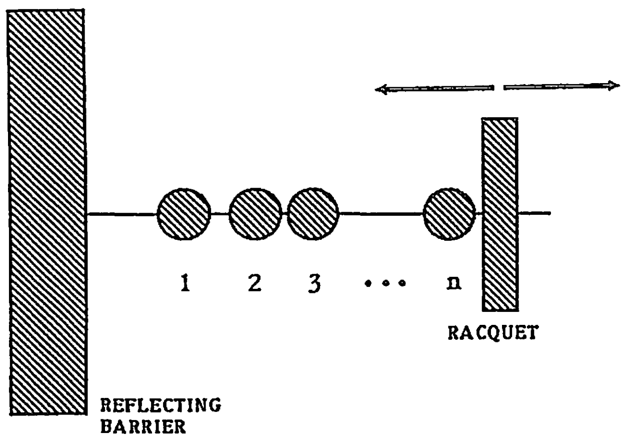

Figure 1: Particles of unit mass slide without friction in one dimension between a reflecting barrier and a racquet which may be moved left and right.

is moving at pre-collision velocity v_p^i. Then the post collision velocity of the particle and racquet are respectively v_p^f and v_r^f given by the formula

$$\begin{pmatrix} v_p^f \\ v_r^f \end{pmatrix} = \begin{pmatrix} \frac{1-M}{M+1} & \frac{2M}{M+1} \\ \frac{2}{M+1} & \frac{M-1}{M+1} \end{pmatrix} \begin{pmatrix} v_p^i \\ v_r^i \end{pmatrix}. \tag{4}$$

The control objective which we pursue for this system is to program sequences of racquet strikes to elicit prescribed stable patterns of motion among the particles. The principal results of this and the next section of the paper are to characterize several achievable patterns. Since we assume control actions are effected by moving the racquet left and right, we must in principle incorporate the dynamics of the racquet into any control strategy. We shall ignore the details of the racquet's dynamics, however, and assume that it may be moved as fast as necessary to any point of the line to strike with velocity v_r. In the next section, we shall also study the dynamics of the racquet and particles system under the assumption that the racquet inertia M may be varied for the purpose of controlling the system. This type of control is used by tennis players who change the effective inertia of a tennis racquet by adjusting their grip to be a greater or lesser distance from the racquet head.

Virtual Particles. If there is more than one particle in our system as depicted in Figure 1, then there are two equivalent ways to view the motions. Physically, the particles remain ordered from left to right, and they constantly exchange velocities by means of elastic collisions. In effect, however, because the particles all have unit mass, each pairwise collision simply results in an exact exchange of velocities. Hence, to describe the overall motion of the system, it is possible to label the particles according to their velocities. From this point of view, the particles "pass through" each other whenever a pairwise collision occurs. We shall refer to these velocity-labeled particles as *virtual particles.* It is convenient to study the motions of the virtual particles since their velocities only change at the reflecting barrier and when they are struck by the racquet.

As mentioned in the preceding section, velocity exchanges resulting from collisions of three or more particles being struck simultaneously by the racquet do not obey a simple algebraic scattering law of the form (3). The following definition thus highlights an important distinction.

Definition 3.1 Collisions simultaneously involving two particles are called *simple.* A racquet strike involving one particle is called a *simple racquet strike.* Collisions simultaneously involving more than two particles and racquet strikes simultaneously involving more than one particle are called *non-simple.*

The first set of controlled behaviors which we wish to study are responses to what may be thought of as constant inputs for the system. Suppose the n particles have initial velocity distribution $v_1^0, \ldots, v_n^0$. We choose coordinates, denoted, say, by x, for the axis along which the racquet and particles move such that the reflecting barrier is located at $x = 0$. Although the racquet may be moved left or right to strike the left-most particle at any point along its trajectory, the nominal rest position of the racquet will be $x = 1$ in our chosen coordinate system. We implement the following simple

CONTROL LAW: the rightmost particle is struck by the racquet moving at velocity $-v$ each time the particle crosses the position $x = 1$ moving in a left to right (positive) direction.

The following characterizes the behavior of the particle motions under this law.

Theorem 3.1 *Assuming that only simple racquet strikes and collisions occur, each virtual particle which is initially in motion will approach a steady state speed Mv under the above control law.*

Proof: Clearly the theorem will hold if it holds for a single particle system. We keep track of the particle's motion by listing its speed just prior to each racquet strike. Under the law (4) , the evolution of these speeds is given by

$$v_p^{k+1} = \left(\frac{M-1}{M+1}\right)v_p^k + \frac{2M}{M+1}v.$$

This mapping has a fixed point: $v_p = Mv$, and this proves the theorem.□

Remark 3.1 In steady state under the above control law, there is an exact momentum exchange between the racquet and particle at each strike.

The following theorem shows that generically only simple collisions and racquet strikes occur.

Theorem 3.2 *For a generic set of initial particle velocities $v_1^0, \dots, v_n^0$ and racquet inertias M, each virtual particle which is initially in motion will approach a steady state speed Mv under the control law of the previous theorem.*

Proof: We prove this theorem in two parts. First we show that if the speeds evolve to within a prescribed threshold of their claimed steady state values, then unless the initial velocities and M have a functional dependency, no two particles approach each other closer than a certain positive distance. It will then be noted that for a generic choice of initial velocities and racquet inertias, no non-simple collisions or racquet strikes occur before the particle speeds have gotten to within the necessary threshold of their steady state values.

To carry out the argument, we shall want to analyze the trajectories of a typical pair of particles. It will be shown that the pair remains separated by a positive distance asymptotically except for regular encounters as directions are reversed at the reflecting barrier and racquet. To eliminate such irrelevant encounters from consideration, we unfold particle motions to occur on a doubly infinite line which is subdivided into subintervals of length two. We lift the dynamics to this unfolded domain by stipulating that all particle motions will be from left to right on the infinite line. The effect of the racquet is modeled by having the particles undergo velocity transitions according to the law

$$v_p \mapsto \left(\frac{M-1}{M+1}\right)v_p - \frac{2M}{M+1}v$$

each time the particle transits one of the length two subintervals.

Consider now the motion of two particles. Assume, without loss of generality, that the left hand particle is slower than the right. (Because the particles are actually moving in a compact interval we may always arrange them such that the slower is on the left in the "unfolded" model.) Suppose also that the initial conditions are such that $x_1(0)$ is a transit point: that is to say the velocity of the left-hand particle has just undergone a transition. Let T denote the amount of time required for the faster particle to subsequently cross the point of velocity transition. In the instants of time immediately after the faster particle has undergone a transition, the positions of the two particles are given by

$$x_1(t) = v_1 \cdot (t+T) + x_1(0), \quad \text{and}$$

$$x_2(t) = \frac{M-1}{M+1}v_2 - \frac{2M}{M+1}v + v_2 \cdot T + x_2(0).$$

The next velocity transition for the slow particle occurs at t satisfying $v_1 \cdot (t+T) = -2$. At this point, we have

$$x_2(t) = \left(-\frac{2}{v_1} - T\right)\left(\frac{M-1}{M+1}v_2 - \frac{2M}{M+1}v\right) + v_2 T + x_2(0).$$

Since T may be expressed in terms of $x_1(0)$, $x_2(0)$, and v_2, we may rewrite this formula as

$$x_2(t) = (\frac{-2}{v_1} - \frac{x_1(0) - x_2(0)}{v_2})(\frac{M-1}{M+1}v_2 - \frac{2M}{M+1}v) + x_1(0).$$

Since $x_1(t) = x_1(0) - 2$, the relative distance between x_2 and x_1 at the next velocity transition for the slow particle, x_1, is

$$x_2(t) - x_1(t) = (\frac{-2}{v_1} - \frac{x_1(0) - x_2(0)}{v_2})(\frac{M-1}{M+1}v_2 - \frac{2M}{M+1}v) + 2.$$

Stated in slightly different notation, suppose x_1^k and x_2^k are the respective positions of the particles when the k-th velocity transition of the slow particle, x_1, occurs, then

$$x_2^{k+1} - x_1^{k+1} = 2(\frac{v_1^k - v_2^{k+1}}{v_1^k}) + \frac{v_2^{k+1}}{v_2^k}(x_2^k - x_1^k),$$

where

$$v_i^{k+1} = (\frac{M-1}{M+1})v_i^k - \frac{2M}{M+1}v.$$

From this we may write

$$x_2^{k+1} - x_1^{k+1} = 2\Big[\frac{v_2^{k+1}}{v_2^{k+1}}\Big(\frac{v_1^k - v_2^{k+1}}{v_1^k}\Big) + \frac{v_2^{k+1}}{v_2^k}\Big(\frac{v_1^{k-1} - v_2^k}{v_1^{k-1}}\Big) + \frac{v_2^{k+1}}{v_2^{k-1}}\Big(\frac{v_1^{k-2} - v_2^{k-1}}{v_1^{k-2}}\Big) + \cdots + \frac{v_2^{k+1}}{v_2^1}\Big(\frac{v_1^0 - v_2^1}{v_1^0}\Big)\Big]$$
$$+ \frac{v_2^{k+1}}{v_2^0}(x_2^0 - x_1^0). \tag{5}$$

Now barring multiple collisions, we have $lim_{k\to\infty}\, v_i^k = -Mv$. Hence suppose we take as our initial velocities $v_1^0 = -Mv + \epsilon + \delta$ and $v_2^0 = -Mv + \epsilon$, where ϵ and δ are small positive numbers. A typical term in the above sum is

$$v_2^{k+1}\Big(\frac{v_1^{j-1} - v_2^j}{v_2^j v_1^{j-1}}\Big).$$

Now if we write $v_i^j = -Mv + \eta$, then

$$v_i^{j+1} = (\frac{M-1}{M+1})v_i^j - \frac{2M}{M+1}v = -Mv + \frac{M-1}{M+1}\eta.$$

Hence a typical term in the sum is the product of

$$-Mv + (\frac{M-1}{M+1})^{k+1}\epsilon$$

and

$$\frac{(\frac{M-1}{M+1})^{j-1}(\delta + \frac{2}{M+1}\epsilon)}{(-mv + (\frac{M-1}{M+1})^j\epsilon)(-mv + (\frac{M-1}{M+1})^{j-1}(\epsilon + \delta))}.$$

These sums are bounded and monotonic and thus clearly form a convergent sequence as k tends to ∞. Hence the relative distance (5) approaches a finite limit d as $k \to \infty$ and this will generally not be equal to zero.

It remains to note that our assumption that the initial velocities were close to the limit $-Mv$ is consistent with the evolution of the system from a generic set of initial velocities. This follows from our discussion of single particle dynamics and the observation that the only way the particles could fail to approach this limit would be if non-simple racquet strikes repeatedly occurred. But the occurrence of a non-simple racquet strike of two or more particles imposes algebraic conditions on the initial position and velocity data and system parameters. Thus, we find it is generic that non-simple racquet strikes do not occur in any pre-specified finite interval of time. Since without non-simple racquet strikes the velocities will get close to the limiting value $-Mv$ in a finite time (which is easily computed for any initial conditions), we see that our assumption holds generically. $\square$

Remark 3.2 Since (generically) any virtual particle having nonzero initial velocity will tend toward steady state speed Mv, while any virtual particle which is initially at rest will remain at rest, it follows from the previous theorem that there are precisely $n+1$ constant-velocity behaviors for our system. These may be indexed by either the number k of particles in motion or the complementary number $n-k$ which are at rest. We note that with even the smallest initial velocity, a virtual particle will eventually approach steady state speed Mv, and hence the only constant-velocity steady state behavior which is stable is the one in which all particles are moving with speed Mv.

Remark 3.3 Suppose we slightly deform the track in our racquet and particle system so that it has a parabolic shape with minimum at $x = 1/2$. This adds a gravitational force which is felt by the moving particles and changes the stability characteristics described above. All virtual particles which are initially sufficiently close to $x = 1/2$ and which have initial velocities too small to make it up the potential well to the strike-point $x = 1$ will remain near the minimum of the potential $x = 1/2$ for all time. On the other hand, any virtual particle which is moving fast enough initially to get to $x = 1$ will be forced toward the steady state speed Mv. Thus for this modified system, there are $n + 1$ constant-velocity motions counted, as in the previous remark, according to the number of virtual particles which are in motion. Each of these motions is stable in the sense that if the rest particles (point masses lying at the minimum of the potential) are perturbed slightly and the velocities deviate slightly from 0, the phase portrait of each rest particle remains in a neighborhood of the point $(x, v_p) = (1/2, 0)$, and those particles moving near the speed Mv tend toward this speed as time evolves.

Remark 3.4 The coexistence of a number of stable modes of behavior as described in the previous remark is an important feature which will also be noted for the periodic racquet motions discussed in the next section. Adding a small amount of friction to the particle motions will result in the steady state motions we have described being asymptotically stable. By making slight planned deviations from the control we have prescribed, it is possible to use the racquet strikes to move the system among the various domains of attraction. Feedback control strategies along these lines will be discussed elsewhere.

4 The response to a periodically moving racquet

Under the control law studied in the preceding section, the dynamics of the individual virtual particles are decoupled, and hence our investigation is reduced to the study of certain iterated scalar mappings. By contrast, even when there is only a single particle in our system, the response to periodic motions of the racquet is governed by two dimensional dynamics. This is because characterization of the response must be given not only in terms of the velocity transitions that occur but also how these are synchronized with the period of the racquet. We again explicitly describe the open loop control law (=periodic racquet motion) we shall study.

PERIODIC CONTROL LAW: The racquet moves back and forth between the positions $x = 1$ and $x = 2$ in the saw-tooth wave form:

$$v(t) = \begin{cases} v & 0 \leq t < h \\ -v & h \leq t < 2h \\ v(t) = v(t \pm 2h) & \text{otherwise} \end{cases}.$$

Note that because we have fixed the amplitude of the racquet's motion, the period and velocity are related by $vh = 1$. The remainder of this section will be devoted to obtaining an understanding of the dynamical response of our system of particles to this forcing.

For the moment, assume there is a single particle. As it moves, it will eventually collide with the racquet when it is moving either left or right. The velocity is changed according to the scattering law (4). To keep track of the particle dynamics over the course of many racquet strikes, as in the preceding section, is is useful to record the particle velocities just prior to each racquet strike. The rules describing the evolution of these quantities will depend on the particle path as it leaves the racquet:

TRANSITION RULES

Type (i): $v_p \mapsto (\frac{1-M}{M+1})v_p + \frac{2m}{M+1}v$

if the particle meets the racquet moving to the right and will again be struck by the racquet on its next left-stroke;

Type (ii): $v_p \mapsto (\frac{M-1}{M+1})v_p - \frac{2m}{M+1}v$

if the particle meets the racquet moving to the right and will next encounter the reflecting barrier before being struck by the racquet again;

Type (iii): $v_p \mapsto (\frac{M-1}{M+1})v_p + \frac{2m}{M+1}v$

if the particle meets the racquet when it is moving left.

For fixed M and v it is somewhat more complicated than what was done in the preceding section to keep track of the evolution of these velocities.

As in the preceding section, the only control variables which may be employed to modify the system's behavior within the structure of the assumed law are the racquet inertia M and the racquet velocity v. In Section 3, varying either M or v could change the steady state velocity of the system, but the dynamics remained qualitatively unchanged for all choices of M and v (both positive). It is not difficult to see that the qualitative response to the periodic control law described above is also unchanged as we vary v. This follows from writing out the explicit dependence of the particle velocity in terms of successive velocity transitions of the above form. As the number of transitions becomes large it is clear that the particle velocity v_p ceases to depend on the initial velocity, and under any change in racquet speed $v \mapsto \alpha v$, we shall have the particle velocity at any instant in time scaled by the same factor. Since the period of the racquet motion will be scaled inversely (i.e. $h \mapsto h/\alpha$ because we keep the amplitude of the racquet trajectory fixed), it follows that the motions of the racquet and particle remain unchanged except for the combined system speeding up or slowing down. Qualitative changes in the particle motion are produced by varying M, however, and it is these changes which we now summarize.

Theorem 4.1 *For $1 < M < 2$, the periodic control law above leads to an asymptotically stable periodic orbit of period $2h$ for the particle.*

Proof: The proof uses a construction which will be more broadly useful than this theorem alone. We note that given any time $0 \le t_0 < 2h$ and particle velocity v_p measured just prior to a racquet strike at t_0, it is possible to write down the next recorded velocity $\bar{v}_p$ (in terms of the velocity transition mappings above) and the time t when the next racquet strike occurs. This defines a function $\bar{F}(v_p, t_o) = (\bar{v}_p, t)$. This is the key to our proof, since we show there is an orbit as claimed by showing that the function $\bar{F}(v_p, t_0) - \begin{pmatrix} 0 \\ 2h \end{pmatrix}$ has a stable fixed point. It is not difficult to show that the only possible velocity value that could be a component of this fixed point is itself a fixed point of the third velocity transition

$$v_p \mapsto (\frac{M-1}{M+1})v_p + \frac{2M}{M+1}v,$$

which is obviously Mv. The t-component of $\bar{F}(v_p, t_0)$ may be explicitly written in this case as

$$t = \frac{(M-1)(v_p + v)t_0 + 8(M+1)}{(M-1)v_p + (3M+1)v}$$

Substituting the steady state velocity Mv for v_p, we wish to solve

$$t = (\frac{M-1}{M+1})t_0 + \frac{8}{M+1)v}$$

and

$$t = t_0 + 2h = t_0 + 2/v$$

for t_0 in the interval $h \leq t_0 < 2h$. Solving, we find $t_0 = \frac{3-M}{v}$, and this will be in the desired range precisely when $1 < M \leq 2$. This shows the period $2h$ orbit exists under the conditions of the formula. To show that it is stable, we need to evaluate the derivative of $\bar{F}$ at the fixed point. This may be done by the obvious explicit calculation in this case, and we find that this derivative has repeated eigenvalues $\frac{M-1}{M+1}, \frac{M-1}{M+1}$ at the fixed point. Since these are both strictly less than 1 in absolute value, the fixed point is stable, and this proves the theorem.□

Remark 4.1 It turns out that this orbit is stable precisely when it exists. Generally, this will be true of all the velocity orbits we study for this system.

Our study of the particle dynamics produced by the periodic racquet motions we have described may be reduced in general to studying the iterated function dynamics of $F : V \times [0, 2h] \to V \times [0, 2h]$, where V is the set of velocities "recorded" just prior to each racquet strike ($V = [-v, \infty)$), and F maps points in this velocity-time space as follows:

If $0 \leq t_0 < h$ then

$$(v_0, t_0) \xmapsto{F} \begin{cases} (\frac{1-M}{M+1}v_0 + \frac{2M}{M+1}, \frac{(M-1)(v-v_0)t_0+2(M+1)}{(1-M)v_0+(3M+1)v}) & v < v_0 \leq \frac{4Mv+(1-M)v^2t_0}{(M-1)(2-vt_0)} \\ ((\frac{M-1}{M+1}v_0 - \frac{2M}{M+1}, s) & v_0 > \frac{4Mv+(1-M)v^2t_0}{(M-1)(2-vt_0)} \end{cases}$$

If $h \leq t_0 < 2h$ then

$$(v_0, t_0) \xmapsto{F} \begin{cases} (\frac{M-1}{M+1}v_0 + \frac{2M}{M+1}, \frac{(M-1)(v+v_0)t_0+8(M+1)}{(M-1)v_0+(3M+1)v} - \frac{2}{v}) & -v \leq v_0 < \frac{(M-1)v^2t_0-(M-5)v}{(M-1)(3-vt_0)} \\ ((\frac{M-1}{M+1}v_0 + \frac{2M}{M+1}, \frac{(M-1)(v+v_0)t_0+2(M+1)}{(M-1)(v_0+v)} - \frac{2}{v}) & \frac{(M-1)v^2t_0-(M-5)v}{(M-1)(3-vt_0)} \leq v_0 \leq \frac{(M-1)v^2t_0+4v}{(M-1)(2-vt_0)} \\ (\frac{M-1}{M+1}v_0 + \frac{2M}{M+1}, \frac{(M-1)(v+v_0)t_0+6(M+1)}{(M-1)v_0+(3M+1)v} - \frac{2}{v}) & \frac{(M-1)v^2t_0+4v}{(M-1)(2-vt_0)} < v_0 \end{cases},$$

where s denotes the time (mod($2h$)) at which the next collision between the racquet and the particle occurs. (We omit the explicit expression for s, since it is in principle straightforward to calculate but roughly doubles the complexity of the explicit formula for $F(v_0, t_0)$.) The idea here is that this function describes simultaneously the velocity transitions of our system together with the sequence of times *modulo the basic period* $2h$ at which the racquet strikes the particle. While a complete characterization of the iterated function dynamics for F cannot be given here, we are able to carry out some elementary calculations which illustrate important qualitative features of the system dynamics.

Thinking of the three basic types of velocity transitions in the above table as letters in an alphabet, we may uniquely identify any trajectory obtained from iterating the function F by the sequence of velocity transitions which it defines. Conversely, if any sequence of velocity transitions is written down, it will define a trajectory of iterates of F provided an appropriate sequence of transition times can be given.

To illustrate what is involved in finding a trajectory which realizes a prescribed sequence of velocity transitions, we shall investigate the existence of several periodic trajectories. Call a trajectory in which there is a repeated pattern of velocity transitions consisting of a type (i) transition followed by $k-1$ type (iii) transitions a *type 1 velocity cycle.* We shall show that type 1 velocity cycles may or may not exist.

Proposition 4.1 *There is no type 1 velocity cycle of length 2.*

Proof: A velocity cycle of length 2 would alternate velocity transitions of type (i) and type (ii). One can explicitly solve for the particle velocities comprising this cycle:

$$v_1 = \frac{2M^2}{(M+1)^2}v \quad \text{and} \quad v_2 = \frac{2M}{(M+1)^2}v.$$

The corresponding 2-cycle of collision times t_1, t_2 would be related according to the above definition of F by the formulas

$$t_2 = \frac{(M-1)(v-v_1)t_1 + 2(M+1)}{(1-M)v_1 + (3M+1)v}$$

and

$$t_1 = \frac{(M-1)(v_2+v)t_2 + 2(M+1)}{(M-1)(v_2+v)} - 2/v.$$

In order for a type 1 2-cycle to exist, we must be able to solve these equations simultaneously for t_1, t_2 in the respective intervals $0 \leq t_1 < h$ and $h \leq t_2 < 2h$. Solving the equations simultaneously for t_1 yields

$$t_1 = \frac{M^6 + 6M^5 + 15M^4 + 56M^3 + 39M^2 + 10M + 1}{(M-1)(M+1)^3(M^2+4M+1)}h.$$

(Recall that $h = 1/v$.) It is not difficult to show that on the interval $1 < M < \infty$, t_1 is monotonically decreasing and always greater than h. Since we therefore cannot have $t_1 < h$, we have shown that no 2-cycle of type 1 exists. □

A tedious but elementary calculation of this type shows that a length 3 type 1 velocity cycle exists with

$$(v_1, v_2, v_3) = \left(\tfrac{3M^2+1}{M^2+3}v, \quad \tfrac{-M^2+4M+1}{M^2+3}v, \quad \tfrac{M^2+4M-1}{M^2+3}v \right)$$

provided we can find a corresponding time cycle (t_1, t_2, t_3) with $0 \leq t_1 < h$ and $h \leq t_2, t_3 < 2h$. Elementary but lengthy arguments along the lines described in proving the proposition show that all required inequalities are satisfied provided

$$3M^2 - 8M + 1 > 0$$

and

$$M^3 - 11M^2 + 3M - 1 < 0.$$

These will simultaneously hold for $2.53518\ldots < M < 10.7291\ldots$.

In principle, the same type of elementary argument may be used to confirm or rule out the existence of an orbit of any type, but the complexity of the formal manipulations places practical limits what can be determined in this way.

We conclude with some remarks based on simulation. Although an asymptotically stable length 3 type 1 cycle exists for the range of inertias M we have indicated, the corresponding domain of attraction may be quite small. Other types of orbits are found to coexist for various values of M, and the observed behavior will depend sensitively on initial conditions. (For instance, for $M \approx 5.4$, a length 3 velocity cycle consisting of a type (ii) and two type (iii) transitions seems to dominate the dynamics.) The possible multiplicities of coexisting stable periodic orbits and the ways in which their geometry may vary with the racquet inertia remains open at the present time. It is precisely by understanding the dependence of the system's dynamics on such control parameters that we hope to develop a control theory for systems of this type.

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