ME421. Viscous Effects.

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1 Viscous Incompressible Flows.

As was shown above, inviscid (zero viscosity) hydrodynamic approximation was able to produce various qualitative and sometimes even quantitative results on various pressure-dominated flows.
The main drawback of the Euler equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + g\mathbf{i}; \qquad \nabla \cdot \mathbf{u} = 0 \tag{1.1}$$

can be demonstrated on a simple example of a flow between two plates at $y = \pm H$ driven by a constant gravity $\mathbf{g}$. The boundary condition for a normal -to-the-wall component of velocity is $v \equiv u_n(\pm H) = 0$. To write an equation for kinetic energy $K = \int_V \frac{\rho}{2}\mathbf{u}\cdot\mathbf{u}$ of a large volume of fluid contained between crossections at $x = -L$ and $x = L$ with $|L| \to \infty$, we multiply (16.1) by $\mathbf{u}$, integrate over entire space occupied by the fluid with the result:

$$\frac{\partial K}{\partial t} + \rho\int_V [\nabla\cdot\mathbf{u}\frac{u^2}{2} + \nabla(p\cdot\mathbf{u})/\rho - \mathbf{g}\cdot\mathbf{u}]dV = 0 \tag{1.2}$$

In the gravity -driven flow, where the pressure $p(L) = p(-L)$ (zero pressure gradient) and the normal component $u_n(\pm H) = 0$, we have :

$$\frac{\partial K}{\partial t} = \rho V g\overline{u} > 0 \tag{1.3}$$

meaning that with time, kinetic energy of a fluid grows to infinity. Since $\overline{u} = \sqrt{2K/\rho}$, we can solve this equation to obtain: $K \propto gt^2$. This is in a gross contradiction with the real-life flows.
Solution to this contradiction is of course found in the estimate (5.3) showing that in a close proximity to the wall $\mathbf{u} = 0$, $y^2 \leq \nu/\frac{\partial u}{\partial y}$, where by the no-slip boundary condition $\mathbf{u} = 0$, $y^2 \leq \nu/\frac{\partial u}{\partial y}$, the viscous contribution cannot be neglected. Thus, the equation for a channel flow must be written as:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + g\mathbf{i} + \nu\nabla^2\mathbf{u}; \qquad \nabla \cdot \mathbf{u} = 0 \tag{1.4}$$

In case $g = 0$, the flow must be driven by a non-zero pressure gradient $\frac{\partial p}{\partial x} = const \neq 0$. In a steady state, integrating the Navier-Stokes equation over y-coordinates in the interval $(-H, 0)$, gives:

$$\frac{\partial p}{\partial x}H = -\nu[\frac{\partial u(-H)}{\partial y} - \frac{\partial u(0)}{\partial y}] = -\nu\frac{\partial u(-H)}{\partial y} \equiv u_*^2 \tag{1.5}$$

Deriving this relation we took into account that in a channel flow $\mathbf{u}\cdot\nabla\mathbf{u} = 0$ and the y-derivatives of the velocity profile $\partial_y u(0) = 0$ due to the symmetry of the flow about the x-axis. The details will be discussed below.

Couette flow.

This a flow between two plate with the one at $y = H$ moving with velocity U and the second, at $y = 0$, at rest. The plates are long and wide with pressure $p_1 = p(x_1)$ and $p_2 = p(x_2)$ with $x_2 > x_1$. Neglecting the gravity force we write equations for components of velocity field:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \tag{1.6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{1.7}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{1.8}$$

These equations are to be solved subject to boundary conditions:

$$u(H) = U;\; u(0) = 0; \qquad v(H) = 0; v(0) = 0; \tag{1.9}$$

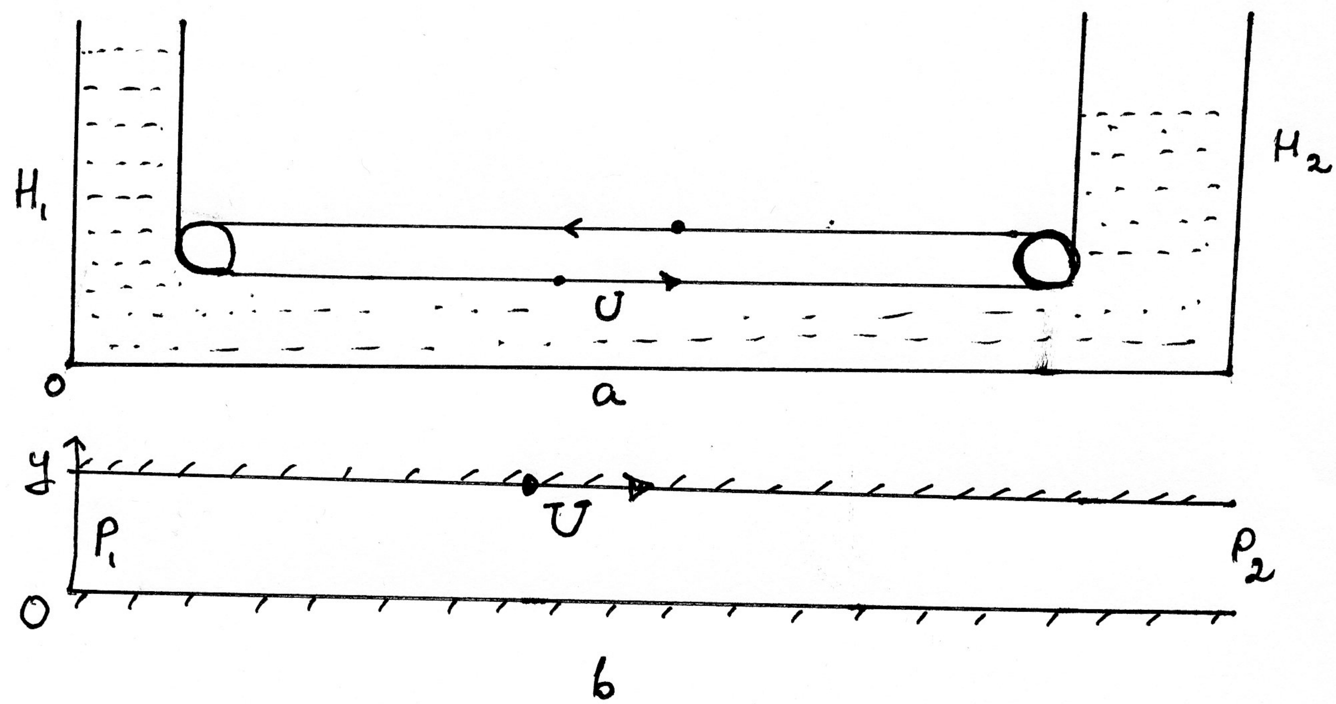


Figure 1: *Couette flow. Schematic.*

The channel can be considered "long and wide" only when the test section (see Figure 9) is located far enough from the inlet/outlet, so that that the flow features are insensitive to the x-ccordinate variation. Therefore, we can set all x-derivatives $\frac{\partial u}{\partial x} = 0$. From the continuity equation (16.6) we have $\frac{\partial v}{\partial y} = 0$. This means that $v(x, y) = const$. Since $v = 0$ at both plates, we have $v(y) = 0$. The solution to (16.8) is:

$$p(x,y) = \rho g y + f(x) \tag{1.10}$$

where $f(x)$ is as yet undetermined function. This solution leads to a method of generating pressure gradient in this flow by keeping different fluid levels at the inlet and outlet (see Figure 41). The momentum equation (16.7) gives:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u(y)}{\partial y^2} \tag{1.11}$$

In a set up we are considering, pressure can be a function of both x and y-coordinates. However, as follows from (16.10), the x-component of the pressure gradient $\frac{\partial p}{\partial x} = \frac{\partial f(x)}{\partial x}$ is y-independent. The balance (16.11) can be valid for an arbitrary pair of coordinates (x,y) only if right and left sides if (16.11) are equal to a constant. Thus, the solution, satisfying the boundary conditions (16.11) is:

$$u(y) = U\frac{y}{H}[1 + \frac{\partial p}{\partial x}\frac{H^2}{2\mu U}(\frac{y}{H} - 1)] \equiv Uz(1 + \kappa(z-1)) \tag{1.12}$$

where $z = y/H$ and $\kappa = \frac{\partial p}{\partial x}\frac{H^2}{2\mu U}$. Thus,

$$\frac{du}{dz} = U(1 - \kappa + 2\kappa z) \tag{1.13}$$

Three possible cases must be considered. In a neutral (zero pressure gradient $\kappa = 0$) case, the velocity field

$$u = U\frac{y}{H} \tag{1.14}$$

When the pressure gradient is negative ($p_1 > p_2$) , it pushes the flow in the direction of a moving wall. This is the case of *favorable pressure gradient.* The velocity profile

$$u(z,\kappa) = Uz(1 + |\kappa| - |\kappa|z); \qquad \partial_z u = 1 + |\kappa|(1 - 2z) \tag{1.15}$$

We see that in this case, in the vicinity of the wall at $z \to 0$, the velocity distribution is a monotonically increasing function $u(z,\kappa) > u(z,0)$.

In a flow with adverse pressure gradient $\kappa > 0$, the situation is different: the pressure force acts in the direction opposite to the velocity vector u. Since $p = p(x)$, independent upon y, it is large and the slow-moving fluid elements in the vicinity of the $y = 0$ plate are most influenced by it. We have

$$u(z,\kappa) = Uz(1 - |\kappa| + |\kappa|z); \qquad \partial_z u = U(1 - \kappa + 2\kappa z) \tag{1.16}$$

Near the wall ($z \to 0$), when pressure gradient is larger than a critical value $\kappa_c = \frac{\partial p}{\partial x}\frac{H^2}{2\mu U}$, so that the critical pressure gradient

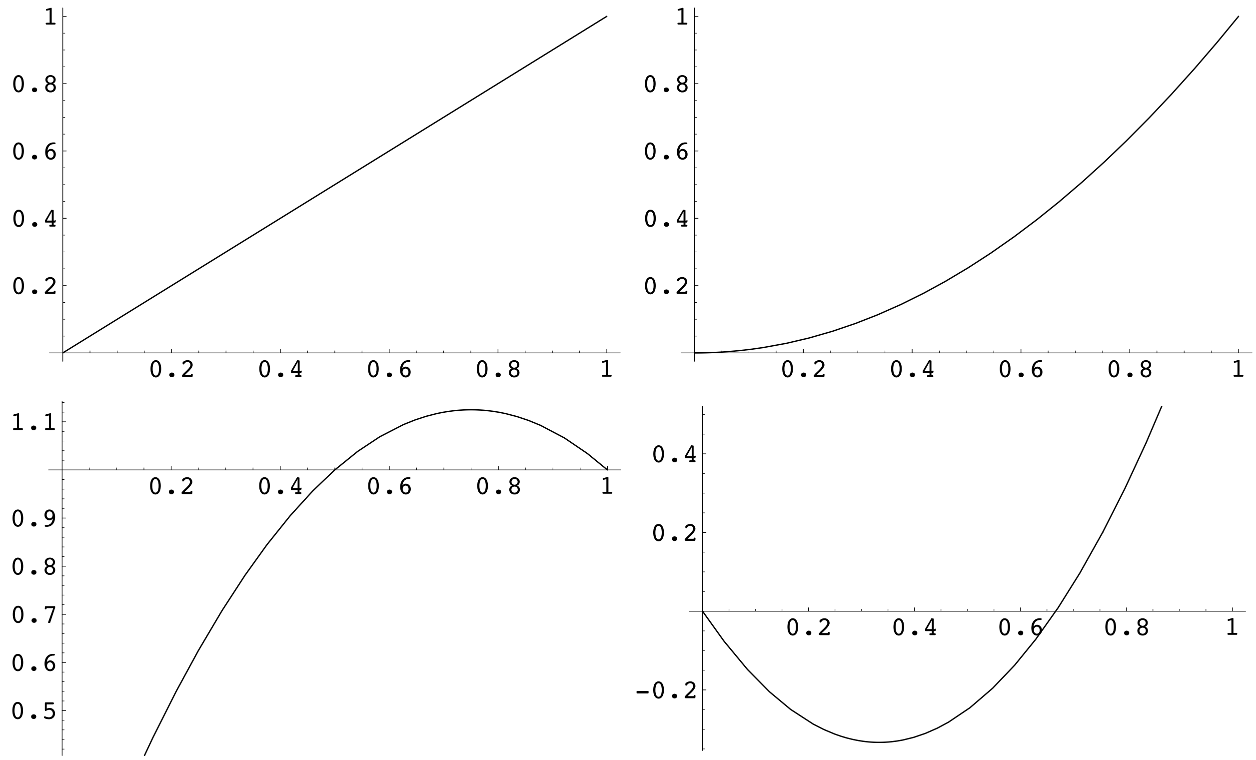


Figure 2: *Couette flow. Normalized velocity distribution* $\frac{u(y)}{U}$ *vs* $\frac{y}{H}$. *a. Zero pressure gradient* $\kappa = 0$. *c. Favorable pressure gradient* $\kappa < 0$. *Critical case* $\kappa = 1$. *d. Adverse pressure gradient* $\kappa > 0$. *Negative velocity indicate flow reversal.*

$$\frac{\partial p}{\partial x}|_c = \frac{2\mu U}{H^2} \tag{1.17}$$

so that $\partial_z u(z) \approx U(1-\kappa) < 0$, the velocity profile $u(z) < u(0) = 0$ and we can notice appearance of a reversal region with negative velocity $u < 0$. The flow reversal is related to a general and very important phenomenon of flow separation which we will discuss below.

Problem.. 1. Find the width δ of the reversed flow layer in Couette flow as a function of pressure gradient $\frac{\partial p}{\partial x}$. Plot the graph δ vs $\frac{\partial p}{\partial x}$. 2. Is the flow reversal possible if the viscosity effects are neglected. Explain.

Infinite channel (Poiselle) flow. Now we consider the flow between the two plates at $y = \pm H/2$, driven be the non-zero pressure gradient. The problem is formulated by equations (16.6)-16.8) with the no-slip boundary condition $\mathbf{u}(\pm\frac{H}{2}) = 0$. From the contunuity equation (16,6) we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, which, combined with the boundary conditions gives $v = 0$. The remaining equation (16.8)

$$\frac{\partial p}{\partial y} = 0$$

tells us that $p = p(x)$. Again, the x-derivative of velocity field is zero, so that the momentum equation reduces to (16.11). Since $p = p(x)$ and $u = u(y)$, we conclude that $\frac{\partial p}{\partial x} = const$. The solution satisfying the boundary condition:

$$u(y) = |\frac{\partial p}{\partial x}|\frac{H^2}{8\mu}(1 - 4(\frac{y}{H})^2) \equiv u_{max}(1 - (\frac{2y}{H})^2) \tag{1.18}$$

The mean velocity is calculated readily (H is the width of the channel):

$$U \equiv \overline{u} = \frac{1}{S}\int_{-H}^{H} dy \int_0^W dz = \frac{1}{2H}\int_{-H}^{H} u(y)du = \frac{2}{3}u_{max} \tag{1.19}$$

and the wall-stress:

$$\tau_w = \tau_{x,y} = \mu\frac{\partial u(-\frac{H}{2})}{\partial y} = \frac{4\mu u_{max}}{H} \tag{1.20}$$

The skin friction:

$$c_f = \frac{2\tau_w}{\rho U^2} = \frac{8\mu}{\rho u_{max} H} = \frac{8\nu}{UH} = \frac{8}{Re_H} = \frac{16}{Re_{2H}} = \frac{12}{Re_{\overline{u}}} \tag{1.21}$$

where the Reynolds number

$$Re_H = UH/\nu \qquad\qquad Re_{\overline{u}} = \frac{2H\overline{u}}{\nu} \tag{1.22}$$

Problem 1. Consider a fluid layer of hight $h = const$ which is bounded from above by a free surface and below by a fixed plane inclined by the angle α to the horizontal.

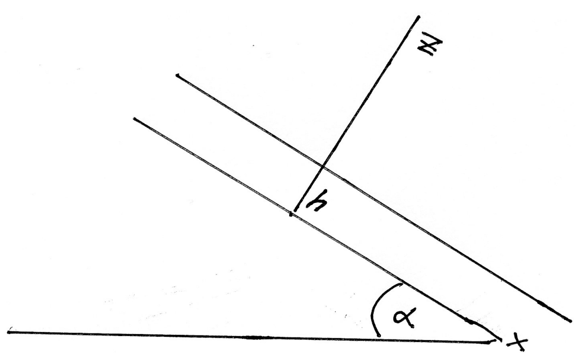


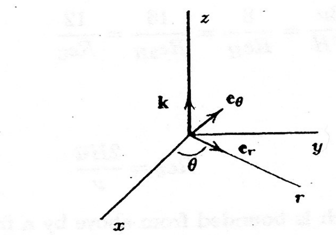
Figure 3: *A gravity - driven fluid layer.*

Determine flow due to gravity. **Prove that**:

$$u = \frac{\rho g \sin \alpha}{2\mu} z(2h - z)$$

$$p = p_0 + \rho g(h - z)\cos \alpha$$

Cylindrical Coordinates.



Velocity vector:
$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + w\,\mathbf{k}$

Position vector:
$\mathbf{r} = r\,\mathbf{e}_r + z\,\mathbf{k}$

Unit vectors:
$\mathbf{e}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$
$\mathbf{e}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$

Figure E.2: *Cylindrical polar coordinates.*

We express the velocity vector, $\mathbf{u}$, in terms of its three components u_r, u_θ and w as shown in Figure E.2. Typical *grad, div, curl* and Laplacian operations are as follows

$$\nabla p = \mathbf{e}_r \frac{\partial p}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial p}{\partial \theta} + \mathbf{k}\frac{\partial p}{\partial z} \tag{E.14}$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial w}{\partial z} \tag{E.15}$$

$$\nabla \times \mathbf{u} = \mathbf{e}_r\left[\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right] + \mathbf{e}_\theta\left[\frac{\partial u_r}{\partial z} - \frac{\partial w}{\partial r}\right] + \mathbf{k}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \tag{E.16}$$

$$\nabla^2 \mathbf{u} = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right](u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + w\,\mathbf{k}) \tag{E.17}$$

The continuity equation and the three components of the Navier-Stokes equation with a body force $\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{k}$ are:

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{E.18}$$

$$\begin{aligned}\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + w\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + f_r \\ &+ \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right)\end{aligned} \tag{E.19}$$

$$\begin{aligned}\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + w\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + f_\theta \\ &+ \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)\end{aligned} \tag{E.20}$$

$$\begin{aligned}\frac{\partial w}{\partial t} + u_r\frac{\partial w}{\partial r} + \frac{u_\theta}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z \\ &+ \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\end{aligned} \tag{E.21}$$

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Figure 4: *Cylindrical polar coordinates.*

2. Fully developed flow in pipe. Consider an infinite pipe. Usually , this means that the pipe length is much larger than its diameter, i.e. $D/L \ll 1$. There is no a well-established theoretical criterion how small the ratio $a = D/L$ has to be. The old experimental data on a friction factor in fully developed turbulent flow, obtained by Nikuradze from a pipe with $a \approx 10$, happened to be in a relatively close agreement with the most recent results by Smits et all observed in Princeton Superpipe with $a \approx 30 - 40$. In any case, it is clear that a test section where the measurements are taken has to be somewhere in the pipe where nonuniversal features of inlet and outlet can be safely neglected neglected. As in the case of a channel flow, we set $\frac{\partial}{\partial x} = 0$. By the symmetry, $\frac{\partial}{\partial \theta} = 0$. We have incompressibility equation in cylindrical coordinates:

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(ru_r) = 0 \tag{1.23}$$

and the Navier-Stokes equations

$$u\frac{\partial u_r}{\partial x} + u_r\frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu[\frac{\partial^2 u_r}{\partial x^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}] \tag{1.24}$$

$$u\frac{\partial u}{\partial x} + u_r\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}] \tag{1.25}$$

The boundary conditions: $\mathbf{u}(r = R) = 0$ and by the symmetry of the problem: $\frac{\partial u}{\partial r}|_{r=0} = 0$. Since $\partial_x u = 0$, it follows from (16.23) that $\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = 0$ and, as a result, $ru_r = const$. Since $u_r(R) = 0$, we conclude: $u_r = 0$. Then, as follows from the equation (16.24) $\frac{\partial p}{\partial r} = 0$, which gives $p = p(x)$. Analysing each term of the equation (16.25) we derive:

$$\frac{\partial p}{\partial x} = \mu(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) = \frac{\mu}{r}\frac{d}{dr}(r\frac{du}{dr}) \tag{1.26}$$

Since we know that $p = p(x)$ and $u = u(r)$,

$$\frac{dp}{dr} = \frac{\mu}{r}\frac{d}{dr}(r\frac{du}{dr}) = const \tag{1.27}$$

and

$$\frac{du}{dr} = \frac{1}{\mu}\frac{dp}{dr}\frac{r}{2} + \frac{A}{r} \tag{1.28}$$

The boundary condition $u < \infty$ at $r = 0$ sets $A = 0$. The final result is:

$$u(r) = \frac{1}{4\mu}|\frac{dp}{dr}|(R^2 - r^2) \equiv u_{max}(1 - \frac{r^2}{R^2}) \tag{1.29}$$

Show that for the pipe of radius R and length L:

$$
\begin{aligned}
U &\equiv \overline{u} = u_{max}/2 \\
\tau_{xr}(r=R) &= \mu\frac{du}{dr} = -\frac{2\mu u_{max}}{R} \\
c_f &= \frac{8}{Re_R} = \frac{16}{Re_D} \\
Q &= \frac{\pi\Delta p}{8\nu L}R^4
\end{aligned} \tag{1.30}
$$

where $Re_R = UR/\nu$ and massflux

$$
Q = 2\pi\rho\int_0^R ru(r)\ dr
$$

Now we derive a a useful relation for a pipe flow. Since $\overline{u}$ and ρgz are independent upon x, we can rewrite (16.24) as

$$
\frac{\partial}{\partial x}(p(x) + \frac{\alpha}{2}\rho\overline{u}^2 + \rho gz) = -\frac{8\mu\overline{u}}{R^2} = const
$$

where $\alpha = \frac{\overline{u^3}}{\overline{u}^3}$. (**Show that for a laminar flow** $\alpha = 2$.) It will be shown below that a fully developed turbulent flow in a pipe $\alpha \approx 1.02 - 1.05$. The above relation can be represented as:

$$
\frac{\partial}{\partial x}(p_1(x) + \frac{\alpha_1}{2}\rho\overline{u_1}^2 + \rho gz_1) = \frac{\partial}{\partial x}(p_2(x) + \frac{\alpha_2}{2}\rho\overline{u_2}^2 + \rho gz_2) + \frac{8\mu\overline{u}L}{R^2} = const
$$

Integrating this over x from $x_1 = 0$ to $x_1 = L$ gives:

$$
(p_2(x) + \frac{\alpha_2}{2}\rho\overline{u_2}^2 + \rho gz_2) - (p_1(x) + \frac{\alpha_1}{2}\rho\overline{u_1}^2 + \rho gz_1) + \frac{8\mu\overline{u}L}{R^2}L = 0
$$

and dividing by $L/\rho g$ leads to :

$$
\frac{p_1}{\rho g} + \alpha_1\frac{\overline{u_1}^2}{2g} + z_1 = \frac{p_1}{\rho g} + \alpha_1\frac{\overline{u_1}^2}{2g} + z_1 + \frac{8\mu\overline{u}L}{R^2}L \equiv \frac{p_1}{\rho g} + \alpha_1\frac{\overline{u_1}^2}{2g} + z_1 + h_L
$$

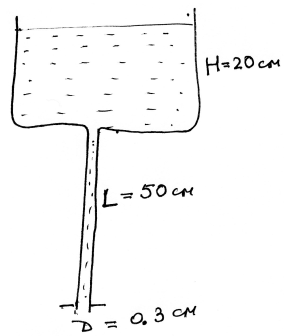
where the head loss

$$
h_L = \frac{8\mu\overline{u}L}{R^2}L = 4c_f\frac{\overline{u}^2}{2g}\frac{L}{D}
$$

.

This energy balance relation tells us that the total energy of a fluid element is spent on a work against both surface (pressure) and viscous forces.

Problem 1. A vertical IV capillary tube of diameter $D = 0.3cm$ and length $L = 50cm$ is connected to a patient blood stream. The parameters of the problem are shown on the Figure. Find amount of liquid supplied to the patient per second if its density $\rho = 1gr/cm^3$ and the bloodstream pressure $p_2 \approx 1atm$.



Solution.

$$\rho g(H+L) = \frac{8\rho\nu\overline{u}}{R^2}L$$

$$Massflux = \rho\overline{u}\pi R^2$$

Problem. 2. Consider a flow in a pipe of annular cross-section, the external and internal radii R_2, R_1. Derive solution:

$$u(r) = \frac{\Delta p}{4\mu L}[R_2^2 - r^2 + \frac{R_2^2 - R_1^2}{\ln(\frac{R_2}{R_1})}\ln(\frac{r}{R_2})]$$

$$Q = \frac{\Delta p}{8\mu L}[R_2^4 - R_1^4 - \frac{(R_2^2 - R_1^2)^2}{\ln(\frac{R_2}{R_1})}]$$

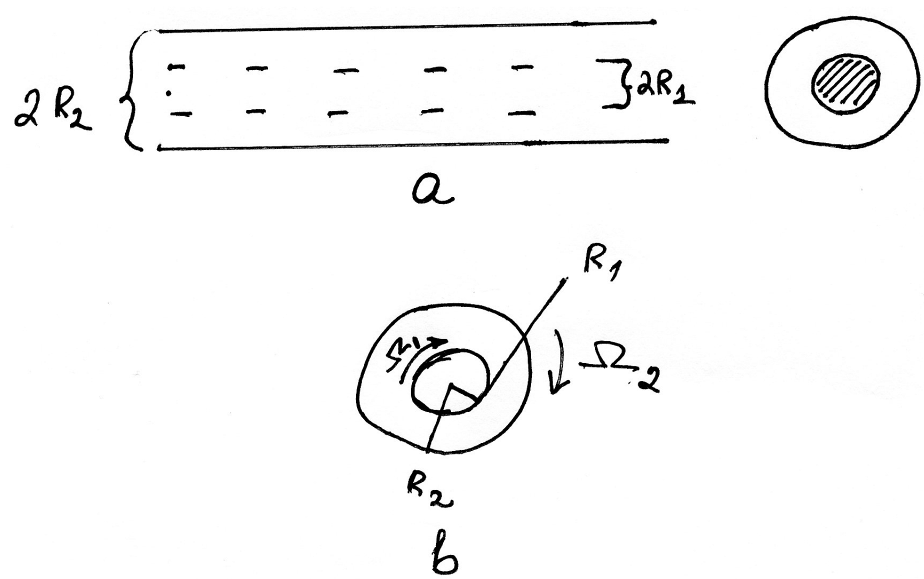


Figure 5: *Flow geometries. a. Problem 2. b. Problem 3.*

Problem 3. Flow between rotating coaxial infinite cylinders. The radii of the cylinders rotating about their mutual x-axis are $R_2 > R_1$. The angular velocities are Ω_1 and Ω_2, respectively. In cylindrical coordinates:$u_x = u_r = 0$ and $u_\theta = u_\theta(r)$. It also clear from equations that $p = p(r)$. In this case:

$$\frac{dp}{dr} = \rho u_\theta^2/r \tag{1.31}$$

and

$$\frac{d^2 u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{u_\theta}{r^2} = 0 \tag{1.32}$$

The boundary conditions: $u_\theta(R_1) = \Omega_1 R_1$ and $u_\theta(R_2) = \Omega_2 R_2$. Seeking the solution to this equation as $u_\theta = Ar^\gamma$, we obtain a simple algebraic equation for γ with solution $\gamma = \pm 1$. Thus,

$$u_\theta = ar + b/r \tag{1.33}$$

The final result:

$$u_\theta(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}\frac{1}{r} \tag{1.34}$$

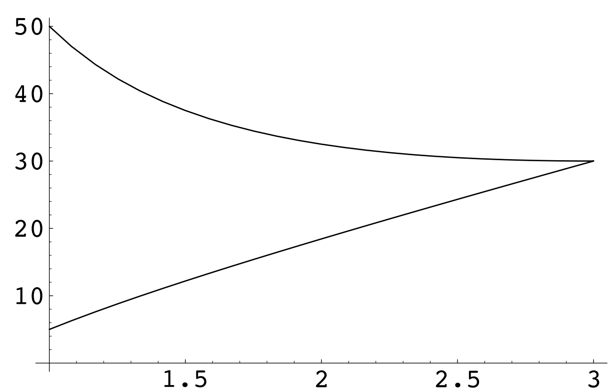


Figure 6: Velocity distributions between to coaxial rotating cylinders. Lower curve; $R_1 = 1;\ R_2 = 3;\ \Omega_1 = 5;\ \Omega_2 = 10.$. Upper curve: same cylinders with $\Omega_1 = 50;\ \Omega_2 = 10$. Arbitrary units.

The frictional force acting on the cylinders is found from the stress:

$$\tau_{r,\theta} = \mu[\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}]_{r=R_1} = -2\mu\frac{(\Omega_1 - \Omega_2)R_2^2}{R_2^2 - R_1^2} \tag{1.35}$$

The force acting on the unit length of the cylinder is $F_1 = 2\pi R_1 \tau_{\theta,r}(1)$. The moment of that force is $M_1 = F_1 R_1$, so that

$$M_1 = -4\pi\mu\frac{(\Omega_1 - \Omega_2)R_1^2 R_2^2}{R_2^2 - R_1^2} \tag{1.36}$$

It is clear that $M_2 = -M_1$.

Infinite plane disk (von Karman flow 1921.

Flow between inclined wall (Hammel 1916)

Stokes' first flow problem.

We consider an infinite plate $y = 0$ which at time $t = 0$ starts moving in the x -direction with velocity $\mathbf{U} = U\mathbf{i}$. Prior to this instant, at $t < 0$, the plate and surrounding fluid were quiescent, i.e. $u(\mathbf{r}, t) = 0$. By the no-slip boundary condition, at $y = 0$, the fluid moves with the velocity of the plate, i.e.

$$\mathbf{u}(y = 0, t) = U\mathbf{i} \tag{1.37}$$

The shear stress generated by the plate motion acts as a driving force applied to the fluid layers in immediate vicinity of the plate, i.e. at $y = 0 + \epsilon$. This layer drives the next one which is further away etc, thus leading to development of the non-zero velocity field at $y > 0$. Our goal is to describe the ftime-dependent low generated by the moving infinite plate.

The Navier - Stokes equations and incompressibility constraint in this case:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.38}$$

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}] \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}] \end{aligned} \tag{1.39}$$

are to be solved subject to the second boundary condition:

$$\lim_{y \to \infty} \mathbf{u}(y, t) = 0 \tag{1.40}$$

The plate is infinite, so that its properties are independent upon coordinate x. Thus, $\partial_x u =$ and as follows from (16.37), the y-component $v(y) = const$. Since $v(y = 0) = 0$, we conclude that $v(y) = 0$ and $u = u(y, t)$. With this result, the equation (16.39) gives $\frac{\partial p}{\partial y} = 0$, meaning that $p = p(x)$. This gives:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2}{\partial y^2}$$

Since $\partial_x u = 0$, differentiating this equation over x gives: $\frac{\partial^2 p}{\partial x^2} = 0$ and

$$p = ax + b$$

For any fixed $y = y_0$, $p(x_1, y_0) = p(x_2, y_0)$ and the above relation for p gives $a = 0$ and $\partial_x p = 0$. The equation governing the flow is thus:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1.41}$$

Subject to boundary conditions $u(y=0,t)=U$ and $u(y\to\infty)=0$. This is a well-known diffusion equation. To simplify the partial differential equation (PDE), we can transform it into an ordinary differential equation (ODE) by introducing a new variable η, so that $u=u(\eta)$

$$\eta = \frac{y}{\sqrt{\nu t}} \tag{1.42}$$

This gives:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\frac{\partial u}{\partial \eta}$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{\nu}{\nu t}\frac{\partial^2 u}{\partial \eta^2}$$

and

$$\frac{\partial^2 u}{\partial \eta^2} = -\frac{\eta}{2}\frac{\partial u}{\partial \eta} \tag{1.43}$$

Introducing $\psi = \frac{\partial u}{\partial \eta}$ transforms (16.43) into a simple equation:

$$\frac{\partial \psi}{\partial \eta} = -\frac{\eta}{2}\psi$$

with a solution:

$$\psi = Ae^{-\frac{\eta^2}{4}}$$

leading to

$$u = A\int_0^y e^{-\frac{\eta^2}{4}}d\eta + B$$

The boundary condition (16.37) gives $B=U$. To evaluate $u(y,t)$ in the limit $y\to\infty$, we have to calculate

$$u(y=\infty) = A\int_0^\infty e^{-\frac{\eta^2}{4}}d\eta + U = 0 \tag{1.44}$$

The integral is evaluated readily:

$$I = \int_0^\infty e^{-\frac{\eta^2}{4}}d\eta = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$$

giving $A = -\sqrt{\frac{1}{\pi}}U$. The result is then:

$$u = U(1-\sqrt{\frac{1}{\pi}}\int_0^\eta e^{-\frac{\xi^2}{4}}d\xi) = U[\int_\eta^\infty e^{-\frac{\xi^2}{4}}d\xi]\mathcal{H}(t) \equiv Uerfc(\frac{y}{\sqrt{\nu t}})\mathcal{H}(t) \tag{1.45}$$

where $\mathcal{H}(t)=1$ for $t\geq 0$ and is equal to $zero$ for $t<0$.

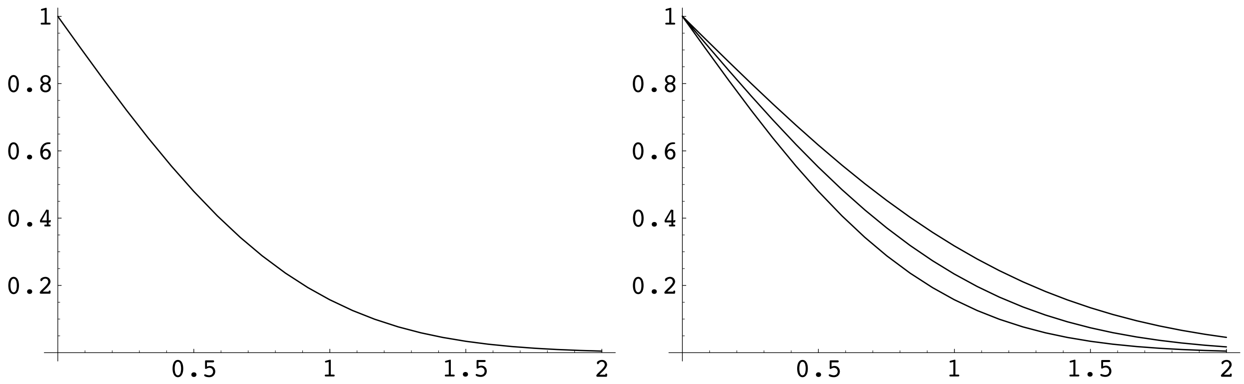


Figure 7: a. Velocity $u(\eta)/U$; b. $u(y,t)/U$ for different times $t = 1; 2; 4;$, respectively.

The curve presented on Fig. 47 is universal and solves the problem. For example, if we are interested in velocity $u(y,t)$ in an air flow ($\nu \approx 0.15cm^2/sec$) at a point $y = y_0 = 0.5cm$ at time $t_0 = 1sec$, all we have to do is: 1. form $\eta_0 = y_0/\sqrt{\nu t_0} \approx 1.25$. 2. From the Fig. 47a we find readily: $u(\eta_0) = u(\eta = 1.25) \approx 0.1U$. At the same point, but at a later time $t_1 = 10sec$, corresponding to $\eta_1 \approx 0.4$, the velocity is: $u \approx 0.7U$. This example illustrate the physics of the flow: the velocity field diffuses from from the source (plate at $y = 0$) into the fluid.

The method of solution, based on a "similarity variable" η, leading to transformation of a PDE into ODE is extremely general and is based on a deep physical insight and properties of space and time.

Qualitative analysis. As we see from the solution (16.45), the velocity distribution $u(y,t) \neq 0$ for any $t > 0$. This means that somehow the entire fluid ($0 \leq y \leq \infty$) is in motion. This unphysical result is an artifact of an incompressibility assumption used in the derivation. Indeed, any fluid perturbation, generated at $t = 0$ by a moving plate, propagates with the speed of sound which in an incompressible flow is infinite. This unphysical result is not too dangerous since, it is clear from Fig. 47, very far from the moving wall the velocity is negligibly small, provided $\eta \gg 1$. We can define width of the Stokes layer δ by a convention, for example $u(y = \delta(t)) = U/2$. Unlike the flows considered above where the channel width H or a pipe diameter D introduced length-scales of the problem, the equation of motion (16.41) defined on an infinite domain, does not contain fixed parameters T and L defining characteristic time and length-scales of the flow evolution. All we are given are the varying time t and viscosity $\nu = const$, which is the only constant of a problem The only physical quantity having dimensionality of a length-scale, which can be constructed from t and ν is

$$\delta \equiv \delta(t) = \gamma\sqrt{\nu t}$$

where γ is a dimensionless constant factor. In accord with our definition, the Figure 47 gives $\gamma \approx 0.53$.

Problem.1 . A butterfly is flying above an infinite plate on a constant hight $h = 5cm$. At $t = 0$ the plates starts moving with velocity $U = 10cm/sec$. How long will it take for a butterfly to "feel the wind" if it is sensitive to the air motion with velocity $u_{air} \geq 1cm/sec$? Find the outcome if $h = 3m$

Problem 2. At t=0, a tiny volume of die is issued at a point (y_0, x_0) above an infinite plate which, exactly at this moment, starts moving with velocity U. Find coordinates of a spot at an instant t. Describe the motion. If needed, use numerics. (**Hint**: analyze the problem analytically if 1. $y_0 \ll \sqrt{\nu t}$ and 2. $y_0 \gg \sqrt{\nu t}$).

Problem 3. What is the major difference between the flow generated by an infinite plate moving with velocity $U\mathbf{i}$, considered above, and that induced by a semi-infinite ($x \leq 0$) plate moving with the same velocity. (**Hint**. Analyze the derivation of equation (16.41)).

Stokes' second flow problem.

Now, let us investigate the flow generated by an infinite plate, described in a previous section but oscillating in time as:

$$u(0,t) = U \sin \omega t \tag{1.46}$$

Repeating considerations leading from (16.38) to (16.41), the problem is reduced to diffusion equation (16.41) subject to the time-dependent boundary condition (16.46) at $y = 0$ and $u = 0$ at $y \to \infty$.

Qualitative considerations. This problem resembles Stokes' previous one with a single, but enormous, difference: the time - scale $T \propto 1/\omega = const$. This means that now, out of viscosity ν and time T, we can construct a parameter δ having dimensionality of a length-scale:

$$\delta \propto \sqrt{\nu/\omega} \equiv \sqrt{\nu T}$$

This parameter can also be associated with the width of the Stokes layer but as we see, unlike in the previous example, now it is finite. This result can be readily understood: due to reversal of the plate speed, in this case, the Stokes layer has only finite time ($\approx T$) to develop. Thus, the above relation, obtained on dimensional grounds, can be derived from (16.32) by substituting $T = 1/\omega$ instead of t.

Since

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

we will be seeking the solution to (16.41), (16.46) as

$$u = \phi(y) Im\{e^{i\omega t}\}$$

Substituting this into (16.41) gives a simple equation:

$$\frac{\partial^2 \phi}{\partial y^2} = i\frac{\omega}{\nu}\phi(y) \tag{1.47}$$

Now, if $\phi(y) = Ae^{iky}$, we find from (16.47):

$$-k^2 = i\frac{\omega}{\nu} \tag{1.48}$$

and

$$k_{i,2} = \pm\sqrt{-i\frac{\omega}{\nu}} \tag{1.49}$$

Since $\sqrt{-i} = (1-i)/\sqrt{2}$, the solution to equation is:

$$u(y,t) = A \exp((1-i)\frac{y}{\delta}) + B \exp(-(1-i)\frac{y}{\delta}) \tag{1.50}$$

where the length-scale (penetration depth)

$$\delta = \sqrt{\frac{2\nu}{\omega}} \tag{1.51}$$

is one of the most important parameters of the theory. Far from the plate ($y \to \infty$, the velocity $u \to 0$. This is possible only is $A = 0$. The solution, satisfying the boundary condition at the wall is thus,

$$u(y,t) = UIm\{\exp(-(1-i)\frac{y}{\delta})\} \tag{1.52}$$

or

$$u(y) = Ue^{-\frac{y}{\delta}} \sin(\omega t - \frac{y}{\delta}) \tag{1.53}$$

This relation describes the so called damped wave of the wave-length $\lambda = \delta$, propagating in the direction perpendicular to the oscillating plate. Due to the exponential factor, this wave decays on the same length-scale δ, which can be called "penetration depth". This fact illustrates an important property of Newtonian fluids: *due to viscosity, no transverse shear waves can propagate there.*

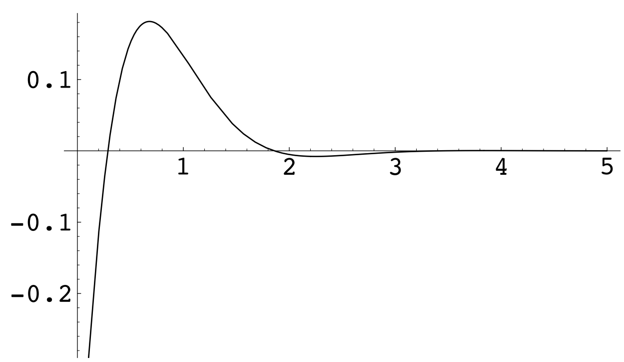


Figure 8: $u(y,t)$ from relation (16.53) for $\omega t = 10$ and $\delta = 0.5$.

It is clear from the Figure 48 that tte wave is damped on a scale $\delta \approx 0.5$. The force acting on a unit area of the plate is calculated readily:

$$\tau_{x,y} = -\mu \frac{\partial u(0,t)}{\partial y} = U\sqrt{\omega\mu\rho} \sin(\omega t + \pi/4) \tag{1.54}$$

The energy dissipated per unit time per unit area of the plate is equal to the mean of the product of τ_{xy} and velocity $u(t)$ evaluated over a cycle $0 \le t \le T = 1/\omega$:

$$-\overline{\tau_{xy}u(t)} = -\frac{1}{T}\int_0^T dt\tau_{xy}U\cos\omega t = \frac{U^2}{2}\sqrt{\frac{\omega\mu\rho}{2}} \tag{1.55}$$

This solution stresses the difference between the two Stokes' problems: in the first one, the time of the boundary layer development $0 < t < \infty$ while in the second flow this time if finite $T \approx 1/\omega$, thus leading to a finite-width boundary layer thickness.

Boundary Layers.

Let us consider a flow over a thin flat plate occupying a half-space $y = 0$ and $0 \leq x \leq \infty$. At $x < 0$, the velocity field is $\mathbf{u}(x, y) = U\mathbf{i}$. It is clear that some features of this idealized problem are present in various finite-size bodies (planes, cars, trains submarines etc) moving in a fluid. In an ideal flow ($\nu = 0$) the solid plate cannot modify the speed of an incoming fluid, so that $u(x, y, t) = U$. In a viscous flow, the situation is dramatically different.

Due to viscosity, on a solid plate $\mathbf{u} = 0$ and far from it ($y \to \infty$) $\mathbf{u} = \mathbf{U} = const$. Thus, there exist a fluid layer where velocity of a flow varies from zero at the plate to, for example, $u(y = \delta) \approx 0.99U$. This interval, dominated by viscous effects, is called boundary layer (BL) and the distance δ is called the width or thickness of the boundary layer. By this definition, in an interval $y > \delta$, the velocity varies very slowly ($0.99U \leq u < \infty$) and the viscous effects are negligibly small. Since at $x < 0$, $\mathbf{u}(y, x, t) = U\mathbf{i} = const$, very close to the inlet at $x = 0+$, the width of the BL is infinitesimally small. It grows with increase of the distance from the inlet and, in general $\delta = \delta(x)$.

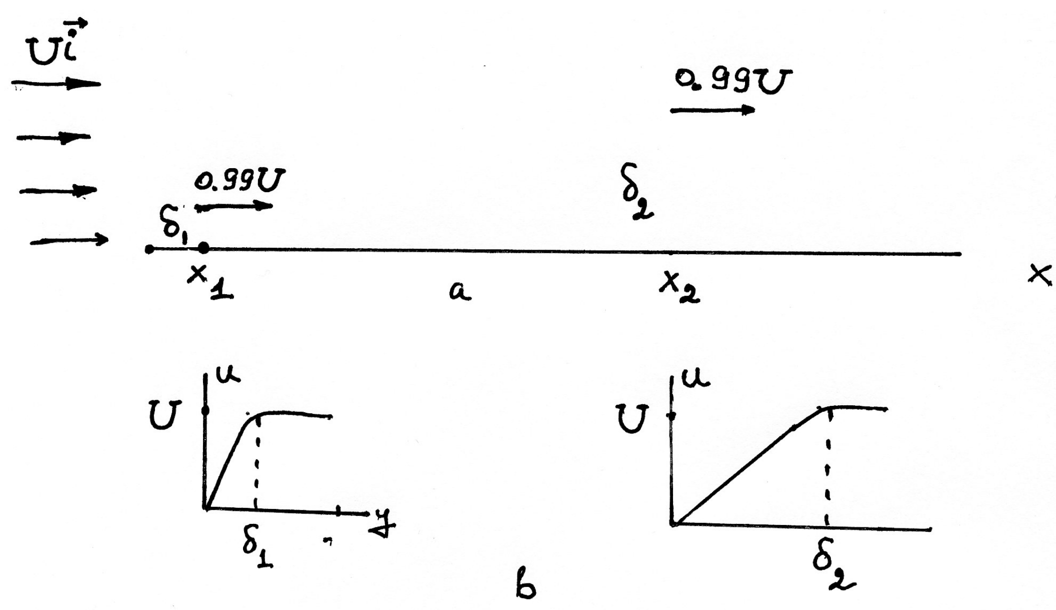


Figure 9: a. Development of the boundary layer in a laminar flow past flat plate. b. velocity distributions in two positions x_1 and x_2.

Studying two Stokes' problems, we discovered that once the plate starts moving with velocity U, it takes some time t for the viscous effects to "diffuse" from the plate at $y = 0$ to a point $y > 0$ in a fluid. This is a general property of viscosity (diffusion) -dominated processes. In a frame of reference moving with a free-stream velocity, the speed of a plate is $-U$ and we can apply the physical insight developed for the Stokes' first problem. It can be seen from Fig. 49 that the fluid element entering the plate (take this as $t = 0$) with velocity U at a distance $y \approx \delta$ from the plate will start "feeling" substantial viscous strain only after time $t \approx y^2/\nu$. In other words, after this time the velocity of this fluid element will "substantially" decrease, for example, by one percent. It is clear that this time $t \approx x/U$, which gives:

$$\delta \approx \sqrt{\frac{\nu x}{aU}} \tag{1.56}$$

It is important that (16.55) is the only flow property having dimensionality of length that can be constructed out of x, ν and U. The coefficient $a = const$ can be found only from detailed theory based on the Navier-Stokes equations. The BL thickness defines the distance from the plate where the viscous effects dominate dynamics and cannot be neglected. Outside the BL, the velocity can be taken as a free stream velocity: $u(y) \approx U$. We see from the relation (16.55) that as $x \to \infty$

$$\frac{\delta}{x} \to 0 \tag{1.57}$$

Mass conservation.

The flows considered in all previous sections were unidirectional i.e. with a single non-zero component of velocity field. This simplifying feature does not exist in Boundary layers. The fact that in this flow the v-component of velocity field is not equal to zero can be understood from the mass conservation law. Consider a control volume shown on the Figure 50.

In the boundary layer $0 \leq u \leq U$, is it clear that in the cross-section number 2, the mean velocity

$$\overline{u} = \frac{1}{\delta}\int_0^\delta u(y)dy \ < U$$

The change of mass in the control volume bounded by the surface 0- 1-2-3 is equal to the flux though the surface 1-2-3. Therefore, since the mass flux though a surface is equal to

$$\int \rho \mathbf{u} \cdot \mathbf{n} ds$$

by considering the figure, we write the total flux across the surface 1-2-3:

$$\dot{m} = \rho \int_0^\delta (u(y) - U)\ dy + \rho \int_0^x \rho \mathbf{u}(y) \cdot \mathbf{n}_3\ dy = 0 \tag{1.58}$$

If no sinks (chemical reactions, bubbles formation etc) present in a chosen control volume, the mass inside a control volume cannot change. Thus, since $\overline{u} - U < 0$, the vertical component perpendicular to the layer

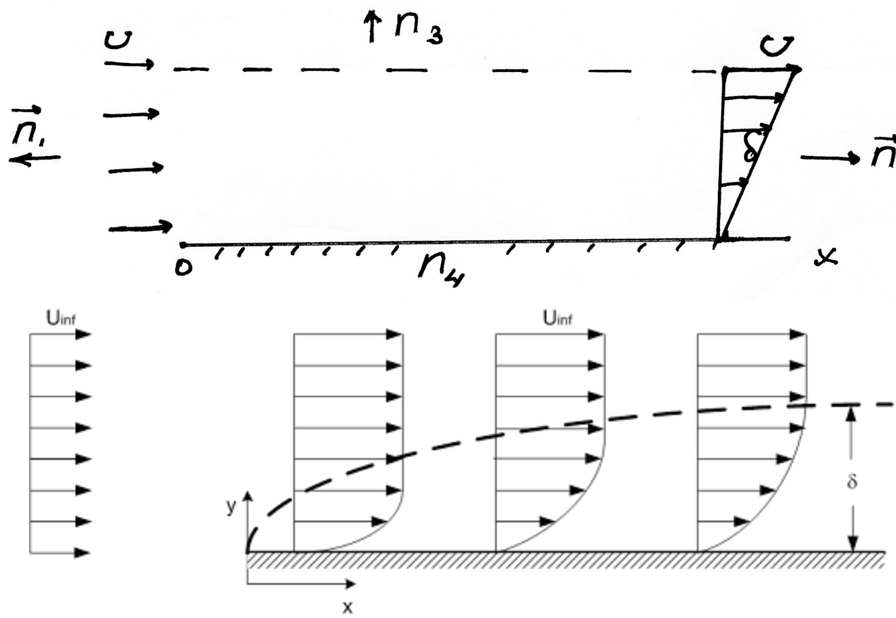
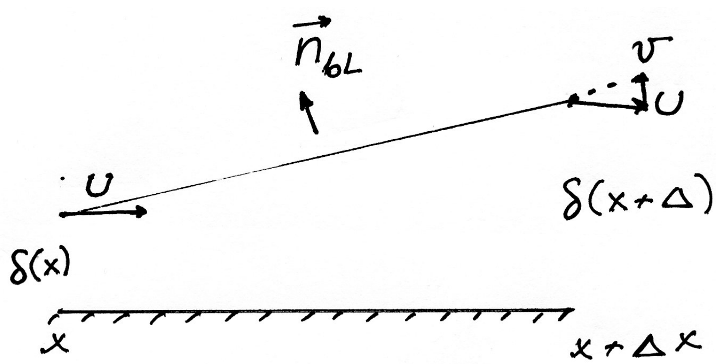


Figure 10: Top: Control volume. Bottom: Boundary layer thickness $\delta(x)$.

boundary is $v = \mathbf{u} \cdot \mathbf{n}_3 > 0$ which means the fluid is pushed away from the plate. To compensate for the loss, the ambient fluid is entrained into the BL. This process leads to broadening of the boundary layer with the distance x to the origin.



Let us consider a fraction of a boundary layer between points x and $x+\Delta x$ where Δ is small, i.e. $\Delta x \ll x$ bounded by a surface 1-2-3-4. (see Fig.). The surface 3 coincides with the shape of the boundary layer and $\mathbf{n_3} \equiv \mathbf{b_{bL}}$. The mass conservation tells us: if no sources or sinks of mass are present within the closed volume, then

$$\dot{m} = \oint \rho \mathbf{u} \cdot \mathbf{n} dS = -\int_0^{\delta(x)} \rho u dy + \int_0^{\delta(x+\Delta x)} \rho u dy + \int_x^{x+\Delta x} (\mathbf{u} \cdot \mathbf{n})_{bL} dx = 0 \tag{1.59}$$

At the edge of a boundary layer, by definition, the x-component of velocity field is $u = U$ where U is a free-stream velocity. However, in the growing boundary layer there must also be a vertical component $v \neq 0$, so that the total velocity is equal to $\mathbf{u} = U\mathbf{i} + v\mathbf{j}$ and (see Fig.):

$$\tan\alpha = \frac{v}{U} = \frac{d\delta(x)}{dx} \tag{1.60}$$

It is clear from the Figure that $\mathbf{n}_{bL} = \mathbf{j} - \mathbf{i}\tan\alpha = \mathbf{j} - \mathbf{i}\frac{d\delta(x)}{dx}$, so that

$$(\mathbf{u}\cdot\mathbf{n})_{bL} = v - U\frac{d\delta(x)}{dx} \tag{1.61}$$

$$\dot{m} = \oint \rho\mathbf{u}\cdot\mathbf{n}dS = -\int_0^{\delta(x)} \rho u dy + \int_0^{\delta(x+\Delta x)} \rho u dy + \int_x^{x+\Delta x} (v - U\frac{d\delta(x)}{dx})dx = 0 \tag{1.62}$$

Dividing this relation by Δx and using, valid in the limit $\Delta x \to 0$, identities $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{df}{dx}$ and $\frac{\int_x^{x+\Delta x} f(\lambda)d\lambda}{\Delta x} = f(x)$, we derive:

$$\frac{d}{dx}\int_0^{\delta(x)} \rho u(y)dy + \rho(v - U\frac{d\delta(x)}{dx}) = 0 \tag{1.63}$$

It is clear from this relation that the **entrainment** velocity $v - U\frac{d\delta(x)}{dx} < 0$. This mean that a fraction of a fluid moves into the boundary layer, thus leading to its broadening.

Momentum conservation..

According to the Navier-Stokes equations , momentum flux from a given control volume enveloped by a surface S is equal to the total force acting on it. Therefore, the balance for the x-component of momentum is:

$$\oint \rho u(\mathbf{u}\cdot\mathbf{n})dS = -\mathbf{i}\cdot\oint(p - p_\infty)\mathbf{n}dS + \mathbf{i}\oint \mathbf{n}\cdot[\tau]dS \tag{1.64}$$

Here $[\tau]$ stand for the shear stress tensor at the wall with components $\tau_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. Since, in the case we are interested in, $\mathbf{n}_4 = -\mathbf{j}$, $\mathbf{i}(\mathbf{n}\cdot[\tau]) = \mu\frac{\partial u}{\partial y}|_{y=0}$.

At the edge of a boundary layer $u(\mathbf{u}\cdot\mathbf{n})_{bL} = U(\mathbf{u}\cdot\mathbf{n})_{bL} = U(v - U\frac{d\delta(x)}{dx})$ and since in this case $p = p_\infty$, we can write:

$$\oint \rho\mathbf{u}\cdot\mathbf{n}dS = -\int_0^{\delta(x)} \rho u^2 dy + \int_0^{\delta(x+\Delta x)} \rho u^2 dy + \int_x^{x+\Delta x} U(v - U\frac{d\delta(x)}{dx})dx = \int_x^{x+\Delta x} \tau_{wall}(\lambda)d\lambda \tag{1.65}$$

Dividing this equation by $\Delta x \to 0$ and repeating considerations leading to (16.62), we obtain:

$$\frac{d}{dx}\int_0^{\delta(x)} \rho u^2(y)dy + \rho U(v - U\frac{d\delta(x)}{dx}) = -\tau_{wall} \tag{1.66}$$

The y-component v can be eliminated using the mass conservation law (16.62) giving:

$$\frac{d}{dx}\int_0^{\delta(x)} \rho u^2 dy - U\int_0^{\delta(x)} \rho u(y) dy = -\tau_{wall} \tag{1.67}$$

Dividing this expression by ρu^2 and recalling the definition of skin friction

$$c_f = \tau_{wall}/\frac{1}{2}\rho U^2 \tag{1.68}$$

we derive:

$$\frac{d}{dx}\int_0^{\delta(x)} \frac{u(y)}{U}(1-\frac{u(y)}{U})dy = \frac{c_f}{2} \tag{1.69}$$

The relations (16.62) and (16.68) define the force and mass balances in a boundary layer.

Qualitative derivation (estimate) of $\delta(x) \propto \sqrt{x}$.

Although the relations derived above are very useful, they do not provide us with any information on the x-dependence of the width of the laminar boundary layer and the velocity distribution $u(y)$. Of course, this can be done analytically only by solving the approximations based on the Navier-Stokes equations or using numerical simulations. However, very important conclusions can be obtained from simple qualitative considerations.

We know that $u(0) = 0$. Thus in the proximity to the wall by the Taylor expansion we have $u(y) \approx \frac{\partial u}{\partial y}|_0 y$. At the edge of a boundary layer $u(\delta(x)) = u_e$. Let us, as a first try, use the linear extrapolation in the interval $0 \leq y \leq \delta(x)$

$$u(y) = U\frac{y}{\delta(x)} \tag{1.70}$$

It is clear that this relation is a rather crude approximation to reality. Indeed, in the interval $y \geq \delta(x)$ the velocity distribution is $u(y) \approx U = const$ or $\frac{\partial u}{\partial y} = 0$ which is in a gross contradiction with the above assumption giving $\frac{\partial u}{\partial y} = u_e/\delta(x) \neq 0$. Still, it is interesting to analyze the consequences of a simplest possible model (16.69).

With the assumed velocity profile, the stress at the wall is

$$\tau_{wall} = \mu\frac{\partial u}{\partial y}|_{wall} = \mu U/\delta(x) = \frac{\rho}{2}U^2 c_f \tag{1.71}$$

Substituting (16.69) and (16.70) into (16.68) gives:

$$\frac{d\delta(x)}{dx} = 6\frac{\nu}{U\delta(x)} \tag{1.72}$$

and

$$\frac{d\delta(x)^2}{dx} = 12\frac{\nu}{U} \tag{1.73}$$

Solution to this equation is:

$$\delta(x) = \sqrt{12}\sqrt{\frac{\nu x}{U}} = \sqrt{12}\frac{x}{\sqrt{Re_x}} \tag{1.74}$$

where $Re_x = Ux/\nu$. This remarkable result, based on a simple qualitative dependence (16.69) gives an absolutely correct functional dependence $\delta \propto \sqrt{x}$, which can be used in various applications.

As a next example, let us consider the velocity distribution (guess) $u(y) = U \sin\frac{\pi y}{2\delta(x)}$. Since in this case $\frac{\partial u}{\partial y}|_{y=\delta} = 0$ this distribution gives a smooth transition from the boundary layer velocity profile to the free-stream distribution. In this case: $\tau_{wall} = \mu\frac{U\pi}{2\delta(x)}$ and $c_f = \frac{\pi\nu}{\delta(x)U}$. Substituting this into (16.68) we obtain after simple integration

$$\frac{4-\pi}{4\pi}\frac{d\delta(x)^2}{dx} = \frac{\pi\nu}{2U} \tag{1.75}$$

and

$$\delta(x) = \pi\sqrt{\frac{2}{4-\pi}}\sqrt{\frac{\nu x}{u_e}} \approx 4.8x/\sqrt{Re_x} \tag{1.76}$$

This relation is very close to the one $\delta(x) \approx 5x/\sqrt{Re_x}$, derived by Blausius (see below) from the Navier-Stokes equations in the boundary layer approximation.

Sensitivity to the details of velocity distribution.

The results obtained above are:

$$\delta(x) \approx 4.8x\sqrt{\nu/U}; \qquad c_f = \frac{\pi}{4.8}\sqrt{\frac{\nu}{Ux}} \tag{1.77}$$

which is to be compared with the "exact" relations:

$$\delta(x) \approx 5x\sqrt{\nu/u_e}; \qquad c_f = 0.664\sqrt{\frac{\nu}{u_e x}} \tag{1.78}$$

Problem. Consider a model delta-wing of the shape presented on the Fig. Find drag and drag coefficient. Choose a strip of the width dz. Using the relation (16.77):

$$dD = 2\int_0^c \tau_w\ dx = 2\ dz\int_0^c \frac{\rho U^2}{2}c_f\ dx = \rho U^2\ dz\int_0^c \frac{0.664}{\sqrt{Re_x}}\ dx = 1.328\mu\sqrt{Re_c}\ dz$$

where $Re_c = u_e c/\nu$. It is clear from the Figure that $c = L(1-\frac{2z}{L})$, and

$$D = 2\int_0^{\frac{L}{2}}\ dD = 2.656\mu U\sqrt{Re_L}\int_0^{\frac{L}{2}}\sqrt{(1-\frac{2z}{L}}\ dz$$

with $\xi = 2z/L$:

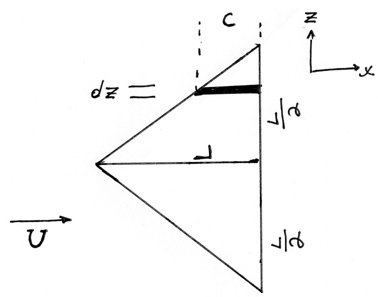


Figure 11: Laminar flow past triangular plate.

$$D = 1.328\mu U \sqrt{Re_L} L \int_0^1 \sqrt{(1-\xi)}\ d\xi = 0.885 \mu U L \sqrt{Re_L}$$

The area of the wing is $A = L^2/2$ and

$$c_D = \frac{D}{\frac{1}{4}\rho U^2 L^2} = \frac{3.54}{\sqrt{Re_L}}$$

This is the result based on exact solution (16.77). With the model (16.76), the derived drag coefficient is smaller by a factor 4.8/5. Taking $\rho \approx 1kg/m^3$, $\nu \approx 0.15 \times 10^{-4} m^2/sec$ and $L \approx 20m$ we derive for a plate cruising velocity $u_e \approx 700km/h$ the drag force $D \approx \frac{\rho}{2} u_e^2 c_f S \approx 800N$. This correpond to the pwer $Du_e \approx 158000N$ which is unreasonably small for the body of such large area traveling with such a speed. The reason for this result will be discussed below.

Prandtl equations for the boundary layers.

Now we show how (1.38) is obtained from the Navier-Stokes equations. The analysis is based on the order of magnitude comparison of various quantities. We will say that in the limit $x \to \infty$ a property $A(x) << B(x)$ if:

$$\lim_{x\to\infty} A(x)/B(x) \to 0$$

.

For example, if $G = a\delta(x) + bx$, then, based on the derived estimate we have $\delta(x)/x \propto x^{-\frac{1}{2}} \to 0$. Thus, in this limit δ can be neglected as small.

If the plate is long and infinitely wide, the flow is two-dimensional and the continuity equation is:

$$\frac{\partial u}{\partial x} \quad + \quad \frac{\partial v}{\partial y} = 0 \tag{1.79}$$

$$\frac{U}{x} \qquad\qquad \frac{V}{\delta} \tag{1.80}$$

We see from the above estimate that the continuity equation can be valid only if $\frac{U}{x} \approx \frac{v}{\delta}$. This gives:

$$v \approx \frac{\delta}{x} U \tag{1.81}$$

It follows from the qualitative energy balance considered above that $v << U$. The momentum equation for the x-component:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{1.82}$$

$$\frac{U^2}{x} \qquad \frac{U^2}{x} \qquad \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \nu(\frac{U}{x^2} + \quad \frac{U}{\delta^2}) \tag{1.83}$$

Based on the relations derived above, we neglect $\nu\frac{\partial^2 u}{\partial x^2}$ as small. Using the relation (43), the terms in equation for the v-component can be estimated as follows:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{1.84}$$

$$\frac{U^2\delta}{x^2} \qquad \frac{U^2\delta}{x^2} \qquad \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \nu(\frac{U\delta}{x^3} + \quad \frac{U}{\delta x}) \tag{1.85}$$

Since $p \approx \rho U^2/2$ and the remaining contributions to (16.85) are small, the equation (16.84) can be balanced only if:

$$\frac{\partial p}{\partial y} = 0 \tag{1.86}$$

which gives $p = p(x)$. This result is very important and useful for aerodynamic applications. By definition, at the distance from the plate $y = \delta$, the velocity $u(y)$ is basically equal to that in a free stream, i.e. $u(\delta) \approx U$. The formula (16.86) tells us that for a fixed value of coordinate $x = X$, for $0 \leq y \leq \delta$, $p(X, y) = p(y = \delta) = const$. However, at $y = \delta$, the viscous effects disappear and potential (inviscid) flow approximation becomes accurate. *Therefore, the pressure distribution within the boundary layer can be obtained from the well-understood and simple potential flow theory.* This result can be used as an input to the full theory based on the Navier-Stokes equations leading to accurate evaluation of velocity and the forces in the BL.

Based on these order-of-magnitude estimates the equations can be simplified:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.87}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{1.88}$$

The boundary conditions;

$$u(y=0) = v(y=0) = 0; \quad u(y \to \infty) \to U; \quad v(y \to \infty) \to 0; \tag{1.89}$$

It follows from the estimate (16.82), (16.83) that non-linear and viscous contributions to the equations of motion are of the same order if $\nu U/\delta^2 \approx U^2/x$. This immediately gives:

$$\delta \propto \sqrt{\frac{\nu x}{U}} \tag{1.90}$$

derived above from the energy balance. To obtain the proportionality coefficient in this important relation, one has to solve the equations (16.87)-(16.88).

These are)are called Prandtl's equations for boundary layer. Some regard these equations as a turning point in mathematics which led to the rise of applied mathematics as a separate branch of knowledge. The major contribution of Prandtl's discovery to our understanding of nature is in the fact that even in the limit of very small viscosity $\nu \to 0$, there exist a region of the flow $(0 < y < \delta)$, where viscous effect dominate dynamics. Similar phenomena have been later been observed in various branches of physics, chemistry and mathematics.

Blausius solution.

As we saw before, the streamfunction representation is very useful for solving two-dimensional incompressible hydrodynamic equations. Thus, let us introduce $\psi(x, y)$, so that $u = \partial_y \psi$ and $v = -\partial_x \psi$, and rewrite (16.87)-(16.88) as:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3} \tag{1.91}$$

Introducing the streamfunction simplified the problem: instead of solving the system of two partial differential equations (16.70), we have to deal with a single equation (16.91). Still, we have to solve a strongly non-linear equation of two variables x and y. The next simplification comes from the assumption that within the boundary layer, the two-dimensional velocity distribution $u(x, y)$ is self-similar, i.e. written in coordinates $u(\eta)$ where $0 \leq \eta = \frac{y}{\delta(x)} \leq 1$, all curves corresponding to different values of x collapse, i.e. are the same.
Defining $\eta = y\sqrt{y\frac{U}{2\nu x}}$ we seek solution as:

$$\psi(x, y) = \sqrt{2U\nu x}G(\eta) \tag{1.92}$$

Keeping in mind that

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = \sqrt{\frac{U\nu}{2x}}[G(\eta) - \eta G'(\eta)] \tag{1.93}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = UG'(\eta) \tag{1.94}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{U}{2x}\eta G''(\eta) \tag{1.95}$$

Simple calculation **pls do it !** gives:

$$G''' + G''G = 0 \tag{1.96}$$

This equation is to be solved subject to boundary conditions easily obtained from (16.89):

$$G(\eta = 0) = G'(\eta = 0) = 0; \qquad G'(y \to \infty) \to 1 \tag{1.97}$$

Numerical solutions $u(\eta)$ and $v(\eta)$ are presented on Fig. The numerical solution gives

$$\delta = 5.0\sqrt{\frac{\nu x}{U}} = \frac{5.0x}{\sqrt{Re_x}} \tag{1.98}$$

where $Re_x = Ux/\nu$. The stress at the wall is

$$\tau_{wall} = \mu\frac{\partial u}{\partial y}|_{wall} = \frac{\mu U G''(0)}{\sqrt{\frac{2\nu x}{U}}} = 0.332\frac{\rho U^2}{\sqrt{Re_x}} \tag{1.99}$$

and the drag per unit span of the plate of length L is:

$$D = \int_0^L \tau_{wall}\ dx = 0.664\frac{\rho U^2 L}{Re_L} \tag{1.100}$$

The drag coefficient:

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \frac{1.328}{\sqrt{Re_x}} \tag{1.101}$$

The entire velocity field is found from these curves as follows:

Problem. A flow with velocity $\mathbf{U} = U\mathbf{i}$ enters a channel of half-width H. Estimate the length L of a developing flow such that at $x > L$, the flow can be considered as developed.

Solution: Close to the inlet, two boundary layers are formed. At the distance L these BLs overlap forming developed velocity profile. Thus, for the estimate we can take: $\delta \approx H = 5.0\sqrt{\nu L/U}$. Thus: $L > \frac{H^2 U}{25\nu}$ and $\frac{L}{H} > \frac{HU}{25\nu} = Re_H/25$. For $Re_H \approx 500$, we have $\frac{L}{H} \approx 20$.

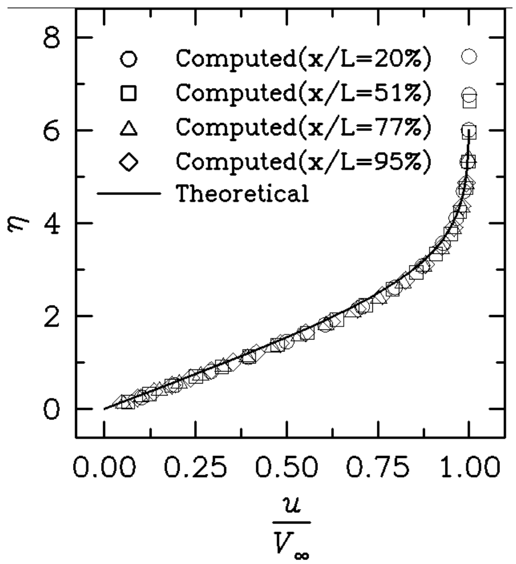


Figure 12: Universal velocity profile in laminar boundary layer. $\frac{u}{U} = G'(\eta)$ vs η. L is a length of a plate in numerical simulations.

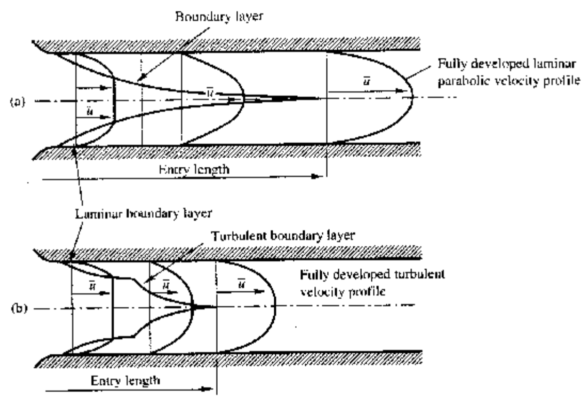


Figure 13: Developing flow in a channel.

This estimate is a subject of an active discussions in the field. Earlier experiments, conducted in 1930's by Nikuradze in Germany in a pipe with $L/H \approx 10$, *were treated with suspicion by the US and European researchers. To resolve the dispute, a remarkable facility, Princeton Superpipe with* $\frac{L}{H} > 30$, *was designed at Princeton University. Interestingly, the obtained data on the friction factor in the range of Reynolds number variation* $10^4 \leq Re \leq 10^7$ *were very close to Nikuradze's. In addition to friction factor, "Superpipe" enabled one to investigate various fine features of velocity distributions in a wide range of Reynolds number variation.*

2 Turbulence.

The velocity distributions for the laminar viscous flows, derived above, are so accurate that they are used in viscometry for an extremely precise determination of viscosity of a fluid. For example, take a capillary of radius R and measure a flux of a fluid driven by a given pressure gradient of gravity field. The flux is found from a time needed to fill a given volume V. It is clear from relation (16.30) that $\nu = \frac{\pi g \sin \alpha R^4}{8Q}$, where α is the angle between the tube and horizontal axis. One can measure the tube parameters, water flux and α very accurately, which immediately leads to the precise determination of viscosity ν.

However, the laminar flow approximation leads to some manifestly incorrect conclusions. Let us consider a thin an very light sheet of metal of mass $M = 20gr$, length and width $L \approx w = 100cm$ and drop it vertically under the action of gravity. The friction force in the developed boundary layer acting on a falling sheet is the only force acting on a body and in a steady state:

$$Mg = \frac{0.664\rho U^2 LW}{\sqrt{Re_L}}$$

Simple calculation gives for terminal velocity $U \approx 85m/sec$ corresponding to $Re_L \approx 6 \times 10^6$. The obtained speed is unreasonably high for such a light body.

To explain this any many other paradoxes we have to consider the Problem of Turbulence, which is one of the most ubiquitous phenomena in Nature and to this day remains one of the last unsolved problems of continuous mechanics. It was described by Leonardo da Vinci 500 years ago, who watching a water flow passing a stone in a creek, wrote in his notebooks that first the phenomenon is born close to the stone, develops slightly behind it and decays (dies) downstream. Being an artist of uncomparable quality, he sketched the entire process and coined the term "turbulence". Since then, it was observed in rivers, atmosphere, stars, bloodstream, various machines and devices.

Transition. First precise investigation of the onset (transition to) of turbulence was performed by O. Reynolds in 1883 (see Fig.54).

In his classic experiment, Reynolds investigated propagation of dye issued from a narrow capillary into a water flow in a pipe. Since water is a very good solvent, the dye trajectory is close to the flow streamlines. At $Re_D < 2300$ the dye, moving with the flow velocity U, formed a narrow line without any signs of broadening. This can be easily understood since the molecular diffusion - dominated law of broadening gives for the time-dependent width of a dye -stream $\delta \approx \sqrt{\kappa t}$ where the diffusivity $\kappa \approx 10^{-6} - 10^{-8} cm^2/sec$. The diameter of the tube in Reynolds experiments was $D \approx 10cm$ and its length $L \approx 1m$. For $Re_D \approx 1000 - 2000$, the mean velocity $U \approx 1-2cm/sec$ giving for a time, the fluid element stays in the flow $T \approx 50-100sec$ corresponding to $\delta \approx 10^{-2} - 10^{-4}cm$, which is impossible to detect by a naked eye.

The picture changed dramatically when the Reynolds number was brought to $Re_D > 2300$ (see Fig.54b). The trace of dye became chaotic and filling the entire tube. It became clear to Reynolds that this effect was

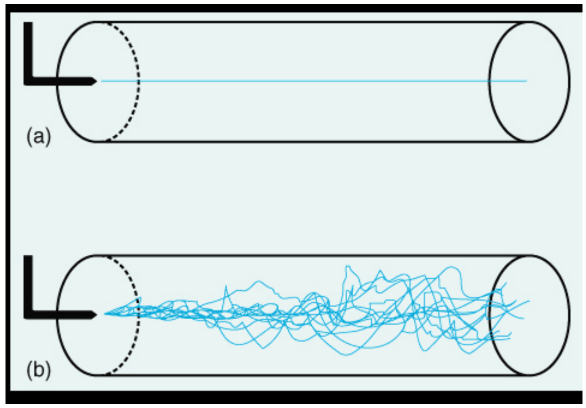


Figure 14: Evolution of dye issued from a narrow capillary into a pipe flow. a: $Re_D < 2300$. b: $Re_D > 2300$. Reynolds 1883.

impossible without generation of u_r, component of velocity field perpendicular to the mean velocity of the flow. The effect of Reynolds number is demonstrated on the next Figure where the left plate shows a flow of Re just above the transition. A slight but clear deviation of the dye trajectory from a strait line is seen. A flow of a higher Reynolds number is shown on the right plate: the dye trace is much more chaotic and the mixing with water is much stronger.

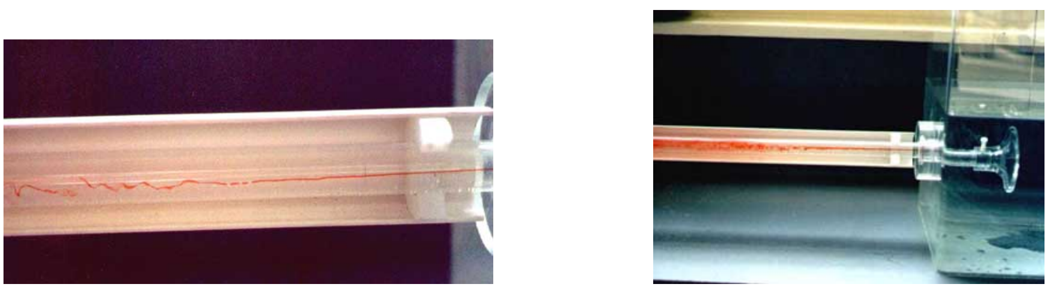


Figure 15: Evolution of dye issued from a narrow capillary into a pipe flow. Left: Re_1 is larger but close to $Re_{tr} \approx 2300$. Right: $Re > Re_1$. Reynolds 1883.

Transition to turbulence is cased by an instability of a high Reynolds number flow to small perturbation. If at $t = 0$, a velocity distribution $\mathbf{U}(\mathbf{x})$ in a stable laminar flow is perturbed by an infinitesimally small perturbation $\mathbf{u}'$ so that $U(\mathbf{x}, 0) + \mathbf{u}'$, due to the viscous effects, the perturbation the state $\mathbf{u} \to \mathbf{U}$. It is also clear that if at $t = 0$ two particles in a laminar flow are separated by a distance $r \to 0$, their trajectories stay close during the entire evolution. In a turbulent flow these trajectories depart exponentially fast and the flow rapidly "forgets" its initial condition. Moreover, the flow never returns to the initial velocity distribution $\mathbf{U}(\mathbf{x})$.

Turbulence Problem. If we measure velocity field at a point in a steady or regular oscillating laminar flow, the obtained signal will be either ($bfu = \mathbf{U} = const$ or $u = \cos\omega t$. I turbulent flows this signal will look differently.

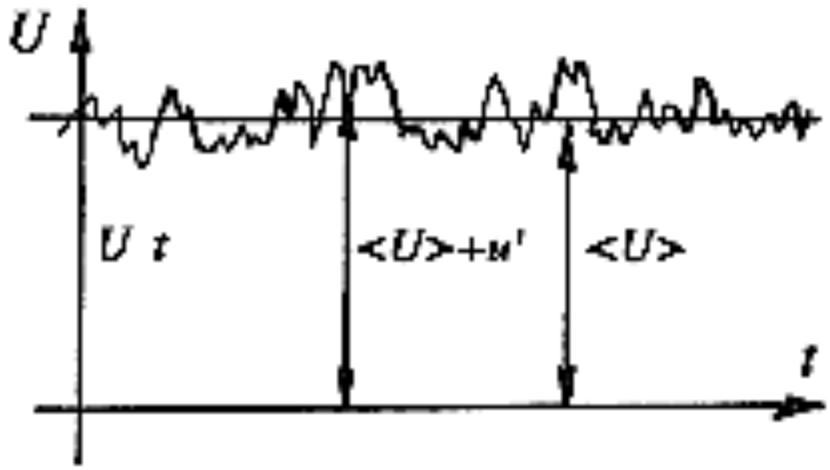


Figure 16: Velocity as a function of time. Dat from Princeton Superpipe (A. Smits.)

We can see that the flow velocity in a pipe can be represented as a sum of a regular or mean $\overline{\mathbf{u}} = \mathbf{U}$ and a random component $\mathbf{u'}$ with the mean value $\overline{\mathbf{u'}} = 0$. Accepting a chaotic concept of turbulence, Reynolds suggested the velocity decomposition $\mathbf{u} = \mathbf{U} + \mathbf{u'}$. This decomposition is simple and accurate in a statistically steady (time-independent $\mathbf{U}$) flow. This can be seen as follows: by definition of the mean

$$\overline{\mathbf{u}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\mathbf{U} + \mathbf{u}')dt = \mathbf{U} \tag{2.1}$$

However, if, for example, $\mathbf{U} = \mathbf{U}_0 \sin\omega t$, this expression gives $\overline{\mathbf{U}} = 0$ leading to the loss of very important information about, for example, periodic vortex shedding behind bluff bodies. Thus, it is desirable to average over time $T \approx 1/\omega$ and if the "random component" varies on a time scale $\tau << 1/\omega$, the averaging, strictly speaking, will be less accurate but the information about time-dependence of the "mean flow will be preserved.

In case of statistically steady flow,

$$\overline{Uu'} = 0; \qquad \overline{(u')^2} > 0; \qquad \overline{u_i'u_j'} = \frac{u'^2}{3}\delta_{ij} \tag{2.2}$$

Using these relations we see that

$$\overline{(\mathbf{u})}^2 = \overline{(\mathbf{U} + \mathbf{u}')}^2 = U^2 + \overline{(u')^2} \tag{2.3}$$

and that the fraction of kinetic energy of a flow is lost into velocity fluctuations not contributing to the mass flux through the pipe. These losses are often huge and one of the principle goals of turbulence theory is to evaluate the mean properties of the flow.

With the Reynolds decomposition the equation for the mean velocity is:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + \overline{\mathbf{u'} \cdot \nabla \mathbf{u'}} = -\frac{\nabla p}{\rho} + \nu \nabla^2 U \tag{2.4}$$

On the first glance this equation looks like the Navier-Stokes equation for the mean velocity $\mathbf{U}$. However, the contribution from the Reynolds stress $\tau = \overline{\mathbf{u'} \cdot \nabla \mathbf{u'}}$ representing the forces due to turbulent component of velocity field, defining the "Turbulence Problem" is very hard to evaluate.

Turbulent Boundary Layers. Physical and numerical experiments on a flow past flat plate, considered above, show an excellent agreement with the Prandtl solution in the range of the distance to the leading edge x, such that $0 \leq Re_x \leq 2 - 3 \times 10^5$. After that the BL becomes irregular with a typical width growing much fast than familiar $\delta \approx 5x/Re_x$. (See Figure).

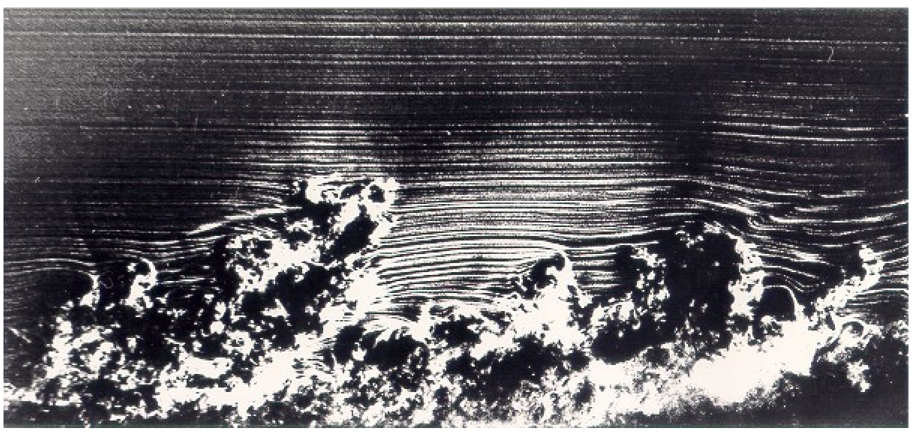


Figure 17: Structure of turbulent boundary layer.

As we see from the Fifure, the single snapshot of the irregular streamlines in turbulent boundary layer barely reveals any coherent velocity profile, BL width and anything else we studied in a laminar case. The photograph taken at a different instant of time will look quite differently. However, making many snapshots, superimposing them and dividing by a number of snapshots, produces an averaged picture of a mean low in a BL in a flow past an airfoil shown on a Fig. 58 below.

In general, the boundary layer consists of a few distinct regions. First, the laminar BL at the leading edge of a plate (body). After a transition point corresponding to $Re_x \approx 2 - 3 \times 10^5$ the BL becomes turbulent and occupies the interval $0 \leq y \leq \delta(x)$. Outside, $y \geq \delta(x)$ the flow velocity is $u(y, x) \approx U \equiv u_\infty$. The structure of BL is not that simple. Very close to the plate, in the interval called "viscous sublayer", the flow is characterized by an interplay of viscous and inertial effects. It is the instability of viscous sublayer that leads to turbulence generation. The structures formed there as powerful bursts are emitted into the outer layer generating irregular pattern which is a mixture of turbulent flow and quasilaminar patches of entrained fluid from the outside. The mathematically rigorous theory of this phenomenon does not exist and below

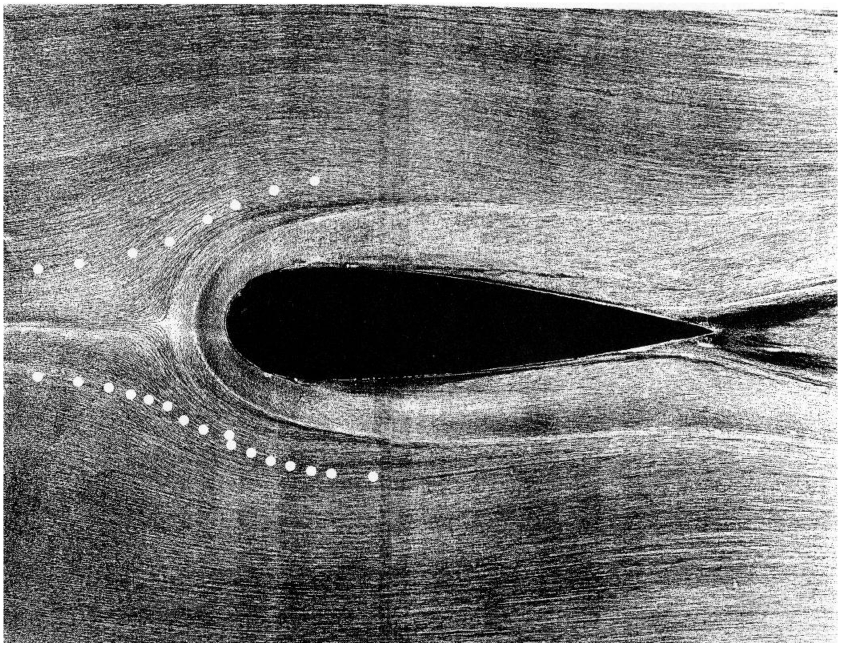


Figure 18: Averaged Boundary Layer in a Flow past Airfoil.

we present a semiqualitative representation of this important flow.

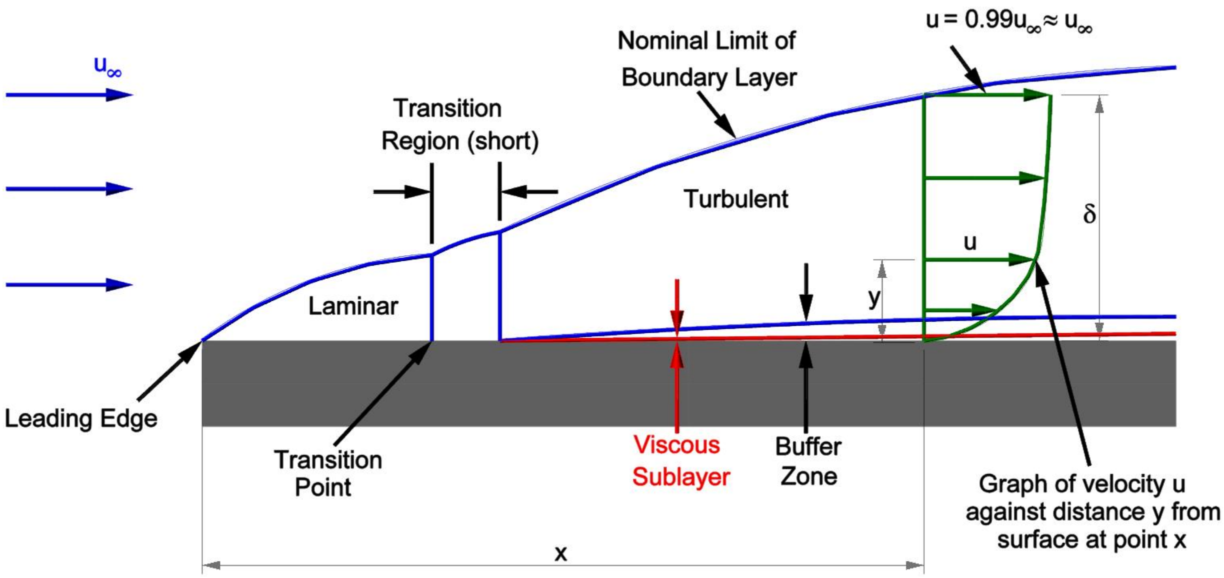


Figure 19: Structure of turbulent boundary layer.

In accord with Kelvin theorem if vorticity $\omega = 0$ on any point of a streamline, it is zero at all points on this streamline. This is so, provided we are not dealing with dividing streamlines on a surface of a solid body where vorticity can be generated. A turbulent flow is vortical, i.e. full of closed streamlines effectively mixing fluid elements from different layers of a flow. This is schematically represented on the Fig.58.

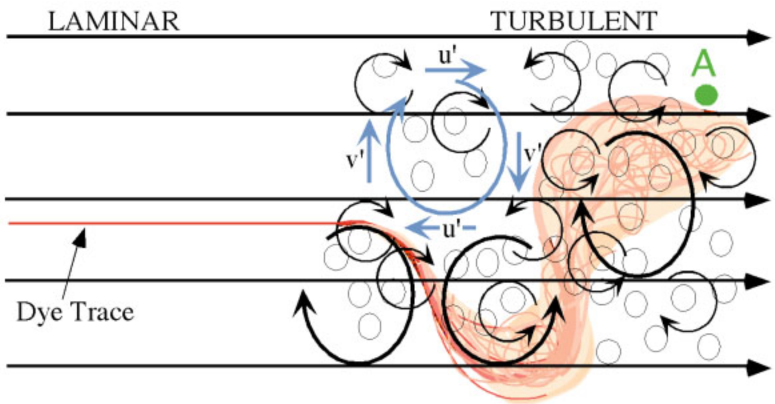


Figure 20: Structures (eddies) in turbulent boundary layer. Schematic.

Mixing length hypothesis.

According to kinetic theory of gases, if two fluid layers, separated by a gap λ, move with different velocities $U(y+\lambda)$ and $U(y)$, then the force per unit surface (shear stress) $\tau = \mu\frac{\partial U}{\partial y}$ where kinematic viscosity $\nu = \frac{\mu}{\rho} \approx v_{rms}\lambda \approx c_s\lambda$ and c_s is a speed of sound. The physics leading to viscous forces was described in detail in Chapter 1. In 1920's, L. Prandtl suggested treating vortices in turbulent flows ("eddies") as hypothetical "molecules of turbulence" leading to efficient mixing. From this qualitative picture, by analogy with kinetic theory, Prandtl assumed existence of a shear force originating from Reynolds stress. To follow his line of reasoning, consider a vortex of a radius l and angular velocity $\Omega \approx \frac{\partial u'}{\partial y}$ (see Figure. 60). Since velocity u' and Ω are fluctuating properties, it is clear that in general there exist an entire spectrum of Ω's and u''s. Analyzing flow structures Prandl introduced a concept of mixing length $l_{mix} = \bar{l}$ and assumed that $\overline{\Omega} \approx \frac{\partial U}{\partial y}$. This led to the celebrated Prandtl mixing length hypothesis:

$$\tau_t \equiv -\rho\overline{u'v'} = \frac{\rho}{2}\rho u_{mix} l_{mix} |\frac{dU}{dy}| \tag{2.5}$$

$$u_{mix} = C l_{mix}\frac{dU}{dy} \tag{2.6}$$

thus

$$\tau_t = \mu_t \frac{dU}{dy} \tag{2.7}$$

$$\mu_t = \rho l_{mix}^2 |\frac{dU}{dy}| \tag{2.8}$$

The equations for a BL flow are:

$$\partial_x u + \partial_y v = 0 \tag{2.9}$$

$$\partial_t U + U\partial_x U + V\partial_y U = -\frac{\partial_x p}{\rho} + \partial_y(\nu\partial_y U + \frac{\tau_t}{\rho}) \tag{2.10}$$

Unlike laminar flows, turbulent flows involving ever-changing velocity fluctuations, is always time - dependent. However, in some cases $\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x}) + \mathbf{u}'(\mathbf{x},t)$, so that while u is time-dependent, the mean flow (U) can be steady, i.e. not varying in time. This class of flows is called statistically steady. Consider a simple statistically steady flow in a channel ($0 \le y \le 2H$) or a boundary layer flow. By the virtue of no-slip boundary condition, $\tau_t = 0$ on a wall. Integrating (17.10) in an interval $y = 0$ to $y = H$ in a channel or $0 \le y \le \delta$ in Bl, we obtain:

$$|\frac{\partial p}{\partial x}|H = -\nu\frac{\partial U}{\partial u}|_{wall} = u_*^2 \equiv u_\tau^2 \tag{2.11}$$

where u_* is called friction velocity. Now we can introduce skin friction in a fully developed turbulent channel flow:

$$f = \frac{\tau_{wall}}{\frac{\rho\overline{U}^2}{2}} = \frac{u_*^2}{\overline{U}^2} \tag{2.12}$$

The equation for a channel flow can be rewritten in dimensionless variables

$$U_+ = U/u_* \qquad\qquad y_+ = yu_*/\nu \tag{2.13}$$

as:

$$\frac{\partial_x p}{\rho} = \frac{u_*}{\nu}\frac{\partial}{\partial y_+}(u_*^2\frac{\partial U_+}{\partial y_+} + \frac{\tau_t}{\rho u_*^2}) \tag{2.14}$$

$$\frac{1}{R_*} = \frac{d}{dy_+}(1 + \frac{\tau_t}{\rho u_*^2})\frac{dU_+}{dy_+} \tag{2.15}$$

where $R_* = u_*H/\nu$. This equation is to be solved subject to boundary conditions: $U_+(0) = 0$ and $\partial_y U_+(R_*) = 0$. In this coordinates half - width of the channel H is equal to $H_+ = \frac{u_*H}{\nu} = R_*$ and $0 \le y_+ \le R_*$.

Combining (17.8) and (17.15) gives:

$$\frac{1}{R_*} = \frac{d}{dy_+}(1 + l_{+,mix}^2|\frac{dU_+}{dy_+}|)\frac{dU_+}{dy_+} \tag{2.16}$$

Below we neglect subscript + in $l_{+,mix}$. In the case of fully developed channel/pipe flows this can be readily rewritten in a simple form:

$$1-\frac{y_+}{R_*}=(1+l_{mix}^2\frac{\partial U_+}{\partial y_+})\frac{\partial U_+}{\partial y_+}$$

valid in a part of a flow where $\partial_{y_+}U_+ \geq 0$. Solving quadratic equation we obtain:

$$\frac{\partial U_+}{\partial y_+}=\frac{1}{2l_{mix}^2}[-1+\sqrt{1+4(1-\frac{y_+}{R_*})l_{mix^2}}] \tag{2.17}$$

Now we need an expression for l_{mix}. Theoretical determination of mixing length is a difficult problem, depending upon flow geometry. For wall flows, in the interval $y+ \ll R_*$, Prandl, suggested

$$l_{mix,+}=\kappa y_+ \tag{2.18}$$

so that:

$$\frac{1}{R_*}=\frac{d}{dy_+}(1+\kappa^2 y_+^2|\frac{dU_+}{dy_+}|)\frac{dU_+}{dy_+} \tag{2.19}$$

This equation defines three layers characterizing an arbitrary wall flow. 1. Viscous sublayer $y_+ \ll \frac{1}{\kappa}\frac{1}{\frac{\partial U_+}{\partial y_+}}$ where:

$$\frac{1}{R_*}\approx\frac{d^2U_+}{dy_+^2} \tag{2.20}$$

and

$$U_+=\frac{y_+^2+2Cy_+ + D}{2R_*} \tag{2.21}$$

Close to the wall where $y_+ \to 0$, by the Taylor expansion $U_+=\frac{dU_+}{dy_+}y_+$ giving: $D=0$ and $C=R_*\frac{dU_+}{dy_+}$. Also, it follows from the definition of u_* (17.11), that $\frac{dU_+}{dy_+}|_{wall}=1$ giving for the viscous sublayer $U_+=y_+$.

2. Universal Log-layer. At $R_* \gg y_+ \gg \frac{1}{\kappa}\frac{1}{\frac{\partial U_+}{\partial y_+}}=\frac{1}{\kappa}$:

$$\frac{1}{R_*}=\frac{d}{dy_+}\kappa^2 y_+^2|\frac{dU_+}{dy_+}|\frac{dU_+}{dy_+} \tag{2.22}$$

and

$$U_+\approx B+\frac{1}{\kappa}\ln y_+ \tag{2.23}$$

3. Outerlayer $y_+ \approx R_*$.

Accurate measurements of velocity distribution in boundary layers, pipes and channel flows showed a rather good quantitative agreement of this description with experimental and numerical data. Still, some unsloved questions remain. Using these or similar relations we can evaluate the mass flux, stress on the wall, energy of the flow etc. This gives for the flow past flat plate:

$$\delta = 0.37x/Re^{0.2}$$

$$c_f = 0.0576 Re_x^{-0.2}$$

van Driest factor. This model was proposed to more accurately account for the viscous sublayer $y_+ \to 0$. In the wall coordinates it is:

$$l_{mix} = \kappa y_+[1 - e^{-y_+/A}] \tag{2.24}$$

where $A \approx 26$.

Numerical Project. 1. Consider fully developed turbulent flow in a channel (pipe). Using two models (17.16) and (17.22) calculate curve $f = f(Re)$ where $Re = \frac{\overline{U}2H}{\nu}$. Compare results. 2. Repeat the same for turbulent boundary layer.

2.1 Free Shear Flows. Wakes, mixing layers, jets.

Now we consider application of the mixing length concept to free shear flows. If the linear dimension of the body which is the source of the wake or a nozzle is L, we are interested the far downstream where $g \gg L$ and the geometric details of the flow generating mechanisms are unimportant. There the flows are assumed **self-similar**, i.e. can be written as

$$U(x, y) = u(x)F(\frac{y}{\delta(x)}) \tag{2.25}$$

where $2\delta(x)$ is the width of the wake (jet etc...). The self-similarity in general means that the y-dependence of a property, in this case velocity field, can be represented as

$$\frac{U(x, y)}{u(x)} = F(\eta)$$

and in each cross-section the velocity dependence upon y-coordinate is represented as a ratio y/δ. Therefore, the velocity distribution in cross–sections x_1 and x_2 can be obtained by a simple rescaling by length-scale $\delta(x)$.

Far wake behind the cylinder.

We assume that the Reynolds number is very large so that $\nu_0 \ll \nu_T$.

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$
$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\nu_T\frac{\partial U}{\partial y} \tag{2.26}$$

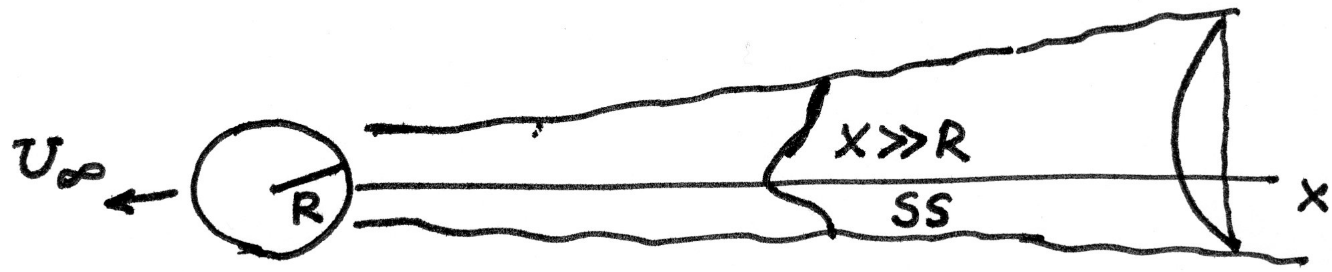


Figure 21: Far turbulent wake behind the cylinder.

These equations can be linearized: writing $\mathbf{U} = \mathbf{U}_\infty + \mathbf{u'}$ and assuming that $U_\infty \gg u'$ we can use continuity equation (17.26) to obtain an estimate $u' \approx V \ll U_\infty$. Then, neglecting $V\partial_y \mathbf{u'} = O((u')^2)$ compared with $U\partial_x \mathbf{u'} = O(u')$, we derive a simplified equation of motion:

$$U_\infty \frac{\partial u'}{\partial x} = \frac{\partial}{\partial y}(l_{mix}^2 |\frac{\partial u'}{\partial y}| \frac{\partial u'}{\partial y}) \tag{2.27}$$

which is to be solved subject to the boundary conditions:

$$u'(x,y) \to 0; \qquad\qquad \partial_y u' = 0 \tag{2.28}$$

as $y \to \infty$ and $y \to 0$, respectively. According to the drag theorem, the drag on a body can be found from momentum theorem: take control volume including a body and obtain

$$H \int_0^\infty U(U_\infty - U)dy = D/2 \tag{2.29}$$

where the factor 1/2 accounts for the integral taken over half-space. For a part of the wake where $\partial_y U > 0$ the equation (17.27) reads:

$$\frac{U_\infty \delta(x) u'(x)}{u(x)^2} F(\eta) - \frac{U_\infty \delta'(x)}{u(x)} \eta \frac{\partial F}{\partial \eta} = \alpha^2 \frac{\partial}{\partial \eta}(|\frac{\partial F}{\partial \eta}| \frac{\partial F}{\partial \eta}) \tag{2.30}$$

As $\eta \to 0$, $F'(\eta) \to 0$ and $F(\eta) \to 0$ for $\eta \to \infty$ following from the boundary conditions (17.28). The global constraint (17.29) is transformed to:

$$\int_0^\infty F(\eta) d\eta = \frac{D}{2\rho U_\infty u(x) \delta(x)} \tag{2.31}$$

The flow is self-similar only if equation (17.30) involves only the η-variable. Dependence upon x must disappear. This happens only when:

$$\frac{U_\infty \delta(x) u'(x)}{u(x)^2} = a_1; \qquad \frac{U_\infty \delta'(x)}{u(x)} = a_2; \qquad \frac{D}{2\rho U_\infty u(x)\delta(x)} = 1 \tag{2.32}$$

o with as yet unknown constants a_1 and a_2. It follows from the last equation (17.32) that

$$\delta(x) = \frac{D}{2\rho U_\infty u(x)}$$

and substituting this expression to the first one gives:

$$\frac{u'}{u^3} = \frac{2a_1\rho}{D} \tag{2.33}$$

and

$$u(x) = \frac{1}{2}\sqrt{\frac{D}{a_2\rho x}}; \qquad \delta(x) = \sqrt{\frac{a_2 Dx}{\rho U_\infty^2}} \tag{2.34}$$

where we used the condition $_2 = -a_1$. The velocity distribution has minimum at $x = 0$, so that $\partial_y F < 0$ in the entire wake. Having this in mind. the equation (17.30) reads

$$\alpha^2 \frac{\partial}{\partial\eta}(\frac{\partial F}{\partial\eta})^2 - a_2(\eta\frac{\partial F}{\partial\eta} + F) = 0 \tag{2.35}$$

which is equivalent to

$$\frac{\partial}{\partial\eta}[\alpha^2(\frac{\partial F}{\partial\eta})^2 - a_2\eta F] = 0 \tag{2.36}$$

Integrating this once gives

$$\alpha(\frac{\partial F}{\partial\eta} = \pm\sqrt{\alpha_2\eta F + const}$$

Since at $\eta = 0$, $F'(0) = 0$, the constant $C = 0$ and taking into consideration that $\partial_\eta F < 0$, the final equation is:

$$\alpha(\frac{\partial F}{\partial\eta} = -\sqrt{\alpha_2\eta F} \tag{2.37}$$

with the solution:

$$F(\eta) = C^2[1 - (\frac{\eta}{\eta_e})^{\frac{3}{2}}]^2 \tag{2.38}$$

where $\eta_e = (3\alpha C/\sqrt{a_2})^{\frac{2}{3}}$. We see that as $\eta \to \infty$ this solution becomes meaningless and there is no way to satisfy the boundary condiiton at infinity $F = 0$. Thus the expression (17.38) must be used in the interval $\eta \le \eta_e$ where $F(\eta_e) = 0$ and assume that it is continued to infinity.

Now we would like to calculate the unknown constant C. The constraint (17.31) combined with the expressions (17.34) reads

$$\int_0^\infty F(\eta)d\eta = 1 = \int_0^1 C^2(1 - \eta^{\frac{3}{2}})^2 = \frac{9}{20}C^2 \tag{2.39}$$

which gives $C = 1.49$ and $\alpha = \sqrt{a_2/20}$.

We still have one undetermined parameter. Experimental data give:

$$\delta(x) \approx 0.805\sqrt{\frac{Dx}{\rho U_\infty^2}} \tag{2.40}$$

which, combined with our solutions yield $\alpha_2 \approx 0.65$ and $\alpha \approx 0.18$. Thus:

$$U(x,y) = U_\infty - 1.38\sqrt{\frac{D}{\rho U_\infty^2}}[1 - (\frac{y}{\delta})^{\frac{3}{2}}]^2 \tag{2.41}$$

where $\delta(x)$ is given by (17.40).

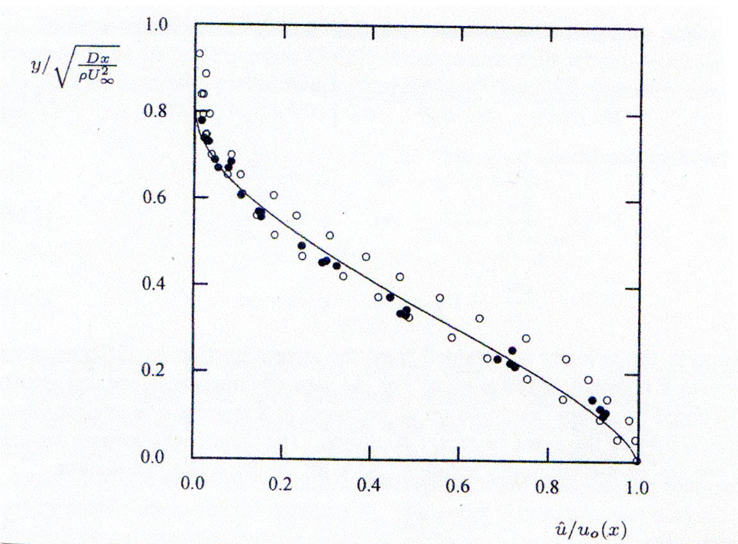


Figure 22: Universal velocity distribution in turbulent wake. Comparison of theoretical prediction with experimental data. .

Plane Jets. In this case, the equations (17.26) are valid with one difference: no linearization of the kind (17.27) is possible.

First, based on the momentum conservation:

$$\int_{-\infty}^{\infty} U\mathbf{u}\cdot\mathbf{n}dS \approx U^2(x)S = U^2(x)\delta(x)H = J = const$$

where $S = \delta(x)H = S$ is the cross sectional area of the jet (H is the dimension perpendicular to the page). It is clear from the Figure that $\delta(x) \approx x\sin\alpha$ where $\alpha = const$ is the jet angle. This relation gives:

$$U \propto \frac{1}{\sqrt{x}}; \qquad Re_\delta = U(x)\delta(x)/\nu_0 \propto \sqrt{x}$$

Since the are of a round jet is $O(\delta^2)$ the momentum conservation leads to $U \propto 1/x$ and $Re_\delta \approx const.$

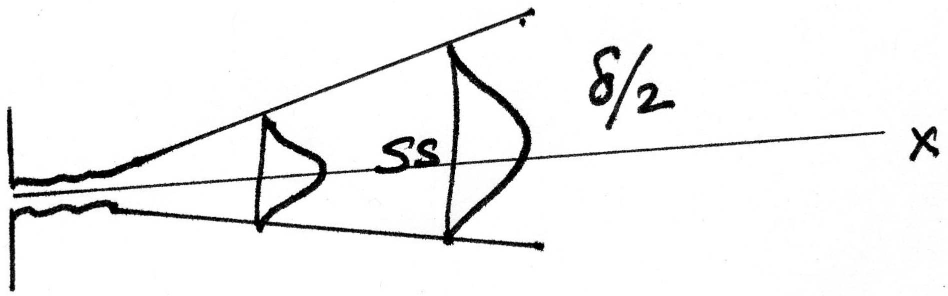


Figure 23: Plane jet. Schematic

It is useful to work with streamfunction ψ, so that: $U = \partial_y \psi$ and $V = -\partial_x \psi$. Again looking for a self-similar solution in the form

$$\psi = \sqrt{\frac{ax}{2}} F(\eta) \tag{2.42}$$

where $\eta = y/\delta(x)$. We have:

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= \sqrt{\frac{a}{2x}}[\frac{F}{2} - \eta F'(\eta)]] \\ \frac{\partial^2 \psi}{\partial x \partial y} &= -\frac{1}{\delta(x)}\sqrt{\frac{a}{2x}}[\frac{F'}{2} + \eta F''] \\ \frac{\partial^n F}{\partial y^n} &= F^{(n)}/\delta(x)^n \end{aligned} \tag{2.43}$$

and , in the mixing length approximation, the equation of motion for thj ψ -function reads:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \alpha^2 \frac{\partial}{\partial y}[\delta^2(x)|\frac{\partial^2 \psi}{\partial y^2}|\frac{\partial^2 \psi}{\partial y^2} \tag{2.44}$$

Using (17.43) we have immediately:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\frac{a}{4\delta^2(x)}[(F')^2 + FF''] = -\frac{a}{4\delta^2(x)}\frac{\partial}{\partial \eta}(FF') \tag{2.45}$$

and since in a jet maximum velocity is at the center plane $\partial_y^2 \psi > 0$ we can write the equation for the half-plane $y > 0$ with the result :

$$\alpha^2 \frac{\partial}{\partial y}[\delta^2(x)|\frac{\partial^2 \psi}{\partial y^2}|\frac{\partial^2 \psi}{\partial y^2} = -\frac{\alpha a \delta^2(x)}{2\delta^4(x)}\frac{\partial}{\partial \eta}(F'')^2 \tag{2.46}$$

Equating (17.25) and (17.46) final result is:

$$2\alpha(F'')^2 = FF' \tag{2.47}$$

Where, due to the boundary condition at $y \to \infty$, the integration constant was set equal to zero. This non-linear equation equation can be solved numerically subject to the following conditions:

$$F(0) = F''(0) = 0; \qquad\qquad F' \to 0; \eta \to \infty$$

and

$$\int_0^\infty (f')^2 d\eta = 1$$

Using experimental result $a \approx 0.25$ gives $\alpha \approx 0.1$ which is reasonably close to the data.

Problem. The plate of the width W and length L moves in the air with velocity U. Neglecting vortex shedding, find the drag. Assume transitional Reynolds number $Re_{x,tr} \approx 250000$.
Solution: in this flow the drag is entirely due to viscosity. The transition point

$$x_{tr} \approx 2.5 \times 10^5 \nu/U$$

If $x_{tr} < L$,

$$D = W\frac{\rho U^2}{2}\int_0^L c_f(x)dx = W\frac{\rho U^2}{2}[\int_0^{x_{tr}} \frac{0.664}{\sqrt{Re_x}}dx + \int_{x_{tr}}^L \frac{0.0576}{Re_x^{0.2}}dx]$$

$$D = W\frac{\rho U^2}{2}[\frac{1.338x_{tr}}{250000} + 0.072(\frac{L}{Re_L^{0.2}} - \frac{x_{tr}}{12.01})]$$

If: $x_{tr} \geq L$, the flow is laminar and

$$D = W\frac{\rho U^2}{2}[\frac{1.338L}{Re_L}]$$

Boundary layer separation.

Often pressure gradient in the boundary layer is not equal to zero. One such example is the problem of Couette flow or a flow past cylinder we discussed above. The origin of pressure gradients in fluids flows are many and vary. They can be externally imposed by pumps and other devices, generated due to non-trivial velocity distributions like in potential flow past cylinder ($c_p = 1 - 4\sin^2\theta$). The flow velocity, which at stagnation ($\theta = \pi, r = R$) is equal to zero, first accelerates away from it, reaches its maximum value at $\theta = \pi/2$ with subsequent decceleration toward next stagnation point at $\theta = 0, r = R$. By Bernoulli law, the pressure distribution along the cylinder surface does not stay the same. Strong pressure gradients occur in the flows with sudden expansions and contractions, flows past the steps etc.

In a statistically steady state the Prandtl approximation for the boundary layer is:

$$u\partial_x u + v\partial_y u = -\frac{\partial_x p}{\rho} + \nu\partial_y^2 u \tag{2.48}$$

Near the wall where $y/\delta \ll 1$ the velocity $u(y)/U \ll 1$. Therefore, the nonlinear terms in (17.21) can be neglected and the remaining equation reads:

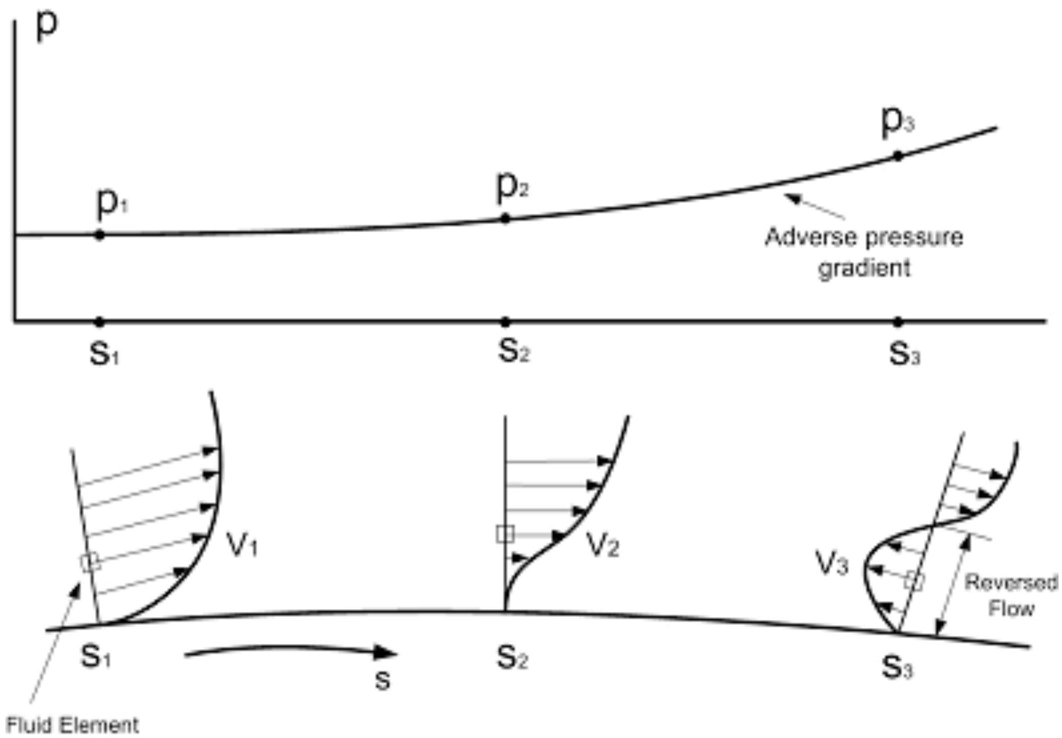


Figure 24: Velocity and pressure distributions in the vicinity of separation point.

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}|_{plate} \tag{2.49}$$

Due to the no-slip boundary condition, this equation is valid in the vicinity of the solid wall for both favorable and adverse pressure gradients. In case of the favorable pressure gradient $\partial_x p(x) < 0$, we find $\frac{\partial^2 u}{\partial y^2}|_{plate} < 0$. In the interval $y \geq \delta(x)$, the velocity $u(y) = U$ has a maximum, and thus $\frac{\partial^2 u}{\partial y^2}|_{\delta} < 0$ and we conclude that at all points inside boundary layer $\partial_y^2 u \neq 0$. In this case viscous term in the NS equations is balanced by the pressure gradient without dramatic increase of the y-component v. This corresponds to the attached BL. Let us consider the case of adverse pressure gradient $\frac{\partial p}{\partial x} > 0$. This situation is depicted on Fig. 61. At the edge of boundary layer $u(\delta) = U$ and, as $y \to \delta$ has its maximum, and as before:

$$\frac{\partial^2 u}{\partial y^2}|_{plate} < 0$$

and the positive pressure gradient cannot be balanced by viscous effects. There, far from the wall, the balance can be achieved accounting for the non-linear contributions. Near the wall, due to (17.22)

$$\frac{\partial^2 u}{\partial y^2}|_{plate} > 0$$

Thus, somewhere in between, there must be an inflection point where

$$\frac{\partial^2 u}{\partial y^2} = 0 \tag{2.50}$$

See point S_3 on Fig. 61. From continuity equation $\partial_x u = -\partial_y v$, we have:

$$v(x,y) = -\int_0^y \frac{\partial u(x,y)}{\partial x}\, dy \tag{2.51}$$

In the deccelerating flow (away from the maximum of velocity at $\theta = \pi/2$ on a cylinder surface, for example) $\frac{\partial u}{\partial x} < 0$ and, by the virtue of (17.24), the vertical component $v > 0$ (out of plate) is larger, the larger the decceleration. *When the positive vertical component becomes too large, the flow separates if*

$$\frac{\partial u}{\partial y}|_{plate} = 0 \tag{2.52}$$

Indeed, at this point any further increase of decceleration will lead to the flow reversal. It can be shown that in the limit of an infinite Reynolds number the component

$$v \propto \frac{1}{\sqrt{x_0 - x}}$$

at a separation point x_0 tends to $v \to \infty$ which leads to a rapid grows of the boundary layer compared to the case $\partial_x p = 0$. From these considerations we see that at the separation point

$$\frac{\partial u}{\partial y} = 0 \tag{2.53}$$

see point S_2 on Fig.61. The relation (17.23) and (17.26) are necessary conditions for the flow separation. Numerical and experimental determination of separation points is a delicate task especially dealing with streamlined bodies where the separation point moves with variation of angle of attack, Reynolds numbers etc.

Problem. A cylinder is moving in the air with velocity U. Find an approximate position of the separation point on a surface of a cylinder.

Solution. In this case we choose the origin at the stagnation point $\theta = \pi$ and, assuming narrow BL ($\delta \ll R$), take coordinate along the BL $l = R\theta$. The direction perpendicular to the BL is $r\mathbf{e}_r$. According to BL equations (16.86) $\partial_r p = 0$, i.e. the pressure distribution within the boundary layer does not vary in the r-direction, we can use the results of potential flow theory giving

$$p - p_\infty = \frac{\rho U^2}{2}(1 - 4\sin^2\theta)$$

thus, since $l = R\theta$, $\partial_l p = \partial_\theta p/R$ we have according to (17.23)

$$\sin\theta\cos\theta = 0$$

and

$$\theta = \pi/2$$

As we can see from Fig.62, this simple estimate is close to the super-computer-based numerical simulations.

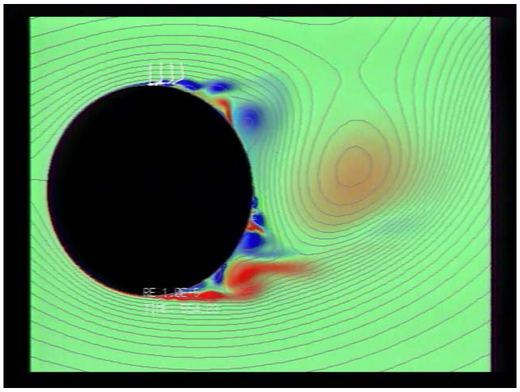


Figure 25: Boundary layer separation on a surface of a cylinder. At the separation point θ_0, $u(\theta_0, R) = const$ $Re \approx 10^5$

The instability of a boundary layer is a complicated dynamics process depending on very fine structure of the wall flow. Many factors effect this process and sometimes it is extremely beneficial to delay it. One of the possibilities is artificially perturb the developing BL by vortices, jets, suction (blowing), creating noisy or regular patterns on a surface etc. One of such attempts is presented on Fig. 63.
We can see, in this flow the separation point coincides with the sharp rear edge of the body of this car.

Pressure drag.

According to the Stokes formula: the drag (friction) force acting on sphere of radius R moving with velocity U is given by the relation:

$$F_f \equiv D = 6\pi\mu RU \tag{2.54}$$

giving skin friction:

$$c_D = \frac{D}{\frac{\rho U^2}{2}\pi R^2} = \frac{12}{Re_R} \tag{2.55}$$

The experimentally determined skin friction is shown on Fig. 65 in an excellent agreement with the relation (17.28) in the range $Re_R < 100$. At $Re_R >> 100$, a completely different dependence of $c_D(Re)$ has been

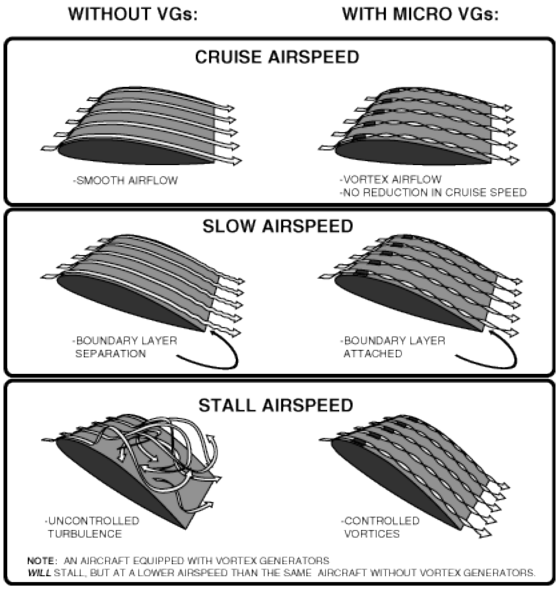


Figure 26: Separation control with an artificial surface vortex generation (VG).

In flows pst bluff bodies like sharp-edged buildings, some cars etc, the pressure gradient behind the edges is so large that the position of separation point is prescribed by geometry. One such example is shown on Fig 63. The separation point is seen right at the rear of the JEEP car.

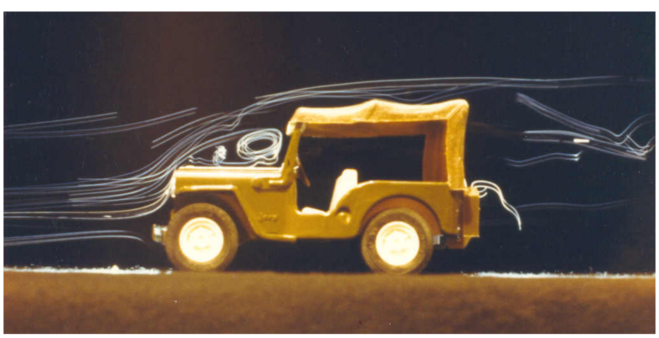


Figure 27: Streamlines in a flow past JEEP.

observed. In one of the problem of this section we, calculating the terminal velocity of sphere ($M = 100kg; R = 2m$ falling in the air under the action of gravity, derive absurdly high number.

The reason for this dramatic disagreement is easily understood from the picture of Figs. 65- 65. The flow separation, accompanied by the vortex formation and shedding is a huge source of energy loss thus acting as an additional drag on a moving body. Moreover, the main action takes place outside the body surface where the viscous effects are unimportant. Thus, the only source of drag is pressure and the pressure force

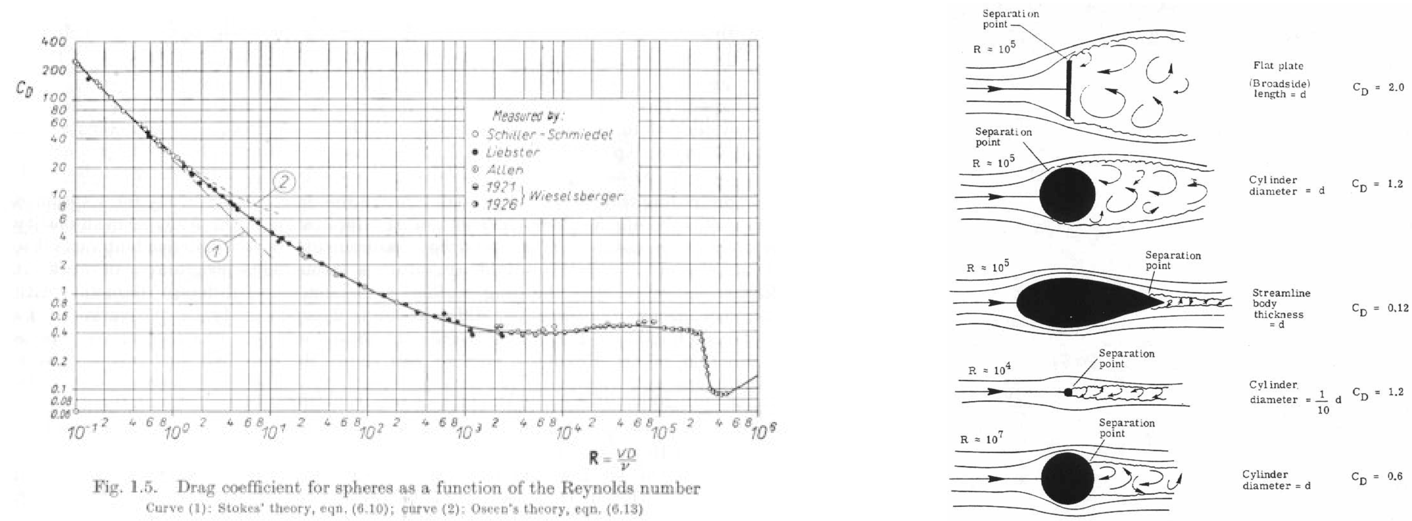


Figure 28: Velocity and pressure distributions in the vicinity of separation point.

acting on a body can be presented as

$$D_p = \overline{p}A = \alpha_p \rho U^2/2$$

where A is the area of a relevant cross-section and α_p is a numerical coefficient. Anayitic evaluation of both α_p, and A is, in general, impossible. This is a subject of Computational Fluid Dynamics, utilizing modern numerical methods and computers to accurately simulate flows past various bodies. During some 50-60 years CFD evolved into a crucial tool revolutionalizing engineering design cycle. An important conclusion from the above relation is that the pressure drag coefficient $C_p = \frac{2D_p}{\rho U^2 A} \approx 2\alpha_p = const$. As we learned, the friction drag $C_f \propto 1/Re$ and we see that as $Re \to \infty$, pressure drag is always dominating factor. This explains general shape of the dependence shown on The Fig. 65.

Problem. A sphere of radius R and mass M falls under action of gravity. Assuming a simple relation $C_D = \frac{12}{Re} + 0.4$ giving correct numbers in two limits $Re \to \infty$ and $Re \to 0$, find terminal velocity when

1.
$$\frac{225\nu^2}{R^2} \gg 3.33gR\frac{\rho_{sphere}}{\rho_{air}}$$

2.
$$\frac{225\nu^2}{R^2} \ll 3.33gR\frac{\rho_{sphere}}{\rho_{air}}$$

Solution: The force balance:

$$M\frac{dU}{dt} = Mg - 6\pi\mu RU - 0.4\frac{\rho U^2 \pi R^2}{2}$$

Time - independent terminal velocity is defined by condition $DU/dt = 0$. Thus, we have an algebraic equation:

$$U = -\frac{15\nu}{R} + \sqrt{\frac{225\nu^2}{R^2} + \frac{Mg}{0.4\rho_{fluid}\pi R^2}}$$

$M = \frac{4}{3}\pi\rho_{sphere}R^3$. This solves the problem for both limiting cases. Consider: a. $\nu = 0.15cm^2/sec$, $R = 10cm$, $\rho_{sphere} = 1gr/cm^3$. Density of the air is $\rho_{air} \approx 10^{-3}gr/cm^3$. 2. The same but $\rho_{sphere} = 10gr/cm^3$ and $R = 30cm$.

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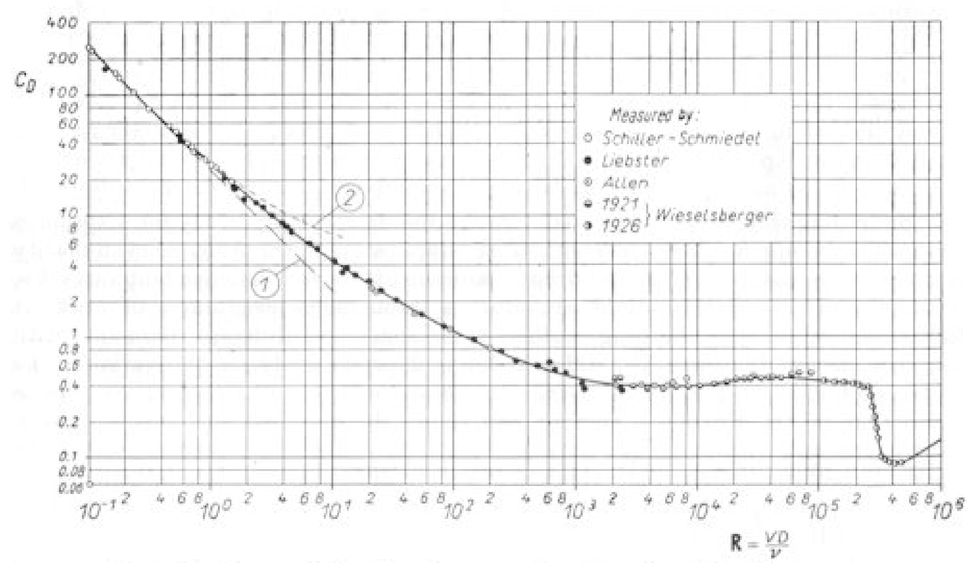

Fig. 1.5. Drag coefficient for spheres as a function of the Reynolds number
Curve (1): Stokes' theory, eqn. (6.10); curve (2): Oseen's theory, eqn. (6.13)

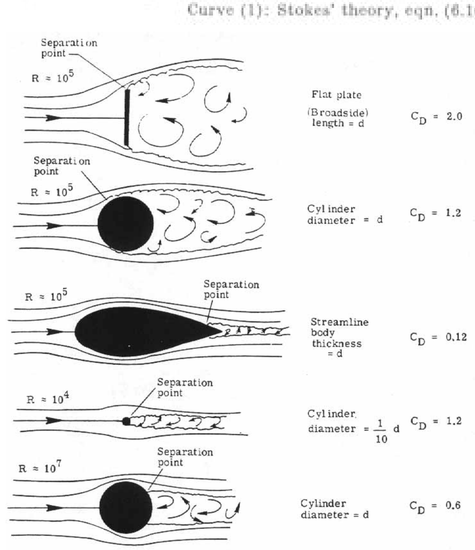

Figure 29: Velocity and pressure distributions in the vicinity of separation point.