Systems Engineering PhD Qualifying Exam

WRITTEN EXAM: Tuesday, May 20th, 2025
9:00AM-1:00PM, 15 Saint Mary's Street, Room 105

- **NO ELECTRONIC DEVICES** (smartphone, iPad, smartwatch) permitted
- Calculators and a ruler are allowed.
- CLOSED BOOK. Only the notes indicated below will be allowed.

INSTRUCTIONS:

1) Write your **EXAM NUMBER** on every sheet of paper

2) Write clearly and legibly as the exam may be scanned to faculty for grading.

3) **Answer 3 out of 5 questions** completely from the five sections below:

Section I: Dynamic Systems Theory (SE 501, Baillieul)
   - CLOSED BOOK, NO NOTES

Section II: Discrete Stochastic Processes (SE 714, Perkins)
   - CLOSED BOOK, NO NOTES

Section III: Nonlinear Systems and Control (SE 762, Wang)
   - CLOSED BOOK, NO NOTES

SE Linear Systems Qualifying Exam - 2025

1. Suppose that the matrix A is $n \times n$, B is $n \times m$, and C is $q \times n$ and that the system

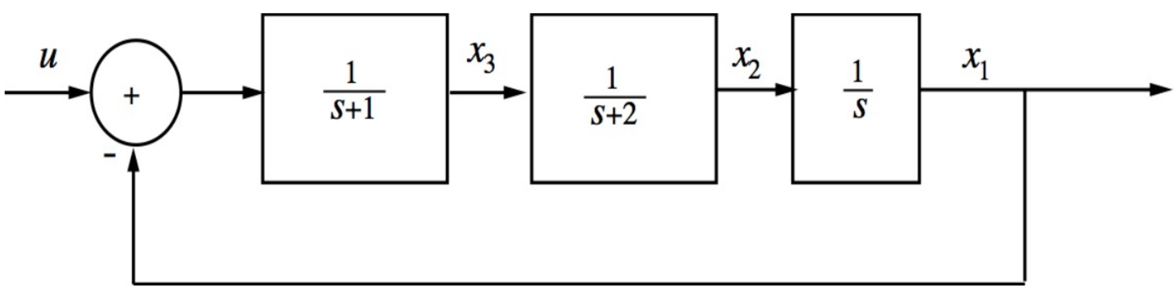
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

is *controllable* and *observable*. Prove that there are an $m \times n$ matrix K and an $n \times q$ matrix E such that the observer system

$$\dot{z} = (A - EC)z + Ey + Bu \tag{2}$$

together with (1) are asymptotically stable under the control $u = Kz$.

2. Design an observer for the system shown in the figure. The observer should be of second order with both eigenvalues equal to -3.



3. (a) Find the Jordan Normal Form $\mathbf{J_A}$ of

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 1 \\ 4 & 4 & -2 \\ 6 & 3 & -1 \end{pmatrix}.$$

(b) Find a matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{J_A}$.

(c) For the $\mathbf{J_A}$ found in part (a), compute $e^{\mathbf{J_A}t}$ in closed form.

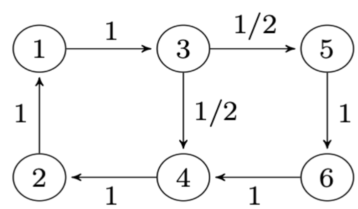
Systems Ph.D. Qualifying Examination
May 2025
Section II: Discrete Stochastic Processes

Problem 1. Let $\{X_n; n \geq 1\}$ be a renewal process such that $X_n \sim U[0, 3]$, i.e., X_n has a continuous uniform distribution on $[0, 3]$. Determine

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P\left(Y(u) > 2\right) du.$$

where $Y(t) = S_{N(t)+1} - t$ is the residual lifetime at time t.

For **Problems 2 and 3**, consider the following discrete-time Markov chain:



Problem 2.

a. Determine $E[T_{1,1}]$ and $E[T_{6,6}]$, where $T_{i,j}$ is the first passage time from state i to state j.

b. Describe all stationary distributions of the chain.

c. For each of the following, state whether the value is or is not an eigenvalue of the one-step transition matrix $P: -1, 0, 1, \sqrt{-1}, \frac{\sqrt{5}}{2}$

d. Let d be the period of state 1. Determine d.

e. Determine $\lim_{n\to\infty} P^{(dn+4)}$, using the value of d determined above.

Problem 3.

For the Markov chain above, a renewal process is defined based on returns to state 1. Let $\{X_1, X_2, \cdots\}$ denote inter-renewal times of this process and $N(n)$ the number of returns to state 1 in at total of n steps.

a. Determine $E[X_2]$.

b. Determine

$$\lim_{n\to\infty} P\left(\text{there is a renewal at}\, nd\right)$$

where d is the value determined in the previous problem.

c. Does

$$\lim_{n\to\infty} \frac{N(n)}{n}$$

exist? If so, determine it. If not, explain why it does not exist.

SE Qualifying Exam
Nonlinear Systems and Control
May 20, 2025

1. Give a precise mathematical definition of the following terms:

   (a) Uniformly asymptotic stability of an equilibrium point x_0 of $\dot{x}(t) = f(t, x(t))$.

   (b) Positive definiteness of a continuous function $W : [t_0, \infty] \times R^n \to R$.

   (c) Positive limit set of a bounded solution $x(t, x_0, t_0)$.

2. Consider Liénard's equation

   $$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

   where g and h are continuously differentiable.

   (a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

   (b) Using $V(x) = \int_0^{x_1} g(y)dy + (1/2)x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the original is asymptotically stable.

3. Consider the following nonlinear control system

   $$\begin{aligned} \dot{x}_1 &= (-1-\alpha)x_1 - 2x_2 + (1+\alpha)u - ux_1(1-\alpha) \\ \dot{x}_2 &= (1-\alpha)x_1 + (1-\alpha^2)u - ux_2(1-\alpha) \end{aligned}$$

   where α is a real number. Determine for which values of α there exists a continuously differential feedback control $u = k(x)$, $k(0) = 0$ such that 0 is an asymptotically stable equilibrium point of the resulting closed loop system.