ME421. Foundations of Aerodynamics.

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1 Introduction. Continuum description. Elements of kinetic theory

Physics of gases and solids is well understood on a microscopic (molecular) level: given the interaction potential $U(\mathbf{r_i} - \mathbf{r_j})$ between the particles , where $\mathbf{r}_n$ defines position of n^{th} particle, we, based on theory or using powerful computers, can explain why at the low temperatures $T \to 0$ the matter tends to a well-ordered state (crystal), while in the opposite limit of high temperatures - everything has to be in a totally disordered state which is called gas. For example, in the temperature interval $T > 373^o\ K$, water is in a gaseous or vapor state, while when $0 < T < 273^o K$, it is a crystal called ice. Only in a tiny temperature interval $273^o K < T < 373^o K$, water is a life supporting liquid. The microscopic nature of a liquid state is not completely understood: we do not have a quantitative theoretical explanation of what makes liquid water flow. However, not understanding microscopic dynamics of the liquid state did not prevent development of hydrodynamics or science of a fluid flow, based on a **macroscopic or continuum** level of description. A gas or liquid can be characterized by the mean-free -path which is a typical distance λ it takes a small perturbation (kick) of a fluid at at a point $\mathbf{r} = 0$ to disappear or relax to zero. In the air at normal pressure $p = 1atm$ and $T = 293K$ this length-scale is $\lambda \approx 10^{-5}cm$. This means that the volume element of linear dimension $\approx 2\lambda$ contains approximately $8n\lambda^3 \approx 10^5$ particles. This number is large enough to conclude that if we take a set of such fluid elements, the mean deviation of number of particles within each element from the estimated $n_v \approx 10^5$, will be smaller that 1%.

Traditionally, one is interested in a fluid flow varying on a length-scale $L >> \lambda$, for example if $L \gg 1\mu m$ the volume of linear dimension $\lambda \approx 10^{-5}cm$, called **fluid element** can be approximately treated as a mathematical point. Indeed, in this flow no characteristics can vary on such a small scale. We must remember that the fluid element, though very small, must contain a large (macroscopic, $n_v \gg 1$) number of particles and pay attention to, for example the gas density. While at the normal conditions the mean -free -path in the air $\lambda \approx 10^{-5}cm$, in a low-pressure (rarified) gas say $p \approx 0.1 - 0.01atm$ $\lambda \approx 10^{-4} - 10^{-3}cm$ and the fluid element must be defined accordingly. In these cases, a flow varying on a length- scale $L \leq 10^{-4} - 10^{-3}cm$ cannot be described using approximation of continuum mechanics.

The goal of continuum mechanics is to describe **macroscopic** fluid flow not dealing with the detailed properties of **microscopic** (atomic or molecular) features. The number of fluid particle is so huge that a detailed accounting for their configurations in both time and space is simply impossible. Thus, continuum mechanics is valid only for description of relatively large and slowly varying in time volumes of fluids. The physical condition of applicability of continuum description can be understood easily. If a flow velocity U is time-dependent, then time-scale characterizing temporal variations is $T \approx \frac{L}{U}$. The microscopic state can be irrelevant only if in the time interval T any molecule in the fluid volume element $\lambda^3 \ll L^3$ undergo a huge number of collisions, so that the state, the liquid had at at the beginning of the time interval T, is totally mixed or forgotten. The microscopic relaxation time in a gas of particles of mass m is $\tau_m \approx \lambda/c_s$ where $c_s \approx \sqrt{\frac{k_B T}{m}}$ is called speed of sound which is approximately equal to the typical (mean) velocity of a gas atom (molecule). Thus, the limit of validity of continnum mechanics is

$$\frac{\tau_m}{T} = \frac{\lambda U}{c_s L} = Kn\ Ma \ll 1$$

where $Ma = U/c_s$ and $Kn = \lambda/L$ are called the Mach and Knudsen numbers, respectively.

It follows from this relation that the micro-flows where $\lambda \approx L$ or the so called rheological flows like solutions of polymers, blood flows in small blood vessels cannot be described in terms of continuum mechanics.

Forces and stresses. Let us define the stress τ as a force per unit area of a surface. If a solid cube of a volume $V = L^3$ is subjected to the normal force $\mathbf{F_n}$, it develops the deformation δL such that the elastic force $\kappa \delta L = F$ exactly compensates the applied external force. This condition also defines the magnitude of the displacement δL. The normal stress $\tau_n = p = F_n/S$ where $S = L^2$ and parameter p is also called *pressure.*

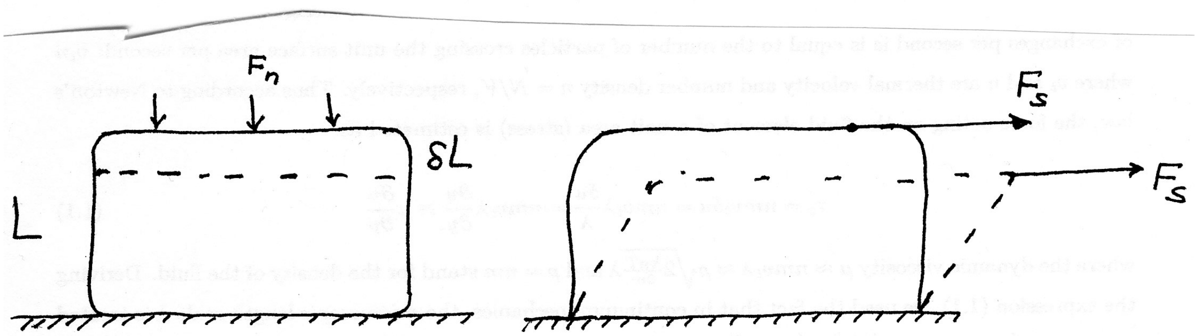


Figure 1: Solids. Response to applied forces.

Similar effect happens when the shear force $\mathbf{F_s}$, which is tangential to the surface and normal to $\mathbf{n}$, is applied : the shear leads to the body deformations and generation of the deformation-resisting force, which eventually compensates the action of the shear force. The value of the deformation is determined from the force balance. In this case the shear stress $\tau_s = F_s/S$.

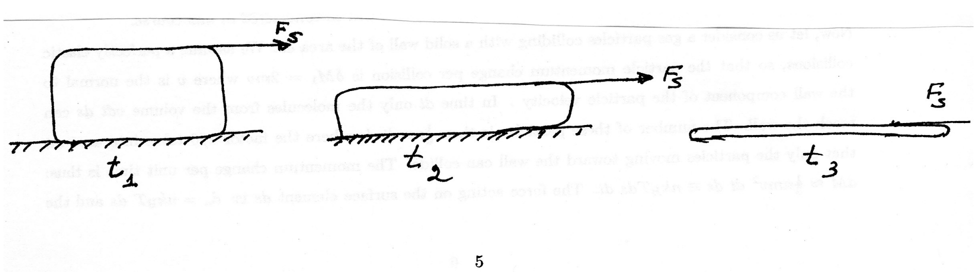


Figure 2: Liquid state. Response to applied shear forces.

In the fluids, the situation is dramatically different: even infinitesimal shear force leads to the shear deformation of the fluid element which is never compensated and, eventually, the element becomes thinner and longer until it forms a mathematical surface.

Problem. Two identical trains of mass M travel along parallel tracks with velocities $U > V$. At the instant of time $t = 0$ the trains exchange two identical bodies of mass m thrown with velocities v perpendicular to the rails. Find the forces acting on the trains if the time the bodies loose their momenta is t_o..

Solution. The change of the momentum projection on a direction of the train motion is $\delta P_{\pm} = \pm m(U - V)$. Thus, in accord with the Newton law, the forces are $F_{\pm} = \frac{\delta P_{\pm}}{t_o} = \pm \frac{m(U-V)}{t_o}$. we see that as a result of the exchange the speed of the faster train tends to decrease while that of the slower one - to increase. The force is zero if $V = U$.

To understand the nature of the shear force in fluids, consider two layers of unit area separated by the distance λ moving in a gas relative to each other with velocity δu. Due to the thermal motion, the fluid molecules moving with the layers exchange their positions, i.e. the molecule from a faster layer jumps onto the slower one and vice versa. As a result, the momentum change per one exchange is $m\delta u$. The number of exchanges per second is is equal to the number of particles crossing the unit surface area per second: $v_t n$ where v_t and n are thermal velocity and number density $n = N/V$, respectively. Thus according to Newton's law, the force acting on the fluid element of a unit area (stress) is estimated as:

$$\tau_s = nmv_t\delta u = nmv_t\lambda\frac{\delta u}{\lambda} = nmv_t\lambda\frac{\partial u}{\partial y} \equiv \mu\frac{\partial u}{\partial y} \tag{1.1}$$

where the dynamic viscosity $\mu \approx nmv_t\lambda \approx \rho\sqrt{2\frac{k_B T}{3m}}$ and $\rho = nm$ stand for the density of the fluid. Deriving the expression (1.1), we used the fact that in continuum mechanics, the *microscopic* length-scale λ is treated as zero and $\frac{\delta u}{\lambda} \equiv \lim_{d\to 0}\frac{u(y+d)-u(y)}{d} = \frac{\partial u}{\partial d}$. We can also define a very useful for the future kinematic viscosity

$$\nu = \mu/\rho \tag{1.2}$$

and $\nu \approx v_t\lambda$. *The relation (1.1) for the shear stress was derived for the case of small velocity gradients, i.e. when the velocity difference between the layers can be represented as $u(y + \lambda) - u(y) \approx \frac{\partial u(y)}{\partial y}\lambda$. When the gradients are large, we cannot limit ourselves by the first term in expansion (1.1) and must take into account the high-order velocity derivatives. The fluids where the estimate (1.1) is reasonably accurate are called Newtonian as opposed to the non-Newtonian ones which will be considered in this course.*

Now, let us consider a gas particles colliding with a solid wall of the area ds. We assume a perfectly elastic collisions, so that the particle momentum change per collision is $\delta M_1 = 2mv$ where v is the normal to the wall component of the particle velocity . In time dt only the molecules from the volume $vdt\ ds$ can reach the wall. The number of these particles is then $\frac{1}{6}nv\ dt\ ds$ where the factor $1/6$ takes into account that only the particles moving toward the wall can collide. The momentum change per unit time is thus: $dM \approx \frac{1}{3}nmv^2\ dt\ ds \equiv nk_BTds\ dt$. The force acting on the surface element ds is: $d_n = nk_BT\ ds$ and the normal stress (pressure) is:

$$P = \frac{dF}{ds} = nk_BT = \rho\frac{k_B}{m}T = \rho\ RT$$

We would like to stress that while the Boltzmann constant k_B is a universal constant, the gas constant R, depending on the molecular weight of the gas molecules, is not.
Since viscosity $\mu \propto \sqrt{T}$ and if pressure $P \approx const$, then $\rho \propto 1/T$ and kinematic viscosity of a gas :

$$\nu \propto T^{\frac{3}{2}} \tag{1.3}$$

Viscosity of air is well approximated by an empirical relation valid in a wide range of temperature variation:

$$\mu = \frac{AT^{\frac{3}{2}}}{T+S} \tag{1.4}$$

where $A = 1.46 \cdot 10^{-6} \frac{kg}{m \cdot sec \cdot K^{\frac{1}{2}}}$. This formula is often referred to as Sutherland's Law. The absolute (Kelvin's) scale of temperature will be used everywhere in these notes.

2 Elements of kinematics.

Consider a particle (material point) of zero linear dimension at a point in space $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at time t. Velocity of the particle is :

$$\mathbf{u} = \frac{d\mathbf{r}}{dt} = \lim_{\tau\to 0} \frac{\mathbf{r}(t+\tau) - \mathbf{r}(t)}{\tau} \tag{2.1}$$

and its acceleration

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \lim_{\tau\to 0} \frac{\mathbf{u}(t+\tau) - \mathbf{u}(t)}{\tau} \tag{2.2}$$

According to Newton's law, the force acting on a particle:

$$\mathbf{F} = m\mathbf{a} \tag{2.3}$$

where the scalar coefficient m is called the particle mass. This differential equation, which is to be solved subject to initial conditions $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{u}(t_0) = \mathbf{u}_0$, describes the entire time evolution of a particle. The force $\mathbf{F}$ can originate from the interaction between the particles, external fields, like gravity or electromagnetic field etc.

On the simplest level of description , we can treat atoms or molecules of a fluid interacting with each other via the pair potential $U(\mathbf{r_i} - \mathbf{r_j})$ and as a result , the force acting on a particle number i is:

$$\mathbf{F_i} = -\sum_j \frac{\partial U(|\mathbf{r_i} - \mathbf{r_j}|)}{\partial \mathbf{r_j}} \tag{2.4}$$

It is in principle possible to describe a flow by following time evolution of velocities and positions of each particle. If the fluid consists of, say 10^{19} particles in a cubic centimeter, this is impractical.

Often, it is much easier to measure the fluid properties at a fixed point $\mathbf{r}$. We define a fluid element of a volume λ^3 occupying vicinity of a point $\mathbf{r}$ and containing large number of particles N. If $\lambda \ll L$ where L is a scale of the macroscopic flow velocity variation, then we can forget about the particles contained in this element and treat it as a mathematical point. Moreover, if $N \gg 1$, the fluctuations of the $O(1/\sqrt{N})$ number of particles in the element can also be neglected. This way we can introduce such mean properties of a fluid as number density $n = N/V$, density $\rho = mn$, momentum $\rho\mathbf{u}$, kinetic energy density $\rho\ \mathbf{u^2/2}$ For computations, it is more useful to introduce the local definitions. If, for example the number of particle in a fluid element $N(\mathbf{r})$ depends on the element position, then the number density is:

$$n = \frac{\partial N(\mathbf{r})}{\partial V} \equiv \frac{\partial^3 N(\mathbf{r})}{\partial x \partial y \partial z}$$

Let at a time t the velocity of a fluid element at a point $\mathbf{r}$ be $\mathbf{u}(\mathbf{r}(\mathbf{t}))$. At an instant of time $t+\tau$, the fluid element moves with a flow to a new position $\mathbf{r}(t+\tau)$, while another one arrives in a point $\mathbf{r}$ from a position $\mathbf{r}(t-\tau)$. Thus, the acceleration measured at a point $\mathbf{r}$ is:

$$\mathbf{a} = \lim_{\tau \to 0} \frac{\mathbf{u}(\mathbf{r}(t), t) - \mathbf{u}(\mathbf{r}(t-\tau), t-\tau)}{\tau} \tag{2.5}$$

Velocity is an analytic function, so we can use the Taylor expansion:

$$f(x+\Delta) \approx f(x) + \frac{f(x)}{dx} + \frac{1}{2}\frac{d^2 f(x)}{dx^2} + ...$$

Thus in the limit $\tau \to 0$:

$$\mathbf{r}(t-\tau) \approx \mathbf{r}(t) - (\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k})\tau = \mathbf{r}(t) - (u_x\mathbf{i} + u_j\mathbf{j} + u_z\mathbf{k})\tau$$

and

$$\mathbf{u}(\mathbf{r}(t-\tau), t-\tau) = \mathbf{u}(\mathbf{r}(t), t) - \frac{\partial \mathbf{u}(\mathbf{r}(t), t)}{\partial t}\tau - \frac{\partial \mathbf{u}(\mathbf{r}(t), t)}{\partial \mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial t}\tau \tag{2.6}$$

Substituting (2.6) into (2.5) gives

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + (u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z})\mathbf{u} \tag{2.7}$$

We see that the acceleration consists of two contributions: the temporal one ($\frac{\partial \mathbf{u}}{\partial t}$) is due to the explicit time-dependence of the velocity field caused by the time variation of boundary conditions , external forces etc and the second one - due to the spatial dependence of the velocity vector $\mathbf{u}$. It is clear that even in a time-independent flow, the acceleration and the force acting on a fluid element is not necessarily zero.

Problem. Consider a one-dimensional flow with velocity :

$$\mathbf{u} = \frac{x}{t}\mathbf{i}$$

Find: a. Convective, unsteady and total acceleration.

b. Repeat the calculation for $u- = -x/t$.

Solution.

Unsteady: $\mathbf{a}_t = \frac{\partial \mathbf{u}}{\partial t} = -\frac{x}{t^2}\mathbf{i}$.

Convective: $\mathbf{a}_c = u_x \frac{\partial \mathbf{u}}{\partial x} = \frac{x}{t}\frac{1}{t}\mathbf{i} = \frac{x}{t^2}$

Total $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_c = 0$.

Problem: Find acceleration if $\mathbf{u} = \frac{U}{H}(x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k})$.

Problem; For the same velocity field, find position $\mathbf{r}$ of a Lagrangian particle and its trajectory. At initial instant $t = t_0$, $x = x_0$, $y = y_0$. Solution:

$$\frac{d\mathbf{r}}{dt} = \frac{x}{t}\mathbf{i}; \qquad \frac{dx}{x} = \frac{dt}{t}; \qquad \frac{dy}{dt} = 0$$

$$\ln\frac{x}{x_0} = \ln\frac{t}{t_0}; \qquad x = x_0\frac{t}{t_0} = u_x(t_0); \qquad y = y_0 = const$$

Trajectory: $y(x) = y_0 = const$

2.1 Fluxes.

The flux or current of any property Ψ across an infinitesimal surface element $d\mathbf{s} \equiv \mathbf{n}ds$

$$dJ_\psi = \psi\mathbf{u} \cdot d\mathbf{s} = \psi\mathbf{u} \cdot \mathbf{n}ds \tag{2.8}$$

where $\mathbf{n}$ is a unit vector normal to the surface element, and ds is its area. The quantity $\psi = \frac{d\Psi}{dV} = \frac{d\Psi}{dxdydz}$ is the density of Ψ. It is clear from this definition that flux or current is equal to amount of Ψ crossing the surface are ds per unit time. Thus:

$$dJ_m \equiv d\dot{m} = \rho\mathbf{u} \cdot \mathbf{n}ds \tag{2.9}$$

$$dJ_p = \rho\mathbf{u}\mathbf{u} \cdot \mathbf{n}ds \tag{2.10}$$

$$dJ_K = \rho\frac{u^2}{2}\mathbf{u} \cdot \mathbf{n}ds \tag{2.11}$$

$$dJ_e = en\mathbf{u} \cdot \mathbf{n}ds \tag{2.12}$$

are the mass, momentum , kinetic energy and electric currents (fluxes) respectively.

Problem. A $V = 1m^3$ aquarium is being filled with water by a circular pipe of diameter $20cm$. *Find: the mean velocity of water in this pipe if this job takes one hour .*

Problem. Consider a flow near stagnation point for which velocity is:

$$\mathbf{u} = Ax\mathbf{i} - Ay\mathbf{j}$$

where $A = const > 0$. Calculate the mass and momentum fluxes across the planes I and II (See figure).

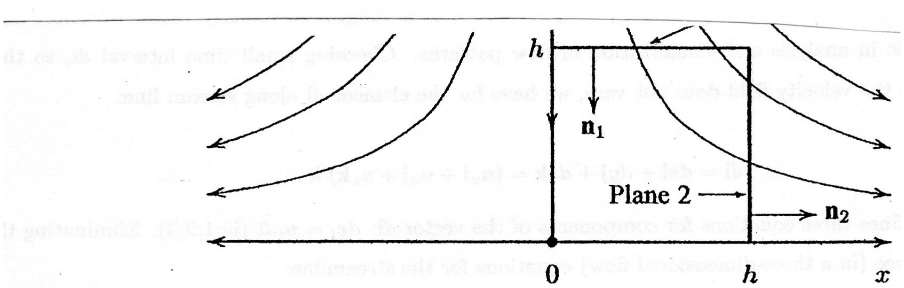


Figure 3: Stagnation point flow.

Solution. On plane 1, $\mathbf{n_1} = -\mathbf{j}$, thus

$$\mathbf{u}\cdot\mathbf{n} = (Ax\mathbf{i} - Ay\mathbf{j})\cdot(-\mathbf{j}) = Ay$$

Denoting the width of the planes W (into the page), the mass flux:

$$\dot{m} = \int_S \rho\mathbf{u}\cdot\mathbf{n_1}dS = W\int_0^h \rho(Ay)_{y=h}dx = \rho WAh^2$$

The momentum flux across the plane I:

$$\dot{M} = \int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n_1})dS = \rho WAh^3(\frac{\mathbf{i}}{2} - \mathbf{j})$$

Calculate a. fluxes across the plane 2.

Problem. The flow in an infinite channel of the width $2H$ and span W is given by the relation $\mathbf{u} = U(1-(\frac{y}{H})^2)\mathbf{i}$. Find mass and momentum fluxes width.

Solution. The vector normal to the crossection is $\mathbf{i}$.

$$\dot{m} = \int_S ds\rho(\mathbf{u}\cdot\mathbf{i}) = UW\rho\int_{-H}^{H}(1-(\frac{y}{H})^2)dy = \frac{4}{3}\rho UH \equiv \overline{u}S$$

This relation defines the mean velocity

$$\overline{\mathbf{u}} = \frac{1}{S}\int_S \mathbf{u}dS$$

Momentum flux:

$$\dot{\mathbf{M}} = \int ds\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{i}) = \frac{16}{15}\rho U^2H\mathbf{i}$$

Mass flux can also be defined in terms of the so called mean velocity: $\dot{m} = \rho\overline{u}S$ where S is the area of the crossection. Comparing this definition with the result obtained above, we get $\overline{u} = \frac{2}{3}U$.

Problem. Consider a flow in a pipe of a radius R. The velocity distribution is $\mathbf{U} = u_{max}(1-\frac{r^2}{R^2})$. calculate mean velocity and mean momentum flux and compare the results with the ones in a channel flow.

2.2 Streamlines and pathlines.

Eulerian description of a flow operates with velocity field $\mathbf{u}(\mathbf{r}, \mathbf{t})$, providing us with information on the instantaneous magnitudes and directions of velocity vectors of all fluid elements (particles) of a flowing fluid. Let us define a line, which at each point ($\mathbf{r}$,t) in space and time is tangent to velocity vector $\mathbf{u}$. These lines are indispensable in analysis and visualization of flow patterns. Choosing small time interval dt, so that during this time the velocity field does not vary, we have for the element $d\mathbf{l}$ along stream line:

$$d\mathbf{l} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})dt$$

This relation defines three equations for components of the vector $d\mathbf{l}$: $dx_i = u_i dt$ (i=1;2;3). Eliminating the time leads to three (in a three-dimensional flow) equations for the streamline:

$$dt = \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} \tag{2.13}$$

Often, one is needs information on coordinates of a particular fluid particle. This is important if we are interested in motion of a particular object (contaminant, chemical reactants etc). The particle coordinates are defined by three differential equations:

$$u_i = \partial_t x_i \tag{2.14}$$

which must be solved subject to initial conditions: $x_i = x_i^0$. Solution to these equations $x_i = x_i(x, y, z, t)$ define the so called pathlines which are tangential to the fluid particle velocity vector at a point (x, y, z), provided initially this particle was at $x_i = x_i^0$. We shall see that in the steady (time independent flow) the pathlines and streamlines are identical. However, in time-dependent flow this is not so.

Problem: The velocity field is $\mathbf{u} = Ax\mathbf{i} - Ay\mathbf{j}$.

Plot: a. streamlines; b. pathlines; c. find coordinates of a fluid particle wihch at the time $t = 0$ was at the point $x =_x 0; y = y_0$;

Solution. a. The equation for the streamline is:

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{y}{x}$$

or $\frac{dy}{y} = -\frac{dx}{x}$. Integrating this equation gives: $\ln y = -ln\ x + C$ with the solution $xy = C_1 = e^C$. The streamlines are plotted bellow.

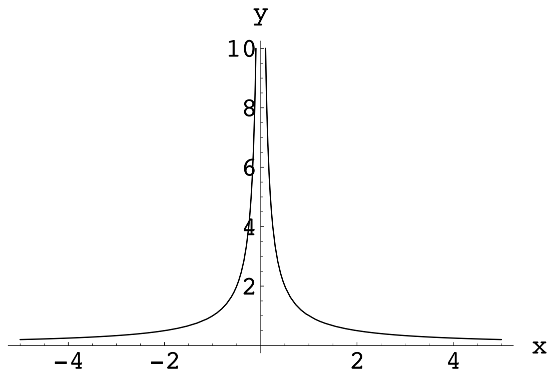


Figure 4: Streamlines for stagnation point flow. $C_1 = 1$.

b. Equations for the particle coordinates: $\frac{\partial x}{\partial t} = u = Ax$ and $\frac{\partial y}{\partial t} = -Ay$. The solutions are: $y = y_0 e^{-At}$; $x = x_0 e^{At}$. Excluding time we have $xy - x_0 y_0$. We see in this case streamlines and pathlines are the same.

Problem. Consider an unsteady velocity field $\mathbf{u} = \mathbf{U}\mathbf{i} + \mathbf{U}\sin\omega t\mathbf{j}$. calculate and plot streamlines and pathlines.

Streamlines are one of the most useful tools for the flow visualization, which is essential in a design process. On an example shown on a Figure one can identify such delicate features of a flow as shedded vortices behind a car and vortices on a hood and roof strongly contributing to the drag and lift. The vortices shedded from a mirror are the source of noise. The information about flow structure is crucial for design of aerodynamically efficient, elegant and economics vehicles. Experimentally, the streamlines can be observed by adding tiny particles into the air flow in the wind tunnel combined with modern optical methods of the particle detection. The surface streamline are found by covering the body (car) surface by a thin film by "dipping" an entire car in a pool filled with oil. In this case, the particle in a flow scratch the film leaving the traces which are very close to the streamlines.

One of the most important processes in fluid mechanics is vortex formation which can be studied, for example, on a well-controlled flow past backward facing step, which is a flow in a channel with sudden expansion. Here the streamlines enable one to follow the vortex generation, study its properties and use the information for scientific research and engineering design.

The vortex dynamics in the wake of a moving body is a complex time-dependent process. Two images of the Figure below demonstrate the streamlines at two different instants of time. We see the vortex formed on a top image separates from a body (shedding), while another one is generated on a "top corner" of a cylinder.(lower image). The shedded vortices form a chain called Karman street.

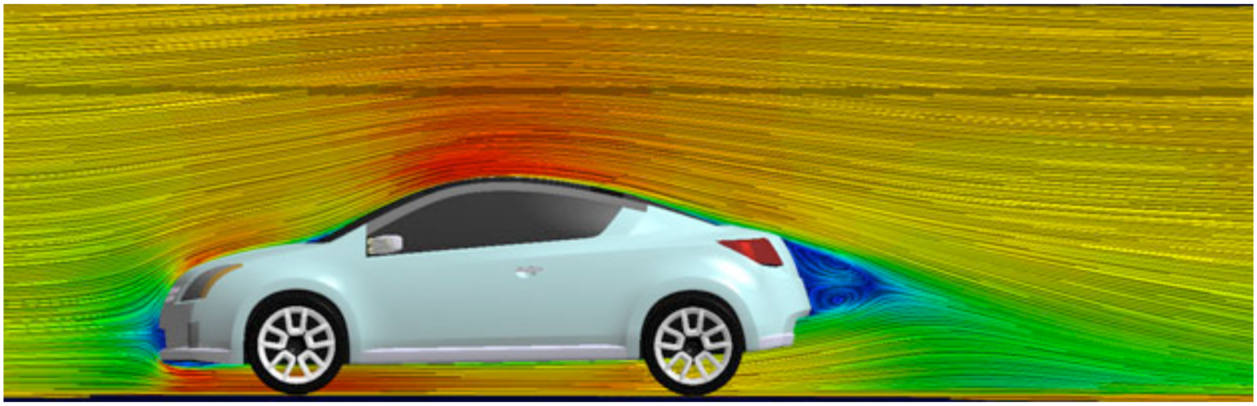


Figure 5: Numerically simulated streamlines in a flow past a car. Courtesy of EXA Corporation.

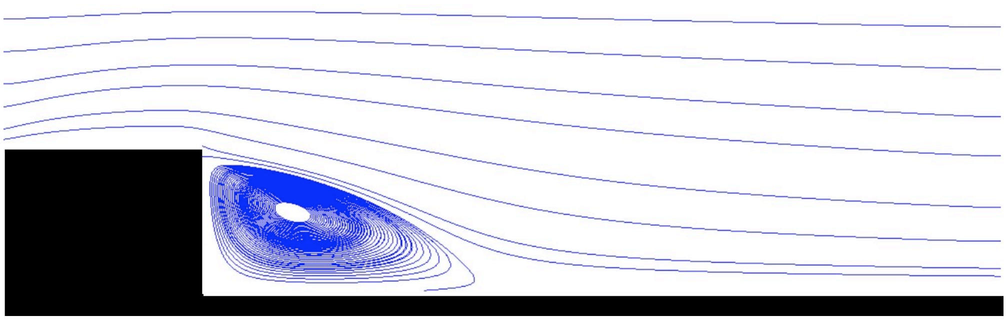


Figure 6: Numerically simulated streamlines in a flow past backward facing step.

2.3 Vorticity.

One of the most important characteristics of a fluid flow is vorticity ω defined as:

$$\omega = \nabla \times \mathbf{u} \equiv curl\mathbf{u} = \begin{vmatrix} \mathbf{i}\ \mathbf{j}\ \mathbf{k} \\ \partial_x \partial_y \partial_z \\ u_x u_y u_z \end{vmatrix} = (\partial_y w - \partial_z v)\mathbf{i} - (\partial_x w - \partial_z u)\mathbf{j} + (\partial_x v - \partial_y u)\mathbf{k}$$

Two-dimensional (2D) flow is defined on a plane, say (x, y), by the velocity vector $\mathbf{u} = u_x\mathbf{i}(x, y) + u_y(x, y)\mathbf{j} \equiv u\mathbf{i} + v\mathbf{j}$. In this case the vorticity vector

$$\omega = (\partial_x v(x, y) - \partial_y u(x, y))\mathbf{k}$$

which is perpendicular to (x, y)-plane is often called pseudo-scalar. In this case The importance of this property can be best illustrated by the fact that if the velocity field is such that $\omega = 0$, this field cannot have closed streamlines. This statement is proved readily: consider the velocity circulation round a closed contour:

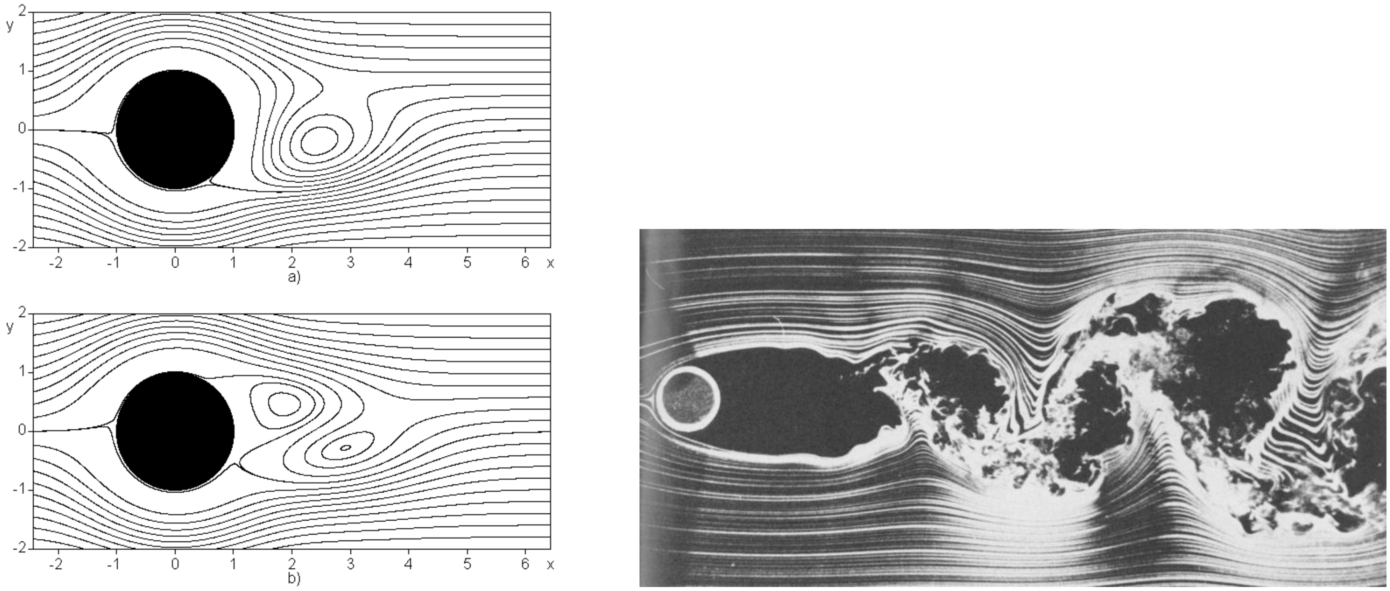


Figure 7: Left:Tme dependent vortex generation and shedding in the flow behind the cylinder. Experimental visualization of Karman street in a flow past behind cylinder.

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} = \int_s \nabla \times \mathbf{u} ds = \int_s \omega \cdot \mathbf{ds} = \int \omega \cdot \mathbf{n} ds \tag{2.15}$$

where s is the area limited by the contour. Let us assume that there exist at least one closed streamline. Then choosing the integration contour along this streamline and taking into account that, since velocity vector is tangential to the streamline at any point and $\mathbf{u} \cdot d\mathbf{l} > 0$ (or < 0), we conclude that $\Gamma \neq 0$. This result contradicts the condition $\omega = 0$.

Now, imagine a get of fuel issued into the non-moving oxygen enviroment. It is clear that the fuel-oxygen interface is a streamline. If it never forms a closed line, the mixing of the fuel with oxygen can happen exclusively due to very slow diffusion process across the interface. In case on a non-zero vorticity, the closed streamlines are rapidly generated on the interface, enabling rapid transport of the oxidizer across the interface. This process is absolutely necessary for efficient combustion and other chemical reactions.

The flow with $\omega = 0$ is called **irrotational** and the one with $\omega \neq 0$ - **rotational**.

Problem. Calculate vorticity of a flow rotating with angular velocity $\mathbf{\Omega} = \Omega\mathbf{k} = const$

Solution. In this case velocity is: $\mathbf{u} = \Omega \times \mathbf{r} = \Omega(-y\mathbf{i} + x\mathbf{j})$ and $\omega = (\partial_x v - \partial_y u)\mathbf{k} = 2\Omega\mathbf{k} = 2\mathbf{\Omega}$.

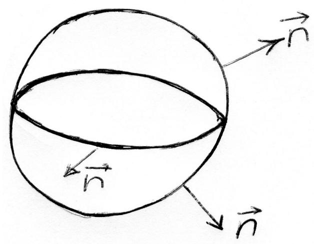
In cylindrical polar coordinates (r, θ, z): $x = r\cos\theta;\ y = r\sin\theta;\ z$ where the unit vectors are defined as: $\mathbf{e}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ and $\mathbf{e}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$:

$$\nabla \times \mathbf{u} = \omega = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) + \mathbf{e}_\theta\left(\frac{\partial u_r}{\partial z} - \frac{\partial w}{\partial r}\right) + \mathbf{k}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \equiv \omega_r \mathbf{e}_r + \omega_\theta \mathbf{e}_\theta + \omega_z \mathbf{k}$$

It follows from this expression that for a potential vortex defined by velocity field $\mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta$, the vorticity $\omega = 0$. Indeed, in this case $u_r = u_z = 0$ and $\omega = \mathbf{k}\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) = 0$.

3 Mass Conservation. Continuity Equation.

Let us consider a fluid volume V inside a closed surface S. Adopting a convention we choose the direction of the unit vector $\mathbf{n}$ out of the volume.



The mass within this volume is

$$M = \int_V \rho(\mathbf{r})dV \tag{3.1}$$

If no sources and sinks are contained inside the surface, the mass variation in unit time is equal to the flux through the surface S:

$$\frac{\partial M}{\partial t} = \int_V dV \frac{\partial \rho}{\partial t} = -\oint_S \rho \mathbf{u} \cdot d\mathbf{s} \tag{3.2}$$

In accord with the divergence theorem the integral over an arbitrary closed surface:

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = \int_V div\ \mathbf{B} dV \equiv \int_V dV \nabla \cdot \mathbf{B} \tag{3.3}$$

where $\mathbf{B}$ is a vector. Using this result, we have:

$$\int_V dV [\frac{\partial \rho}{\partial t} + div\ \rho \mathbf{u}] = 0 \tag{3.4}$$

This integral is equal to zero independently of a particular choice of the volume V. This can happen only if the integrand in (3.4) is equal to zero. This leads to a *continuity equation*:

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + div\ \rho \mathbf{u} = 0 \tag{3.5}$$

which is one of the basic equations of theoretical physics. The importance of (3.5) becomes clear since the derivation presented above can be literally repeated for any conserved scalar space-time -dependent property of a physical system. As a result, the continuity equation is one of the most general and important equations of physics.

In incompressible flows, where the density $\rho = const$, this equation gives *the incompressibility condition:*

$$\nabla \mathbf{u} \equiv div \mathbf{u} = \partial_x u + \partial_y v + \partial_z w = 0$$

Problems. Consider a two-dimensional velocity field: $\mathbf{u} = Axy\mathbf{i} + v(x,y)\mathbf{j}$ *with* $A = const$. *If the flow is irrotational* ($\omega = 0$) *and incompressible, find the* y *-component* v.

Solution.

1. Incompressibility: $\partial_x u + \partial_y v = Ay + \partial_y v = 0$;

2. Irrotational: $\omega = \partial_x v - \partial_y u = \partial_x v - Ax = 0$;

This gives: $v = -\frac{Ay^2}{2} + f_1(x) = \frac{Ax^2}{2} + f_2(y)$. It follows from the second equation that $\partial_y v = \partial_y f_2 = -Ay$. Thus, $f_2(y) = -\frac{Ay^2}{2} + const$. The y component of the velocity field is thus $v = \frac{A}{2}(x^2 - y^2) + const$.

4 Navier- Stokes Equation. Qualitative Derivation.

Now, we again choose a volume V inside a closed surface S and using Newton's law, evaluate the force acting on a fluid element of mass ρdV is equal to momentum chance per unit time:

$$d\mathbf{F} = \frac{d}{dt}(dV\rho\mathbf{u})$$

and, using the expression (2.7) for $\mathbf{a} = \frac{d\mathbf{u}}{dt}$, the total force acting on a volume V:

$$\mathbf{F} = \int_V dV \frac{d}{dt}\rho\mathbf{u} = \int_V dV[\mathbf{u}\frac{d\rho}{dt} + \rho(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u})] \tag{4.6}$$

This force is a sum of all forces acting on a fluid element at point $\mathbf{r}$ and time t. We will subdivide the forces in a few groups. 1. The surface forces $\mathbf{F}_S$ from the surrounding fluid acting on a chosen volume through the interface. We define them in terms of pressure field $p(\mathbf{r})$:

$$\mathbf{F}_S = -\oint_s p(x,y,z)\mathbf{n}ds \tag{4.7}$$

Body forces due to acceleration of the frame of reference, gravity, electromagnetic fields etc:

$$\mathbf{F}_b = \int_V dV\rho\mathbf{g} \tag{4.8}$$

The last force which we would like to introduce here in a *qualitative way* is the viscous force, defined in terms of the viscous stress in the introduction.

$$\mathbf{F}_{visc} = \oint_s \mu\nabla : \mathbf{u}ds \tag{4.9}$$

where $\nabla : \mathbf{u}$ is the vector directed opposite to $\mathbf{u}$ and the derivative is taken in the direction perpendicular to $\mathbf{u}$. Summing up (4.6)-(4.9) and using the divergence theorem we have:

$$0 = \int_V dV \frac{d}{dt}\rho\mathbf{u} = \int_V dV[\mathbf{u}\frac{d\rho}{dt} + \rho(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p - \mu\nabla^2\mathbf{u}] \tag{4.10}$$

By continuity equation (3.5) $\frac{d\rho}{dt} = 0$, which gives:

$$\int_V dV[\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{u}] = 0$$

The force balance is independent upon our choice of the volume V , only if:

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} \tag{4.11}$$

which together with the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\rho\mathbf{u} = 0 \tag{4.12}$$

form the set of the Navier-Stokes equations, governing a vast variety of hydrodynamic phenomena. In the steady flow, where $\rho\mathbf{u} = \mathbf{U}(x, y, z)$, the equation (4.12) gives:

$$\nabla \cdot \rho\mathbf{u} \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{4.13}$$

In the incompressible flow ($\rho = const$), $\nabla \cdot \mathbf{u} = 0$.

4.1 Boundary conditions.

The Navier-Stokes equations are to be solved subject to boundary conditions which, being constraints imposed on fluid mechanics, must often be obtained from microscopic considerations. Typically, interested in a flow over a body, the inlet and outlet boundary conditions are readily defined as a given velocity field of a free flow far from a body. Much more interesting and involved is the problem of boundary conditions at various interfaces. If we define a solid body as a surface , the fluid cannot penetrate, the zero flux boundary condition is reduced to a constraint on a velocity component u_n normal to the solid wall:

$$u_n = 0 \tag{4.14}$$

A more difficult condition is that on a tangential to the solid surface component of velocity field. Let us consider a process of a fluid interaction with a wall. As a result of collision, a microscopic particle (atom/molecule) is adsorbed by a solid, gives away part of its momentum and energy, rapidly coming to thermal equilibrium with it . It means that a particle "forgets" all information about its state prior to the collision and its kinetic energy becomes equal to $K \approx \frac{3}{2}k_B T$. After some time τ, which depends upon the nature of the atom-wall interaction, the particle is randomly emitted back into the flow with a typical velocity $u \approx \sqrt{k_B T/m}$. Since the direction of the particle velocity vector is random and the number of particles in a close proximity to the wall $N >> 1$, the velocity vectors of different particles belonging to a fluid element sum up to a close-to-zero number, defining the so called no-slip boundary condition on a solid surface:

$$\mathbf{u}_{ss} \equiv \mathbf{u}_t = 0 \tag{4.15}$$

If the fluid number density n is small, so that the number of particles in the volume $V \approx \lambda^3$ is not too large, then the no slip condition is violated leading to generation of the slip velocity on a solid wall. Similar effects happen when the velocity of the flow is very large, so that the particles bounce from a wall so rapidly that the thermodynamic equilibrium is not established. The slip velocity is also often formed in biological flows and flows of polymer fluids or solutions, where the relaxation time of achieving the equilibrium is very long and the particle is emitted back to the flow before reaching thermodynamic equilibrium with the solid..

On a force - free surface (for example interface of water with the air) the viscous forces are so weak that can be neglected. This leads to the the following boundary condition:

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \tag{4.16}$$

where the derivative is taken in the direction normal to the interface.

To demonstrate the importance of boundary conditions we consider two flows governed by the equations:

$$\frac{\partial u}{\partial t} = g + \nu \frac{\partial^2 u}{\partial y^2} \tag{4.17}$$

with two different boundary conditions: a. $u(\pm H) = 0$ and b. $u(-H) = 0$; $\frac{\partial u}{\partial y}|_{y=H} = 0$, respectively. The velocity distributions, corresponding to these equations are:

$$u(y) = \frac{gH^2}{2\nu}(1 - (\frac{y}{H})^2)$$

and

$$u(y) = \frac{gHy}{\nu}(-\frac{y}{2H} + 1) + \frac{3gH^2}{2\nu}$$

Compare mass and momentum fluxes in these flows.

To conclude this section we would like to mention that the no-slip boundary condition implies vorticity formation on a solid boundary. Indeed, locally choosing the x axis parallel to the wall, we have at $y = 0$ the velocity $u(0) = 0$ and $u \neq 0$ outside. As a result, $\omega_z = \frac{\partial u}{\partial y} \neq 0$.

5 Ideal flow.

According to the argument presented above, at a solid boundary $\mathbf{u} = 0$. Let us consider a flow over a body See Fig. 1.

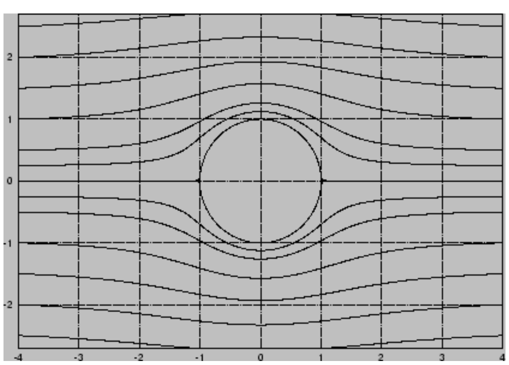


Figure 8: Flow past cylinder of dimensionless radius $R = 1$. Incoming (free-stream velocity $\mathbf{U} = U\mathbf{i}$).

Far from the boundaries, where the velocity difference between the points separated by the distance $L \approx R$ is $\approx \mathbf{U}$, the spatial derivative can be estimated as $\frac{\partial u}{\partial y} \approx U/L$. As a result, the terms in the equation (4.11) (see (5.1)) can be estimated as:

$$\frac{U}{T} + \frac{U^2}{L} \approx \frac{U^2}{L} + \nu \frac{U}{L^2} \tag{5.1}$$

The typical time of the flow variation is of course $O(L/U)$. We find that the viscous term in (5.1) is small when the Reynolds number

$$Re = \frac{UL}{\nu} >> 1 \tag{5.2}$$

In the vicinity of the wall, where $u = 0$, the local value of the Reynolds number is small and the relation (5.2) cannot be correct. We can ask the following question: how close to the wall the inviscid ($\nu = 0$) approximation breaks down? To answer this question, we can use the Taylor expansion near the wall and write a simple expression for the velocity component parallel to the wall:

$$u(y) \approx u(0) + \frac{\partial u(0)}{\partial y} y = \frac{\partial u(0)}{\partial y} y = S_{1,2}(0) y$$

With $y \approx L$, one can introduce a local Reynolds number $Re_y = \frac{S_{x,y}(0) y \times y}{\nu}$ and define the viscous (boundary layer) $Re_y \leq 1$. Thus, *if the normal distance to the solid surface*

$$y \leq \sqrt{\frac{\nu}{|\frac{\partial u}{\partial y}|_{wall}}} \tag{5.3}$$

the flow are dominated by the viscous effects. The wall rate of strain $S_{x,y}(0)$ must be found as a solution to the Navier-Stokes equations subject to boundary conditions and initial conditions. For example in a flow between parallel plates, separated by the gap $2H$, considered above, the velocity derivative at the wall is:

$$|\frac{\partial u}{\partial y}|_{wall} = \frac{g}{\nu H}$$

This general relation can be obtained by integrating (4.17) in the interval $0 \leq y \leq H$ taking into account that at the centerline of a channel ($y = 0$) the derivative $\partial_y u(0) = 0$. Substituting this into (5.3) gives:

$$y \leq \nu \sqrt{\frac{H}{g}} \tag{5.4}$$

This relation tells us that as $\nu \to 0$, the width of the "sublayer" dominated by the viscous effects tends to zero. We will see below that despite this extremely important fact, even in this limit the sublayer plays a crucial part in a flow and viscosity cannot be neglected.

Problem. Introducing a characteristic velocity $U = const$ and the length-scale L, write the Navier-Stokes equations for the dimensionless velocity $V = u/U$ in dimensionless coordinates $\mathbf{X} = r/L$. (For example, to *describe a flow over a body (car, plane..) , it is convenient to take U as a speed of a body and the typical linear dimension of a body as L.*

Solution. The equation for $\mathbf{V}$ is:

$$\partial_\tau \mathbf{V} + \mathbf{V}\nabla_X \cdot \mathbf{V} = -\nabla_X \mathcal{P} + \frac{1}{Re}\nabla_X^2 \mathbf{V}$$

Here $\tau = tU/L$, $\mathcal{P} = \frac{p}{\frac{\rho U^2}{2}}$ and $\nabla_X = \frac{x}{L}\mathbf{i} + \frac{y}{L}\mathbf{j} + \frac{z}{L}\mathbf{k}$ are dimensionless variables. The Reynolds number is defined as: $Re = UL/\nu$. This equation tells us that the similar geometry flows of various fluids differ by the Reynolds number only. This is important from the engineering view-point: we can use the small-scale models in the low viscosity fluids to understand the flow physics of the large- scale scale, provided the Re is kept unchanged. This statement is called *law of similarity* and is summarized as $\mathbf{u} = \mathbf{U}f(\frac{r}{L}, Re)$.

6 Euler Equation.

When the Reynolds number is large enough and far from solid walls, the viscous terms in (5.1) can be neglected and the dynamics are approximately described by the Euler equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{6.5}$$

Using the vector identity:

$$\frac{1}{2}\nabla u^2 = \mathbf{u}\times\nabla\times\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} \tag{6.6}$$

the Euler equation can be rewritten as:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla u^2 - \mathbf{u}\times\nabla\times\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{6.7}$$

If the motion is isentropic (entropy s=const), then the well-known thermodynamic equality for the heat function per unit mass of fluid (enthalpy) w :

$$dw = Tds + Vdp = Vdp = \frac{dp}{\rho} \tag{6.8}$$

where $V = 1/\rho$ is the specific volume, gives:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla u^2 - \mathbf{u}\times\nabla\times\mathbf{u} = -\nabla w \tag{6.9}$$

The Euler equation (6.4) for a fluid in a gravity field reads:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} \tag{6.10}$$

The equation of motions for rotating fluid can be written in a non-moving frame of reference with the boundary conditions including the rotational component to the velocity field. In a rotating frame of reference, this can be avoided. In this case though, the fluid is under the action of the centrifugal acceletaion $\mathbf{a} = -\Omega \times \mathbf{u}$ where Ω is a rotation rate. The Euler equation in this case is:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g} - \Omega \times \mathbf{u} \tag{6.11}$$

To conclude this section, let us derive the equation for vorticity. Taking *curl* of equation (6.6), gives:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u}$$

Problem. Consider an unsteady flow in an incompressible fluid in the gravity field $\mathbf{g}$. The velocity vector is: $\mathbf{u} = U\mathbf{i} + U\cos[k(x - Ut)\mathbf{k}$ and $\mathbf{g} = -g\mathbf{k}$, where U, k and g are constants. Find pressure $p(x, y, z)$ distribution in the flow. *Solution..* The Euler equation in this case is:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - g\mathbf{k}$$

1. $\partial_t \mathbf{u} = U^2 k \sin[k(x - Ut)]\mathbf{k}$.

2. $\mathbf{u} \cdot \nabla \mathbf{u} = (u\partial_x + v\partial_y + w\partial_z)\mathbf{u} = -U^2 k \sin[k(x - Ut)]\mathbf{k} + 0 + 0$

Substituting this into the Euler equation gives:

3. $-\nabla p/\rho - g = 0$.

The pressure distribution is thus: $p = -\rho g z + f(t)$ where $f(t)$ is the function to be found from the boundary conditions.

6.1 Hydrostatics.

Ther hydrostatics $\mathbf{u} = 0$ condition following the Euler equation is:

$$\nabla p = \rho \mathbf{g} \tag{6.12}$$

This equation is to be solved subject to boundary conditions on a pressure field $p(\mathbf{r})$. If $\mathbf{g} = 0$, the pressure field is: $p = const$. If density $\rho = const$, we, choosing the gravity field anti-parallel to the z-axis ($\mathbf{k} \cdot \mathbf{g} < 0$), write the equation:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0; \quad \frac{\partial p}{\partial z} = -\rho g \tag{6.13}$$

with the solution

$$p = -\rho g z + const \tag{6.14}$$

Let us consider a fluid at rest with a surface height denoted by h, to which a constant pressure p_0 is applied. The example is a pond under the normal atmospheric pressure. This condition is sufficient to find a constant in the solution (6.13). Indeed, on a surface we have we have :

$$p_0 = -\rho g h + const; \qquad h = (-p_0 + const)/(\rho g) \tag{6.15}$$

and

$$p = p_0 + \rho g(h - z) \tag{6.16}$$

We see that for an arbitrary bottom topology the surface is a horizontal plane $h = z$.

Problem. Find the shape of water layer surface (an aquarium) moving with acceleration $\mathbf{a} = a\mathbf{i}$. Initial height of the layer when $a = 0$ is H. The fluid is incompressible. *Solution.* Placing the origin at $x = 0; y = 0$, the Euler equation is:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g} - \mathbf{a} \tag{6.17}$$

The hydrostatic condition means :

$$\nabla p = -\rho(\mathbf{g} + \mathbf{a}); \quad \frac{\partial p}{\partial y} = 0; \quad \frac{\partial p}{\partial z} = -\rho g; \quad \frac{\partial p}{\partial x} = -\rho\ a; \tag{6.18}$$

Solution to these equations is:

$$p = f(x, z); \quad p = -\rho g z + f_1(y, x); \quad p = -\rho a x + f_2(y, z) \tag{6.19}$$

where the integration functions are yet to be found. Differentiating (6.18) gives:

$$\frac{\partial f_1(x, y)}{\partial x} = -\rho a; \quad f_1(x, y) = -\rho a x + \varphi(y); \quad p = -\rho g z - \rho a x + \varphi(y) \tag{6.20}$$

Substituting the expression for $f_1(x, y)$ with the integration function $\varphi(y)$ derived above, into (6.18) and differentiating over y gives $\partial_y \varphi(y) = 0$ and $\varphi(y) = const$. Thus,

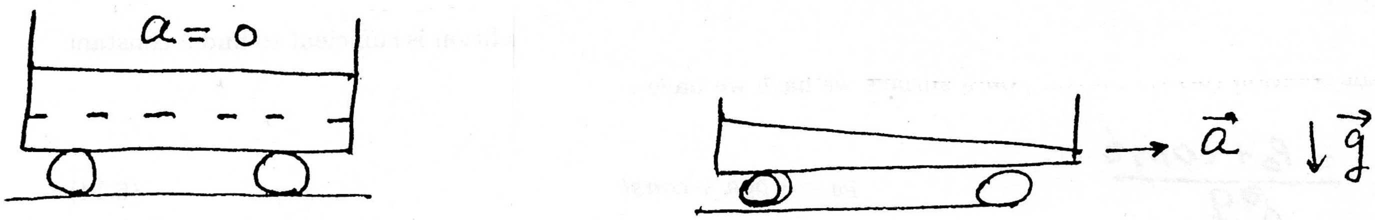
$$p = -\rho g z - \rho a x + const \tag{6.21}$$

Now we have to use the boundary condition on a surface $z = h(x)$: $\quad p(h(x)) = p_0$. This gives:

$$h(x) = -\frac{p_0}{\rho g} - \frac{a}{g} x + const \tag{6.22}$$

The value of the constant is found from the mass conservation: the mass of an accelerating fluid layer is the same as the mass of the layer of the hight $h(x) = H$ with $a = 0$. Substituting $a = 0$ and $h(x) = H = const$ into (6.21) we derive the value of the integration constant and the final expression for the layer surface:

$$h(x) = H - \frac{a}{g}x \tag{6.23}$$



Problem. Find the shape of a surface of an incompressible fluid in a gravitational field, rotating around vertical axis (z-axis) with constant angular velocity Ω.

Solution. The velocity in this case is: $\mathbf{u} = \Omega \times \mathbf{r}$, so that $u = -\Omega y$ and $v = \Omega x$. The flow is steady incompressible. Substituting this into the Euler equation we have:

$$(u\partial_x + v\partial_y)(-\mathbf{i}\Omega y + \mathbf{j}\Omega x) = -\nabla p/\rho + \mathbf{g}$$

or

$$-\Omega^2 y\mathbf{j} - \Omega^2 x\mathbf{i} = -\nabla p/\rho + \mathbf{g}$$

and

$$-\rho\Omega^2 x = -\partial_x p; \quad -\rho\Omega^2 y = -\partial_y p; \quad -\rho g = \partial_z p$$

Solution to these equations is found exactly as in the previous example:

$$\frac{p}{\rho} = \frac{\Omega^2}{2}(x^2 + y^2) - gz + const \tag{6.24}$$

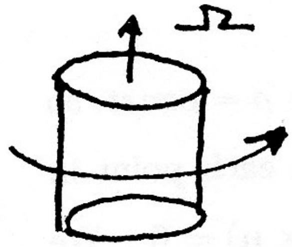
On a free surface $z_s = h(x, y)$ the pressure $p = p_0$ and the equation for the surface is:

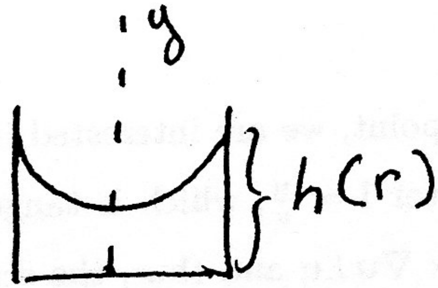
$$z_s = h(x, y) = \frac{\Omega^2}{2g}(x^2 + y^2) \tag{6.25}$$

where the integration constant $const = -p_0/(\rho g)$ is chosen to place the coordinate origin at the lowest point of the surface.

Problem. A sphere of a radius R and mass M is hanging (height $H \gg R$ above surface of water filling a swimming pool. Find shape of the water surface.*

Problem. Consider a constant temperature perfect gas (the simplified model of atmosphere). Find the density variation with height from the ground. *Solution.*





Choosing the vertical z-axis in such a way that $\mathbf{k} \cdot \mathbf{g} < 0$, where $\mathbf{k}$ is the unit vector in the z-direction, the hydrostatic equation reads:

$$\nabla\, p = RT\nabla\rho = -\rho\mathbf{g}$$

The equations for components are:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0; \quad RT\frac{\partial \rho}{\partial z} = -\rho g$$

The solution to this equation satisfying the boundary condition $p(z=0) = p_{at}$

$$\rho = \rho_{at} e^{-\frac{g}{RT}z}$$

is called barometric formula.

Obtaining this result we used an approximation $T = const$. The accepted data on the temperature variation in the atmosphere $0 \le z \le 11000m$ can be approximated by the relation

$$T \approx T_0 - \alpha z; \qquad T_0 \approx 288K; \qquad \alpha \approx 0.006\frac{K}{m}$$

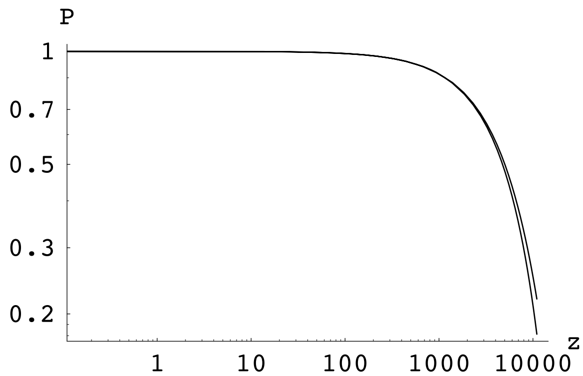
meaning that at $z = 11km$, the temperature is $T \approx 233K$. In this case, the equation for density variation reads

$$\frac{\partial \rho}{\partial z} = -\frac{\rho g}{T_0 - \alpha z}$$

$$\rho(z) = \rho_0 (1 - \frac{\alpha z}{T_0})^{\frac{g}{\alpha R}}$$

The constant $R = k_B/m$ where $k_B \approx 1.38 \times 10^{-23} m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}$. Thus, taking $T_0 \approx 288K$, $g \approx 9.8 m/sec^2$ and the accepted value of mass of the air molecule $m \approx 5.6 \times 10^{-26} kg$, we have $\frac{mg}{k_B T_0} \approx 1.38 \times 10^{-4}$ and $\frac{mg}{k_B}\alpha \approx 6.6$.

The pressure distribution at a constant temperature $T = T_0 \approx 288K$ can be written as $p = p_{atm}e^{-0.000138z} \approx 0.25p_{atm}$ at $z = 11km$. Accounting for the temperature variation with height yields $p = p_{atm}(1 - 2 \times 10^{-5}z)^{6.6} \approx 0.18p_{atm}$ on the same height. As we see, even at the maximum height $z \approx 11km$, the approximation $T = T_0 = const$ is accurate within some 20%. The two pressure distributions are compared on the Fig. xx.



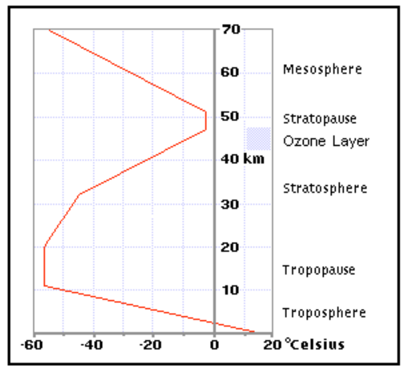


Figure 9: Left: Normalized pressure $P = p/p_{atm}$ variation in normal atmosphere. The two curves corresponding to $T = T_0 = const$ and accepted $T = T(z)$ (see text) are almost indistinguishable differing by some 20% at the hight $z \approx 11km$. Right: Temperature variation in atmosphere $h \leq 11km$

Problem. A body of a volume V and density ρ_c is placed into a pool of fluid of density ρ. If the gravity is g, find the force acting on a body in the hydrostatic conditions. The fraction of a volume submerged in a fluid is $\lambda \leq 1$, so that the underwater volume is $V_i = \lambda V$.

Solution. The gravity force is $\mathbf{F}_g = -\rho_c V \mathbf{k}$. The buoyancy force acting on a body is:

$$\mathbf{F_b} = -\int_S p(x,y,z)\mathbf{n}ds$$

where the integration is carried out over the underwater body surface. In accord with the Gauss theorem:

$$\mathbf{F_b} = -\int_S p(x,y,z)\mathbf{n}ds = -\int_{V_i} \nabla p dV$$

The pressure distribution in a liquid is given by: $p = p(z = 0) - \rho g z$ and as a result $\nabla p = -\rho g \mathbf{k}$.

$$\mathbf{F}_b = \rho g V_i \mathbf{k}$$

Thus, the buoyancy force is equal to the weight of the fluid displaced by the body. The total force is: $\mathbf{F} = (\rho W - \rho_b V_i)g\mathbf{k}$. The hydrostatic condition is $F = 0$. If the entire body is submerged, then $V_i = V$.

7 Bernoulli Equation.

Consider the Euler equation (6.6). At this point, we are interested in an incompressible flow $\rho = const$, so that $w = p/\rho$. Let us introduce a unit vector $\mathbf{l} = \frac{\mathbf{u}}{u}$, which is tangent to the streamline at each point in space and time. It is easy to check that $\mathbf{u} \times \nabla \times \mathbf{u} \perp \mathbf{u}$ and thus, the scalar product $\mathbf{l} \cdot (\mathbf{u} \times \nabla \times \mathbf{u}) = 0$. We also have:

$$\mathbf{l} \cdot \nabla \equiv \mathbf{l} \cdot (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}) = \frac{\partial}{\partial \mathbf{l}}$$

where $\frac{\partial}{\partial \mathbf{l}}$ is a derivative along streamline. In a steady flow where the time-derivative is equal to zero, the scalar product of the vector $\mathbf{l}$ with the Euler equation (6.6) gives:

$$\frac{\partial}{\partial \mathbf{l}}(\frac{1}{2}u^2 + w) = 0 \tag{7.1}$$

This equation means that on a streamline of a steady flow:

$$\frac{1}{2}u^2 + w = const \tag{7.2}$$

The gravity field is accounted for in the Euler equation (6.6) in a following way: Choosing the z-axis with coordinate z-increasing upward, the acceleration due to gravity is:

$$\mathbf{g} = -g\mathbf{k} = -g(\partial_x \mathbf{i} + \partial_y \mathbf{j} + \partial_z \mathbf{k}) = -g\nabla z$$

and $\mathbf{l} \cdot \mathbf{g} = -g\mathbf{l} \cdot \nabla z = -g\frac{\partial}{\partial \mathbf{l}}$. Adding this to (7.1), we find that

$$\frac{\partial}{\partial \mathbf{l}}(\frac{1}{2}u^2 + w + gz) = 0 \tag{7.3}$$

and **on a streamline**:

$$\frac{1}{2}u^2 + w + gz = const \tag{7.4}$$

Problems. Applications. Pitot tube. Venturi. Pressure gauges.

Problem 1. Find velocity of a fluid at the outlet os a straw. The gravity $\mathbf{g} = -g\mathbf{k}$.

Solution. We put the open end of a straw at $z = 0$. Consider a streamline starting on a free surface $z = h_0$, entering a tube at $z = h_0 - d$ and ending at the outlet of a straw ($z = 0$). The Bernoulli equation for the points $z = 0$ and $z = h_0$ on this stream line is:

$$p_{atm} + \rho g h_0 = p_{atm} + \frac{\rho U^2}{2}$$

$$U = \sqrt{2gh_0}$$

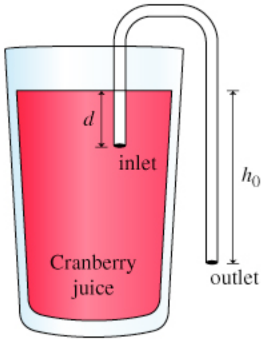


Figure 10: Glass of cranberry juice and a straw (siphon tube)

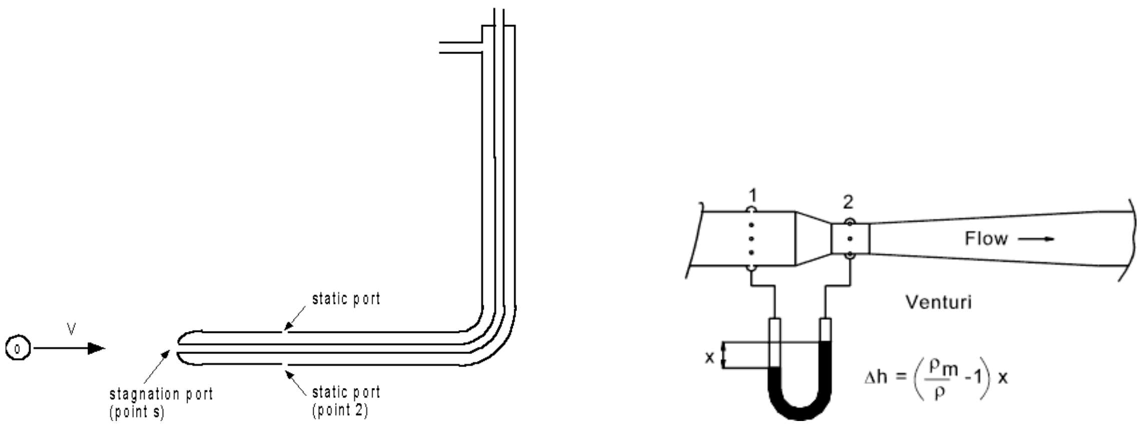


Figure 11: Pitot tube and Venturi meter

Pitot tube, Venturi meter.

On the above Figure you see Pitot tube for determination of the flow velocity and a Venturi meter. The physics of the devices is based on the Bernoulli equation. Read about both and show that if the crossectional areas of the Venturi meter at the points 1 and 2 are A_1 and A_2 , pressure readings are p_1 and p_2, show that the massflux

$$\dot{m} = \rho A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(1 - \frac{A_2^2}{A_1^2})}}$$

8 Kelvin's Theorem.

Consider a closed contour in a flow (Fig. 12). The integral

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} \tag{8.5}$$

taken along this contour is called velocity circulation round that contour. This contour includes fluid particles, moving with a flow. Let us see what happens to the circulation moving with this evolving contour. Circulation can be represented in a discrete form:

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} = \sum \mathbf{u}(i) \cdot \delta \mathbf{r_i} = \frac{(\mathbf{u(i+1)} + \mathbf{u(i)}) \cdot (\mathbf{r(i+1)} - \mathbf{r(i)})}{2} \tag{8.6}$$

where $d\mathbf{l} = \delta\mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i$ is defined in terms of a position vector $\mathbf{r}$ on a chosen contour. The time-derivative:

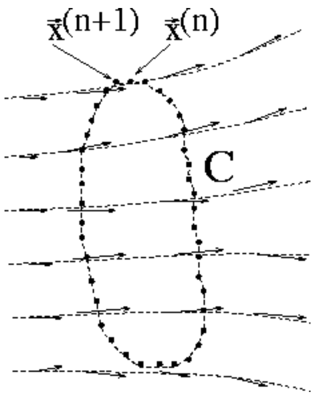


Figure 12: Closed contour used to prove Kelvin's theorem . Here $\mathbf{r} = \mathbf{x}$.

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint \mathbf{u} \cdot d\mathbf{l} = \sum (\frac{d\mathbf{u(i)}}{dt} \cdot \delta \mathbf{r_i} + \mathbf{u(i)} \cdot \frac{d\delta \mathbf{r_i}}{dt}) \tag{8.7}$$

Since

$$\frac{d\delta\mathbf{r}_i}{dt} = \delta \frac{d\mathbf{r}_i}{dt} = \delta \mathbf{u(i)},$$

$\mathbf{u(i)} \cdot \frac{d\delta\mathbf{r}_i}{dt} = \mathbf{u} \cdot \delta\mathbf{u} = \frac{1}{2}\delta u^2 = \frac{1}{2} du^2$. The integral of a total differential of any function is:

$$\int_a^b du^2 = u^2(b) - u^2(a)$$

and as a result the integral of a total differential over any closed contour ($a = b$) is equal to zero. This gives the second term in the right side of (8.7) is equal to zero.
Substituting the Euler equation

$$\frac{d\mathbf{u}}{dt} = -\nabla w = -[\nabla \frac{p}{\rho}] \tag{8.8}$$

into (8.7) gives by the virtue of the Stokes theorem $\oint \mathbf{f} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{f}) \cdot \mathbf{n} dS$:

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint \mathbf{u} \cdot d\mathbf{l} = \oint \frac{d\mathbf{u}}{dt} \cdot d\mathbf{l} = -\oint \nabla w \cdot d\mathbf{l} = -\int_S \nabla \times \nabla w \cdot d\mathbf{S} = 0 \tag{8.9}$$

This result is clear since $\nabla \times (\nabla w) = 0$. Finally, we have:

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} = const \tag{8.10}$$

meaning that *in ideal flow governed by the Euler equation, the velocity circulation round any closed fluid contour is constant in time.*

9 Potential flow.

Consider a flow where at some point vorticity $\omega = \nabla \times \mathbf{u} = 0$. Take a streamline passing through this point and encircle it with an infinitesimally small closed contour of the area dS. If the contour is small enough, the variation of the vorticity vector within the contour can be neglected and, according to the Stokes theorem, the velocity circulation round this contour $\Gamma \approx \omega dS = 0$. Since at each point velocity is tangent to the streamline, this contour will move along the streamline staying small. In accord with Kelvin's theorem, the circulation must stay constant (equal to zero) and we conclude that if vorticity is zero at any point on a streamline, it is zero at all points of this streamline. One may think that if vorticity is equal to zero in the far field (inlet) flow, where all streamlines are generated , then it must be zero at each and every point in the flow.

The flow with $\omega = 0$ is called potential or irrotational, contrary to rotational flow with $\omega \neq 0$ at least somewhere in a flow. Based on the above consideration and vorticity equation derived in Section 6, we may conclude that if the flow is potential ($\omega = 0$) at some instant of time, it must stay potential forever.
This statement is correct only if the fluid does not flow along the boundaries of a solid body where the normal component of velocity field is equal to zero and as a consequence, the velocity vector is tangent to the surface. Thus, the boundary of a solid body is a surface of streamlines which we cannot encircle even by an infinitesimally small closed contour. On this streamline the tangential velocity component, which in the direction perpendicular to the body surface jumps from a finite value to zero, is a discontinous function of space coordinates. For these streamlines the Kelvin theorem and the consideration presented above, are not applicable.

In reality, at a certain point on the surface the dividing streamline separates from the body and continues into the fluid. This allows appearence of the non-zero vorticity regions in a flow. The line of tangential velocity discontinuity separates moving from stationary regions of the fluid or irrotational from rotational regions. An important consequence of these considerations is that the boundary of a solid body corresponds to a closed streamline. In addition,it has been shown above, due to the no-slip conditions on a solid wall, the vorticity at the solid boundary is never equal to zero.
Still, for the so called streamlined bodies, far enough from the surface the Reynolds number is large and vorticity generation can be neglected. This makes the study of the potential flows where vorticity $\omega = 0$ at each and every point an interesting and important subject.

First we notice that due to the Stokes theorem, the integral over any closed contour in a potential flow

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} = \int_S \omega \cdot d\mathbf{S} = 0 \tag{9.1}$$

where S is an arbitrary surface bounded by this contour. This means that in the absence of singularities, potential flow cannot have closed streamlines. This can be proved as following: assume that there exist at least one closed streamline. We can choose the integration contour along this line and, since on a streamline $\mathbf{u} \| d\mathbf{l}$, the integral along the contour cannot be equal to zero. This contradicts the relation (9.1).
Any vector with zero *curl* can be represented as a gradient of a scalar potential ϕ , so that

$$\mathbf{u} = \nabla \phi \tag{9.2}$$

In this case the Euler equation (6.8) can be written as:

$$\nabla(\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + w) = \nabla(\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho}) = 0 \tag{9.3}$$

and

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + w = \frac{\partial \phi}{\partial t} + \frac{\rho}{2}u^2 + \frac{p}{\rho} = f(t) = const \tag{9.4}$$

The velocity field (9.2) is not modified if an arbitrary function of time $F(t)$ is added to the potential. Choosing this function as $F(t) = \int f(t)dt$ removes $f(t)$ from the solution (9.4) without any loss of generality. In the steady flow $\partial_t \phi = 0$ and expression (9.4) becomes similar to Bernoulli's equation (7.4):

$$\frac{1}{2}u^2 + w + gz = \frac{1}{2}u^2 + \frac{p}{\rho} + gz = const \tag{9.5}$$

It is clear from the above derivation that in a potential flow the equation (9.5) is valid at any point, while in general, the Bernoulli equation is satisfied on a streamline only.

9.1 Incompressible fluids.

The above expressions are simplified for potential flows of incompressible fluids where density $\rho = const$. It follows from (9.2) that $div(\mathbf{u}) \equiv \nabla \cdot \mathbf{u} = \nabla \cdot \nabla \phi = 0$ which means that potential function ϕ is a solution to the Laplace equation:

$$\nabla^2 \phi \equiv \triangle \phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\phi = 0 \tag{9.6}$$

and, since for $\rho = const$, $\nabla w = \frac{\nabla p}{\rho}$, Bernoulli's equation reads:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} = f(t) \tag{9.7}$$

These equations must be solved subject to initial and boundary conditions, typically written in terms of prescribed properties of the velocity field at infinity and at the boundary of a solid body where the normal component of the velocity field:

$$u_n = \mathbf{n} \cdot \nabla\phi = \frac{\partial \phi}{\partial n} = 0 \tag{9.8}$$

As we see, if the potential flow approximation is valid, we have, instead of solving the system of four non-linear differential Euler and continuity equations, to solve the Laplace equation with initial and boundary conditions. This task is infinitely simpler since the most powerful methods to solve the Laplace equation have been developed during last two hundred years. Then, given the potential function ϕ, the velocity field is evaluated by a simple differentiation and after that the pressure distribution is found from the Bernoulli equation. With known pressure field, the forces acting on a body are calculated readily. It is interesting that the time variable does not explicitly enters the equation (9.6) but appears only through the time-dependent boundary conditions.

Since the differential operations commute, i.e. $\nabla\triangle\phi = \triangle\nabla\phi = \triangle\mathbf{v}$, the velocity field is a solution of three Laplace equations:

$$\triangle\mathbf{u} = 0 \tag{9.9}$$

The equation for potential is linear and sum of particular solutions to it is also a solution. This is extremely useful for analysis of fluid flows.

10 Two-dimensional potential incompressible flows.

In two dimensions, the potential flow theory can be formulated in terms of the theory of analytic functions. Consider a stream function $\psi(x, y, t)$ defined in a following way:

$$u_x = \frac{\partial \psi}{\partial y};\ u_y = -\frac{\partial \psi}{\partial x} \tag{10.1}$$

The velocity field defined this way automatically satisfies the incompressibility condition $\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$
.

Since $u_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $u_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$, the differential equation for the stremfunction is derived readily:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} = -(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}) \equiv -\nabla^2\psi = 0$$

Thus, two-dimensional potential flow of incompressible fluids satisfies the following relations:

$$\omega = 0;\ \nabla \cdot \mathbf{u} = 0;\ \nabla^2 \phi = \nabla^2 \psi = 0$$

The equation

$$\psi(x, y, t) = const \tag{10.2}$$

defines streamlines. This follows from the relation:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -u_y dx + u_x dy = 0$$

which is a two-dimensional version of a general kinematic relation (2.14) obtained from the geometric considerations. It is easy to show that in a two-dimensional flow

$$\mathbf{u} = \nabla \times \mathbf{\Psi} \tag{10.3}$$

where $\Psi = \psi \mathbf{k}$ is the vector perpendicular to the (x,y)-plane.

Problem. The velocity field is $\mathbf{u} = (x - 4y)\mathbf{i} - (y + 4x)\mathbf{j}$. Show that: a. This flow is incompressible and potential. b. Find potential and stream function; c. Plot a few streamlines.

Solution: $u = \partial_y \psi = x - 4y$ so that $\psi = xy - 2y^2 + f(x)$. We also know that $v = -\partial_x \psi = -(y + 4x) = -(y + \partial_x f)$. The equation for an unknown function $\partial_x f = 4x$ and $f = 2x^2 + C$ and $\psi = xy + 2(x^2 - y^2) + C$. c. The equation for streamlines $\psi = const$ which gives:

$$y^2 - \frac{xy}{2} - x^2 + C = 0$$

and

$$y_{1,2} = \frac{x}{4} \pm \sqrt{\frac{17}{16} x^2 + C}$$

The dividing streamline (C=0) corresponds to a wedge bounded by two strait lines corresponding tp $\pm$ in the above relation..

11 Fundamental solutions to the Laplace equation in two dimensions

. Since the Laplace equation (LE) is a linear equation, the sum of various fundamental solutions is also a solution. This fact will be used below for quantitative description of various potential flows. We begin with two-dimensional flows. In rectangular coordinates:

$$\nabla^2\phi \equiv \triangle\phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\phi = 0 \tag{11.1}$$

and a general solution can be written as $r = \sqrt{x^2 + y^2}$:

$$\phi = A + Bx + Cy + Dxy + Gln(\sqrt{x^2 + y^2}) \tag{11.2}$$

In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$; z; $\theta = \ tan^{-1}\frac{y}{x}$:

$$\triangle\phi \equiv \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{11.3}$$

and

$$\phi = A + Bln(r) + C\theta + D\theta ln(r) \tag{11.4}$$

The velocity field is defined as:

$$u_r = \frac{\partial\phi}{\partial r}; \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{11.5}$$

The integration constants A;B;C;D and G are determined from the boundary conditions for velocity field. Without loss of generality we can set $A = 0$. Below we analyze a few simple cases and broaden the class of fundamental solutions to the laplace equation..

11.1 Uniform Flow

. First we consider the simplest case $A = C = D = 0$, so that

$$\phi = Ux \tag{11.6}$$

The velocity field is found readily from relation ((9.2):

$$u_x = \partial_x\phi = B; \quad u_y = \partial_y\phi = 0 \tag{11.7}$$

If at any point (x_0, y_0) in a flow, the velocity field $\mathbf{u}(x_0, y_0) = U\mathbf{i}$ and $U = const$, then we determine the constant $B = U$ and the solution for the velocity field is:

$$\mathbf{u}(x, y) = U\mathbf{i} \tag{11.8}$$

The velocity field is independent upon coordinates, therefore this solution describes the so called uniform flow. The steramfunction for this flow is found easily:

$$u_x = U = \partial_y \psi; \quad u_y = -\partial_x \psi = 0 \tag{11.9}$$

Solutions to these two equations is:

$$\psi = Uy + p(x); \quad \psi = q(y) \tag{11.10}$$

where the integration functions $p(x)$ and $q(y)$ are found from the following consideration: the solution (11.10) is valid for all values of coordinates x and y. Fixing $x = x_0 = const$ and varying y, gives

$$q(y) = Uy + p(x_0) \tag{11.11}$$

Thus:

$$\psi = Uy + const \tag{11.12}$$

The equation $\psi(x, y) = const$ for the streamlines give $\frac{\psi}{U} = y = const$. The streamlines of a homogeneous flow are the strait lines parallel to the x-axis.

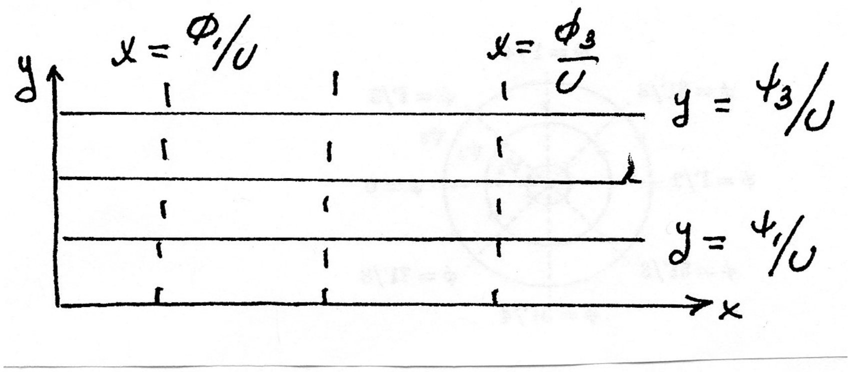


Figure 13: Uniform flow. Streamlines and equipotential lines

11.2 Potential vortex

.

Set $A = B = D = 0$ in (1.4) and, as a result, the potential ϕ is:

$$\phi = C\theta \tag{11.13}$$

The components of the velocity vector are:

$$u_r = \frac{\partial \phi}{\partial r} = 0; \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{C}{r} \tag{11.14}$$

The solution (11.14), describes the so called potential vortex. The streamfunction, found from the relations:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \qquad u_\theta = -\frac{\partial \psi}{\partial r} \tag{11.15}$$

$$\psi = -Cln(r) \tag{11.16}$$

The constant C can be found in a following way. Using the fact that the velocity field is independent on the angle θ , we choose a circular contour of an arbitrary radius r and evaluate the circulation

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{l} = \oint (u_\theta dl + u_r dr) = \int_0^{2\pi} u_\theta r d\theta = \int_0^{2\pi} \frac{C}{r} r d\theta = 2\pi C \tag{11.17}$$

Calculating the integral, we used a simple geometrical fact that on a chosen circular contour the radial coordinate $r = const$ and the length of infinitesimal arc $dl = rd\theta$. Thus,

$$\phi = \frac{\Gamma}{2\pi}\theta; \qquad \psi = -\frac{\Gamma}{2\pi}ln(r) \tag{11.18}$$

and

$$u_r = 0; \qquad u_\theta = \frac{\Gamma}{2\pi r}; \tag{11.19}$$

so that:

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta = \frac{\Gamma}{2\pi r}(-\mathbf{i}\sin\theta + \mathbf{j}\ cos\theta)$$

The streamfunction is independent on the angle θ therefore, the equation for the streamlines $\psi = -ln(r) = const$ is reduced to a simple relation $r = \sqrt{x^2 + y^2} = cost$, which represents circles on the (x, y) plane.

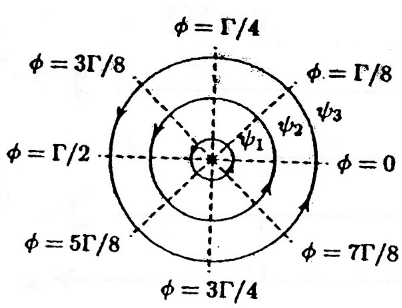


Figure 14: Potential vortex. Streamlines and equipotential lines. The radii of the circular streamlines are $r_i = e^{-2\pi\Psi_i/\Gamma}$

Let us calculate the mass flux through the surface $r = const$ and $0 \leq z \leq H$:

$$\dot{m} = \int \mathbf{u} \cdot d\mathbf{S} = \int \mathbf{u} \cdot \mathbf{n} dS = \int \mathbf{u} \cdot \mathbf{n} dl dz = \int_0^{2\pi} u_r r d\theta \int_0^H dz = 0 \tag{11.20}$$

where the unit vector $\mathbf{n} = \frac{\mathbf{r}}{r}$. The mass flux through cylindrical surface is zero. Indeed,
One feature of the solution (11.19) deserves discussion. While in potential flows the vorticity $\omega = 0$, the circulation Γ generated by potential vortex is not. On the first glance, this contradics the Stokes theorem. The angle-independent vorticity vector in cylindrical coordinates is:

$$\omega = \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) \mathbf{k} \tag{11.21}$$

With u_θ from (11.18), this expression gives $\omega = 0$ everywhere, except at the origin $r = 0$ where the velocity field is divergent. As a result, the Gauss theorem with $\omega = 0$ is applied to any closed countour not containing the singularity at the origin. This singularity will be of crucial importance in aerodynamic applicationsof potential flow theory.

11.3 Source and Sink

Now we discuss the case C=D=0 in (11.14). The potential ϕ is (again A=0):

$$\phi = B ln(r) \tag{11.22}$$

and the velocity field is:

$$u_r = \frac{\partial \phi}{\partial r} = \frac{B}{r}; \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0 \tag{11.23}$$

and

$$\mathbf{u} = u_r \mathbf{e}_r = u_r (\mathbf{i} \cos\theta + \mathbf{j} \sin\theta$$

Evaluation of a massflux through a surface of constant radius r per unit height $H = 1$ gives:

$$Q = \int \mathbf{u} \cdot d\mathbf{S} = \int \mathbf{u} \cdot \mathbf{n} dS = \int u_r \mathbf{e_r} \cdot \mathbf{e_r} dl dz = \int \mathbf{u} \cdot \mathbf{n} dl dz = \int_0^{2\pi} u_r r d\theta = B \int_0^{2\pi} \frac{1}{r} r d\theta = 2\pi B \tag{11.24}$$

and

$$u_r = \frac{Q}{2\pi r}; \qquad u_\theta = 0 \tag{11.25}$$

Using this velocity field, the two equations for the streamfunction

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}; \quad u_\theta = -\frac{\partial \psi}{\partial r} \tag{11.26}$$

yield:

$$\psi(r, \theta) = B\theta + f(r); \quad \psi(r, \theta) = g(\theta) \tag{11.27}$$

, where the integration functions f and g are to be determined from the physics of the problem. It follows from these solution that $B\theta + f(r) = g(\theta)$ thus, these two solutions can coexist at any point $(r, \theta\ )$ only if $g(\theta) = f(r) = const$. This gives:

$$\psi = \frac{Q}{2\pi}\theta + const \tag{11.28}$$

with $\theta = tan^{-1}\frac{y}{x}$. Streamlines following the equation $\frac{2\pi\psi}{Q} = \theta = const$ are the set of radial rays with the origin at $r = 0$. The flow direction is defined by the sign of the amplitude Q. We will call $Q > 0$ and $Q < 0$ sourse and sink, respectively.

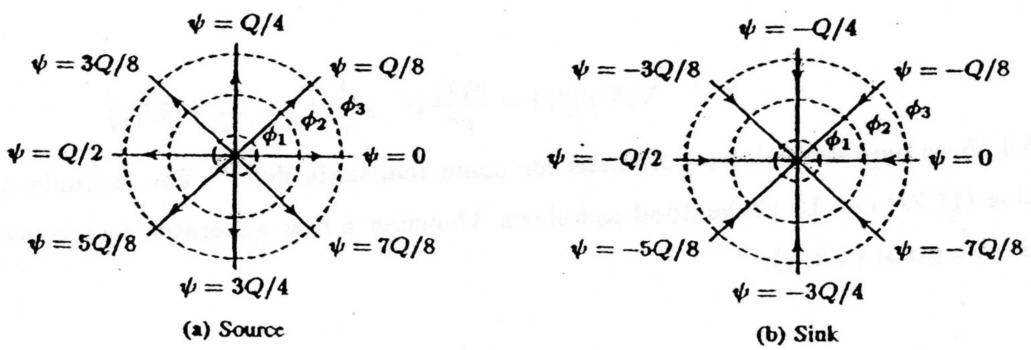


Figure 15: Source and sink. Streamline and equipotential lines

This result can be obtained directly from the mass conservation law. Indeed, consider a z-independent line source at $(\mathbf{r} = 0, z)$. Since mass is not created or annihilated outside this line, the mass flux QH through the surface of a cylinder of a radius $r = const$ and hight $H \to \infty$ is independent upon cylinder radius$r$. The mass flux $QH = \int_0^{2\pi} u_r r d\theta \int_0^H dz = 2\pi H$ leading to the expression (11.23) for the velocity field.

11.4 Doublets. Multipole expansion

Potential ϕ is a solution to the Laplace equation $\triangle\phi = 0$. Since the differential operators commute and if $\Phi(x, y, z, t)$ is a solution to the Laplace equation, for example considered above, , then

$$\Phi_{i,j,k,..}(\mathbf{r}, t) = \nabla_i\nabla_j\nabla_k\cdots\Phi$$

is also a solution to the Laplace equation : $\triangle\Phi_{i,j,k...} = 0$. Let us consider the θ-independent potential $\Phi = \Phi(r)$. Then,

$$\frac{\partial\Phi(r)}{\partial x} = \frac{\partial\Phi(r)}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial\Phi(r)}{\partial r}\frac{x}{r} \tag{11.29}$$

and in general:

$$\nabla\Phi(r) = \frac{\partial\Phi}{\partial r}\frac{\mathbf{r}}{r} = \frac{\partial\Phi}{\partial r}\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r} \equiv \frac{\partial\Phi}{\partial r}\mathbf{n}_r$$

.

Then,

$$\frac{\partial^2\Phi(r)}{\partial x\partial y} = \frac{xy}{r^2}(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r})\Phi$$

and

$$\frac{\partial^2\Phi(r)}{\partial x^2} = \frac{1}{r^2}(x^2\frac{\partial^2\phi(r)}{\partial r^2} + (r - x^2/r))\frac{\partial\phi}{\partial r}$$

and

$$\nabla_i\nabla_j\Phi(r) = \frac{x_i x_j}{r^2}\Phi'' + \frac{1}{r}(\delta_{ij} - \frac{x_i x_j}{r^2})\Phi'(r)$$

All these expressions are potentials for some realizable flows. For example, the physical meaning of solution (11.29) can be understood as follows. Consider a flow generated by a source at a point $(-a, y)$ and a sink at a point $(+a, y)$.

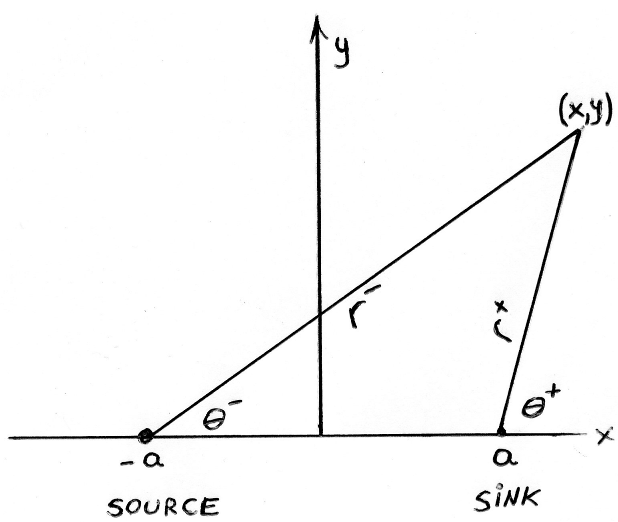


Figure 16: Source and sink separated by a distance $2a$.

The distances from a source and sink to an arbitrary point $\mathbf{r}$ are: $r^- = \sqrt{(x+a)^2 + y^2}$ and $r^+ = \sqrt{(x-a)^2 + y^2}$, respectively. Based on the solutions derived for a previous problem, the potential for this flow is:

$$\phi = \frac{Q}{2\pi}(ln(r^-) - ln(r^+)) = \frac{Q}{2\pi}(ln\sqrt{(x+a)^2 + y^2} - ln\sqrt{(x-a)^2 + y^2}) \tag{11.30}$$

In the limit of a very small displacement $a \to 0$ the Taylor expansion leads to:

$$\phi = \frac{Q}{2\pi} lim_{a\to 0}(ln\sqrt{(x+a)^2+y^2} - ln\sqrt{(x-a)^2+y^2}) \approx \frac{Q}{2\pi}\frac{\partial}{\partial x} ln(r)2a \equiv \frac{D}{2\pi}\frac{\partial}{\partial x} ln(r) \tag{11.31}$$

The derivative $\frac{\partial}{\partial x} ln(\sqrt{x^2+y^2}) = \frac{x}{r^2} = \frac{cos(\theta)}{r} \equiv \nabla_x ln(r)$ and we say that the flow having potential:

$$\phi = \frac{D}{2\pi r} cos(\theta) \tag{11.32}$$

is generated by a doublet. Recalling that $x = rcos\ \theta$, we see that (11.32) is equivalent to (11.29) obtained by a simple differentiation of a source-sink logarithmic potential (11.22).

The same result is obtained far from the sourse and sink when $a = O(1)$ but $a/r \to 0$. Indeed, $r^{\pm} = \sqrt{(x \pm a)^2 + y^2} \approx r(1 \pm \frac{ax}{r^2})$. Substituting this into the expression for potential and recalling that $ln(r(1 \pm \frac{ax}{r^2})) \approx lr(r) \mp \frac{ax}{r^2}$ gives (11.31)-(11.32). This means that the expression (11.32)) (dipole approximation) accurately describes the velocity field far from system sources and sinks, provided the combined strength of the system $Q_1 + Q_2 = Q = 0$.

The streamfunction is derived in a similar manner. It is clear that

$$\psi = \frac{Q}{2\pi}(\theta^- - \theta^+) = \frac{Q}{2\pi}(tan^{-1}(\frac{y}{x+a}) - tan^{-1}(\frac{y}{x-a})) \tag{11.33}$$

In the limit $a \to 0$ this is simply:

$$\psi = \frac{Q}{2\pi}(\theta^- - \theta^+) = \frac{Q}{2\pi}\frac{\partial}{\partial x} tan^{-1}(\frac{y}{x})2a = -\frac{D}{2\pi}\frac{y}{x^2+y^2} = -\frac{D}{2\pi}\frac{sin(\theta)}{r} \tag{11.34}$$

Since $sin(\theta) = y/r$, the streamlines are found from equation:

$$\frac{2\pi\psi}{D} = const = -\frac{y}{x^2+y^2} \tag{11.35}$$

and

$$x^2 + y^2 + \frac{D}{2\pi\psi} y = (y + \frac{D}{4\pi\psi})^2 + x^2 - (\frac{D}{4\pi\psi})^2 = 0 \tag{11.36}$$

The streamlines are the circles of radius $|\frac{D}{4\pi\psi}|$ with the center at $y_c = \frac{D}{4\pi\psi}$. Thus, all these circles touch the point $(x = 0, y = 0)$.

The considerations which led to the above results can be readily generalized to the case of the so called quadruplet or quadrupole used for various important applications:

$$\Phi_{ij} = q\frac{\partial^2 ln(r)}{\partial_{x_i}\partial_{x_j}} = q\frac{(r^2\delta_{ij} - 2x_i x_j)}{r^4} \tag{11.37}$$

It can be shown that flow having this potential is generated in the far field by the two sources and two sinks having combined strength equal to zero.

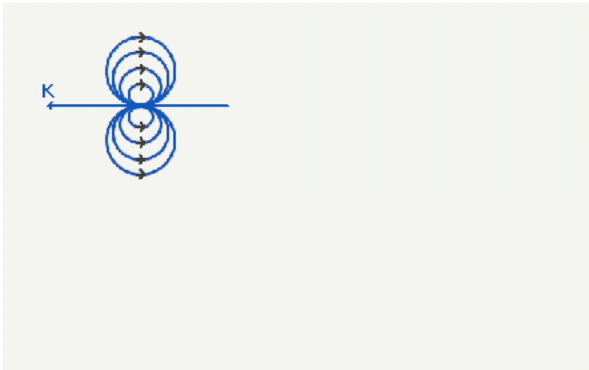


Figure 17: Streamlines in a flow past doublet.

Problem. Consider two sources of the same intensity Q at points $(-a, +a)$; $(+a, -a)$ and two sinks $(-Q)$ at the points $(+a, +a)$; $(-a, -a)$. In the limit $\frac{a}{r} \to \infty$, find potential, stream function, velocity field etc.

Wedge.

It is easy to see that the streamfunction

$$\psi = \psi_0 r^n \sin n\theta$$

with $\psi_0 = const$ and $n = const$ is a solution to Laplace equation and, thus, can describe a fluid flow past a body. To see this, we notice that:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \psi}{\partial r} = \frac{n^2\psi}{r^2}$$

$$\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = -\frac{n^2\psi}{r^2}$$

The Laplacian which is a sum of these two terms is indeed equal to zero. From this we find:

$$u_r = n\psi_0 r^{n-1} \sin n\theta; \qquad u_\theta = -n\psi_0 \sin n\theta$$

Body shape. The dividing streamline $\psi(r, \theta) = 0$, gives the following solutions:

$$\theta = \frac{m\pi}{n}$$

where m is an arbitrary integer. This solution describes strait lines with the slopes $\theta = m\pi$. Consider $m = 0$ and $m = 1$ so that $\theta = 0$ and $\theta = \pi/n$. If $n = 2$, then the flow is confined to the interval $0 \leq \theta \leq \pi/2$. Since, $u_\theta = 0$, the lines $\theta = 0$ and $\theta = \pi/4$ form a wall, the fluid cannot penetrate. See Fig.[xx]. If $n = 4$, the flow is in the domain $0 \leq \theta \leq \pi/4$.

11.5 Flow past doublet.

Let us consider a homogeneous at the inlet $x \to -\infty$ flow over a doublet placed at the origin $r = 0$. The inlet velocity field is: $\mathbf{u} = U\mathbf{i}$ with $U = const$. In this case the potential and streamfunction are readily constructed from the basic solutions (11.32) and (11.6):

$$\phi = Urcos(\theta) + \frac{D}{2\pi}\frac{cos(\theta)}{r} \tag{11.38}$$

and

$$\psi = Ursin(\theta) - \frac{D}{2\pi}\frac{sin(\theta)}{r} \tag{11.39}$$

The velocity field corresponding to this potential is:

$$u_r = Ucos(\theta) - \frac{D}{2\pi}\frac{cos(\theta)}{r^2}; \quad u_\theta = -Usin(\theta) - \frac{D}{2\pi}\frac{sin(\theta)}{r^2} \tag{11.40}$$

This expression describes a flow over doublet of a given strength D.

12 Flow past a cylinder.

The most interesting application of the above result is the case of imposed boundary condition $u_r(R, \theta) = 0$. This happens when $\frac{D}{2\pi} = UR^2$, so that

$$u_r = Ucos(\theta)(1 - \frac{R^2}{r^2}); \quad u_\theta = -Usin(\theta)(1 + \frac{R^2}{r^2}) \tag{12.1}$$

and potential

$$\phi = Urcos(\theta)(1 + \frac{R^2}{r^2}) \tag{12.2}$$

The streamfunction is then:

$$\psi = Ursin(\theta)(1 - \frac{R^2}{r^2}) \tag{12.3}$$

The equation $\psi = 0$ has a solution $r = R$ and $\sin(\theta) = 0$, so that both halves of the x-axis ($x \leq -R$ and $x \geq R$) are on this streamline. Since a closed streamline $r^2 = x^2 + y^2 = R^2$ defines a boundary of a body, we conclude that the flow (12.1) is a flow past two-dimensional cylinder of the radius R with the center at the origin $r = 0$. The flow has two stagnation points ($r = R; \theta = \pi$) and ($r = R; \theta = 0$) where the velocity vector $u = u_\theta = u_r = 0$. It is clear that in this case the normal velocity at the solid boundary $u_n(R) = u_r(R) = 0$. The force on the cylinder must not be generated since the dividing streamline does not break the symmetry of the problem between right and left, up and down etc.

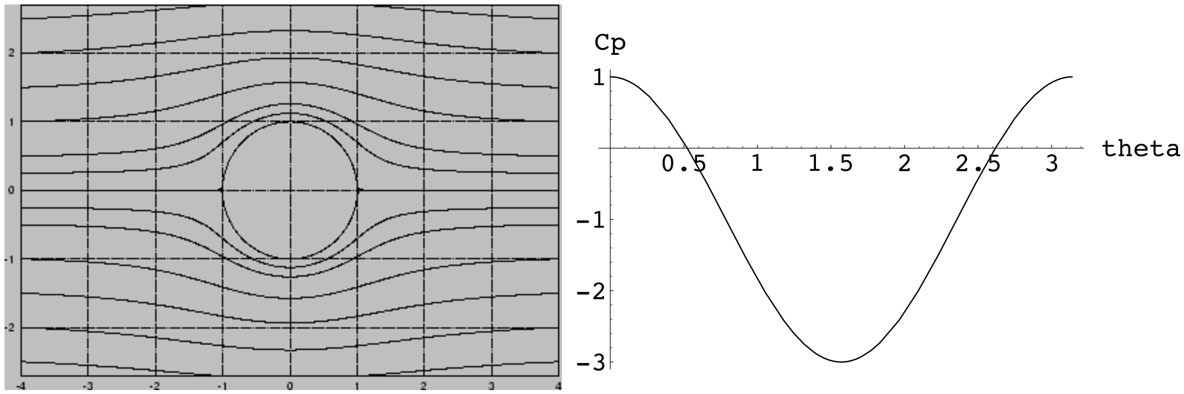


Figure 18: Left: Streamlines for potential flow past cylinder. The stagnation points are at $\theta = 0;\ \pi$. Right: Pressure coefficient for $C_p = 1 - 4\sin^2\theta$

Now, let us calculate the force acting on a cylinder. From the Bernoulli equation (9.5) we derive the surface pressure distribution in a steady flow ($\partial_t \phi = 0$):

$$p - p_\infty = \frac{\rho U^2}{2} - \frac{\rho u(R)^2}{2} = \frac{\rho}{2}(U^2 - u_r^2(R) - u_\theta(R)^2) = U^2\frac{\rho}{2}(1 - 4sin^2(\theta)) \tag{12.4}$$

The force acting on a cylinder is found from the surface integral:

$$\mathbf{F} \equiv D\mathbf{i} + L\mathbf{j} = -\oint_S (p - p_\infty)\mathbf{n}dS = -\oint_S (p - p_\infty)(\mathbf{i}cos(\theta) + \mathbf{j}sin(\theta))dS \tag{12.5}$$

where the drag (parallel to the incoming velocity vector $\mathbf{U}$) and lift (perpendicular to it) forces $\mathbf{D}$ and $\mathbf{L}$, respectively are defined by (12.5). If the length of the cylinder is very large ($H \to \infty$), the integrand is independent upon coordinate z and:

$$\mathbf{F} \equiv D\mathbf{i} + L\mathbf{j} = -\frac{1}{2}\rho U^2 HR \int_0^{2\pi} d\theta(1 - 4sin^2(\theta))(\mathbf{i}cos(\theta) + \mathbf{j}sin(\theta)) = 0 \tag{12.6}$$

The integrls are calculated easily.

$$\int_0^{2\pi} cos\ \theta d\theta = \int_0^{2\pi} sin\ \theta d\theta = 0 \tag{12.7}$$

$$\int_0^{2\pi} sin^2\theta cos\ \theta d\theta = \int_0^{2\pi} d\ sin^3\theta = 0 \tag{12.8}$$

$$\int_0^{2\pi} sin^3\theta d\theta = 0 \tag{12.9}$$

Thus, the force is indeed is equal to zero.

The pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - 4sin^2(\theta) \tag{12.10}$$

often used in engineering, shown on Fig.

This example illustrates the method of using combinations of basic solutions to the Laplace equation to describe potential flows over various solid bodies. To establish the geometry of a body, all one has to do is to find a closed streamline from the equation $\psi(x, y) = 0$. On this streamline (body surface) the normal velocity boundary condition $u_n = 0$ is automatically satisfied. To illustrate this point we discuss the next example.

Three-dimensional potential flow past sphere.

This problem is mathematically more involved. The results are presented here without derivation.

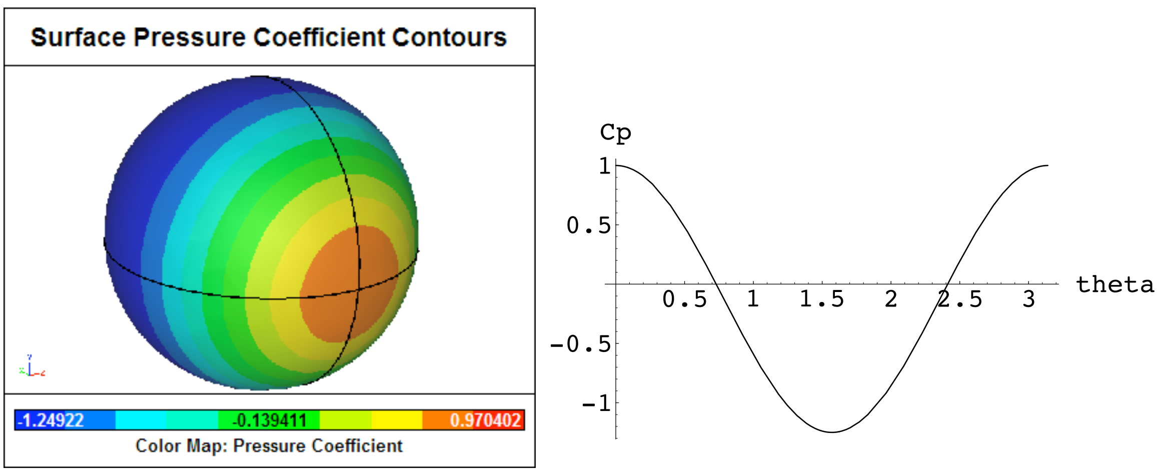


Figure 19: Left: Equipressure surfaces for 3D-flow past sphere. Right: Pressure coefficient $C_p = 1 - \frac{9}{4}\sin^2\theta$.

As we saw above, the total force acting on a cylinder in a potential flow $\mathbf{U} = const$ is equal to zero. Now, we will study a few important cased of potential flows with the non-zero lift and drag, defined asa force parallel and perpendicular to the velocity vector $\mathbf{U}$, respectively.

12.1 Accelerating cylinder.

We consider again an accelerating cylinder moving along the x-axis. In the frame of reference moving with a body, this problem is equivalent to the one of a uniform flow of the time-dependent velocity $\mathbf{U}(t) = U(t)\mathbf{i}$. The potential is exactly the one discussed above :

$$\phi = U(t)(r + \frac{R^2}{r})cos\ \theta \tag{12.11}$$

The velocity components are:

$$u_r = U(t)(1 - \frac{R^2}{r^2})cos\ \theta; \quad u_\theta = -U(t)(1 + \frac{R^2}{r^2})sin\ \theta \tag{12.12}$$

In this case, however, one has to apply the time-dependent Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}u^2 = f(t) \tag{12.13}$$

Since

$$\frac{\partial \phi}{\partial t} = \frac{\partial U}{\partial t}(r + \frac{R^2}{r})cos\ \theta \tag{12.14}$$

we see that at $y \to \infty$ and finite x, where $cos\theta = \frac{x}{r} \to 0$, the time-derivative of potential is zero. Thus, it follows from (12.20)

$$f(t) = \frac{p_\infty}{\rho} + \frac{1}{2}U^2 \tag{12.15}$$

and

$$p = p_\infty + \frac{\rho}{2}(U^2(t) - u_R^2 -_\theta^2) - \rho\frac{\partial \phi}{\partial t} \tag{12.16}$$

Combining this with (12.19) gives:

$$p = p_\infty + \frac{\rho U^2(t)}{2}(1 - 4sin^2\theta) - \rho\frac{\partial \phi}{\partial t} \tag{12.17}$$

On a surface of the cylinder we have then:

$$p = p_\infty + \frac{\rho U^2(t)}{2}(1 - 4sin^2\theta) - 2\rho R\frac{\partial U}{\partial t}cos\ \theta \tag{12.18}$$

The force on a cylinder of unit height $H = 1$ is evaluated as before: we see that the contribution from the term proportioanl to $1 - 4sin^2\theta$ is equal to zero and

$$\mathbf{F} = -\int_0^{2\pi}(-2\rho R\frac{\partial U}{\partial t}cos\ \theta)(\mathbf{i}cos\ \theta + \mathbf{j}sin\ \theta)Rd\theta = 2\pi\rho R^2\frac{\partial U}{\partial t}\mathbf{i} \tag{12.19}$$

In this example we in fact, discussed a flow past a doublet with intensity chosen to mimic a flow over a cylinder of the same density as that of the incoming flow. In this case, the result (12.26) is interesting this it gives

$$\mathbf{F} = 2M\frac{dU}{dt} \tag{12.20}$$

with the mass $2M$ instead of the cylinder mass equal to M. On the first glance this expression contradicts the Newton law . It is not so since the accelerating body has to push the fluid of mass $m_v = M$ around it , which requires an additional force. The mass m_v is called added or virtual mass. In a general case of a flow past a cylinder of density ρ_b, drag force is evaluated from the relation (12.26) with $M = \pi(\rho + \rho_b)R^2$.

12.2 *Cylinder problem revisited.*

The problem considered above has been solved by constructing a solution with subsequent demonstration that it corresponded to the flow past cylinder. It is clear that in flow past a body of general geometry, this procedure is not practical. Indeed, the initial guess we used above was a ”lucky strike, and, in general, one cannot rely upon such luck. So, we have to solve the following problem: Find the velocity field $\mathbf{v}$, generated by a cylinder moving with, in general time-dependent, velocity $\mathbf{u}(\mathbf{t})$. The boundary condition at the wall, not allowing fluid penetration through the cylinder surface at $r = R$ is: the normal to the cylinder surface component of the relative flow velocity $(v - u)_n = (\mathbf{v} - \mathbf{u}) \cdot \mathbf{n} = 0$, where we defined the unit vector $\mathbf{n} = \frac{\mathbf{r}}{r}$. By the Newton law, the force needed to drive the cylinder in the vacuum is $\mathbf{F}_c = m_c \frac{du}{dt}$. Below we would like to calculate the force in the presence of the generated flow.
Since $\ln r$ is a solution to the Laplace equation $\nabla^2 \phi = 0$, the gradient $\nabla \ln r$ is also a solution. However, potential is a scalar and that is why it is easy to see that $\phi = \mathbf{A} \cdot \nabla \ln r$, with the unknown vector $\mathbf{A} = const$ is also solution. Thus: $\nabla r = \frac{\mathbf{r}}{r} = \mathbf{n}$ and:

$$\phi = \mathbf{A} \cdot \mathbf{n}/r$$

The velocity field

$$\mathbf{v} = \nabla \phi = -\frac{2(\mathbf{A} \cdot \mathbf{n})\mathbf{n} - \mathbf{A}}{r^2}$$

The boundary condition on a surface $r = R$ gives:

$$u_n = \mathbf{u} \cdot \mathbf{n} = -\mathbf{A} \cdot \mathbf{n}/R^2$$

So that $\mathbf{A} = -R^2\mathbf{u}$. Therefore, the solution is:

$$\mathbf{v} = \frac{R^2}{r^2}[2\mathbf{n}(\mathbf{u} \cdot \mathbf{n}) - \mathbf{u}]$$

This expression solves the problem. For example, we have:

$$v_r = \mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{r}/r \equiv \mathbf{v} \cdot \mathbf{e}_r = \frac{R^2}{r^2} u_r$$

and

$$v_\theta = \mathbf{v} \cdot \mathbf{e}_\theta = -\frac{R^2}{r^2} u_\theta$$

and

$$v^2 = \frac{R^2}{r^2} u^2$$

so that on the cylinder surface: $v^2 = u^2$.

These relations give the velocity field $\mathbf{v}$ generated by a cylinder moving with velocity $\mathbf{u}$. To find a flow field V generated by a homogeneous flow $u\mathbf{i}$ past a steady cylinder we have to make a transformation to a frame of reference moving with speed $-\mathbf{u}$, i. e. $\mathbf{V} = \mathbf{v} - \mathbf{u}$. The result

$$V_r = u_r(\frac{R^2}{r^2} - 1); \quad V_\theta = -u_\theta(\frac{R^2}{r^2} + 1)$$

which is the case $u = -U\mathbf{i}$ coincides with the solution, we guessed at the begining of this chapter.

The pressure distribution on the cylinder surface is found from the Bernoulli equation:

$$\frac{\partial \phi}{\partial t} + \frac{\rho v^2}{2} + p = f(t) = p_\infty = const$$

Since far from the cylinder $v = 0$ and $p = p_0 \equiv p_\infty$. Now we have to calculate $\frac{\partial \phi}{\partial t}$. In the frame moving with the cylinder (velocity $\mathbf{u}$), we are conducting our calculation

$$\frac{d\phi}{dt} = \frac{d\phi}{d\mathbf{u}} \cdot \frac{d\mathbf{u}}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \mathbf{v}$$

These relations give:

$$\frac{\partial \phi}{\partial t} = -R\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial t} - 2u^2 \cos^2 \theta + u^2$$

and taking into account that $-4\cos^2\theta + 3 = -1 + 4\sin^2\theta$, we obtain finally:

$$p = p_\infty - \frac{1}{2}\rho u^2(1 - 4\sin^2\theta) + \rho R\mathbf{n} \cdot \frac{d\mathbf{u}}{dt}$$

For a cylinder moving along x-axis, $\mathbf{u} = U(t)\mathbf{i}$. The fluid- force acting on a cylinder per unit hight, is:

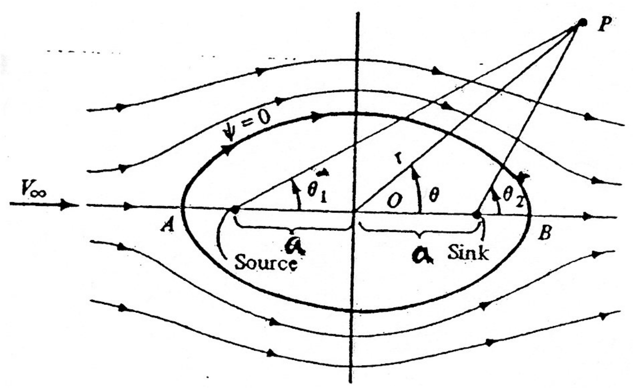
$$F_f = -\int_0^{2\pi} p\mathbf{n}Rd\theta = -\rho R^2 \frac{dU}{dt} \int_0^{2\pi} \mathbf{n}(\mathbf{n} \cdot \mathbf{i})d\theta = -\pi\rho R^2 \frac{dU}{dt}$$

We have calculated the force acting on a fluid by a moving cylinder. The force needed to move both the cylinder and surrounding fluid is: $\mathbf{F}_c = m\frac{d\mathbf{u}}{dt} = \rho\pi R^2 \frac{d\mathbf{u}}{dt}$, so that the total force acting on a cylinder is:

$$\mathbf{F} = \pi R^2(\rho + \rho_c)\frac{d\mathbf{u}}{dt}$$

12.3 Rankine oval.

Consider a flow past a body defined by a source and sink considered above, but separated by a finite displacement a.



Superposition of a uniform flow and a source-sink pair; flow over a Rankine oval.

In this case the streamfunction is given by a combination of (11.12) and (11.28) for both $\pm Q$

$$\psi(x,y) = Uy + \frac{Q}{2\pi}(\theta^- - \theta^+) = Uy + \frac{Q}{2\pi}(tan^{-1}(\frac{y}{x+a}) - tan^{-1}(\frac{y}{x-a})) \tag{12.21}$$

The geometry $y = y(x)$ of the body is found from the equation:

$$\psi(x,y) = Uy + \frac{Q}{2\pi}(\theta^- - \theta^+) = Uy + \frac{Q}{2\pi}(tan^{-1}(\frac{y}{x+a}) - tan^{-1}(\frac{y}{x-a})) = 0 \tag{12.22}$$

The expression $y = y(x)$, defining the shape of the body can readily be found numerically. The equation (12.12) can be simplified. We have:

$$tan(\theta^- - \theta^+) = \frac{sin(\theta^- - \theta^+)}{cos(\theta^- - \theta^+)} = \frac{sin(\theta^-)sin(\theta^+) - cos(\theta^-)cos(\theta^+)}{cos(\theta^-)cos(\theta^+) + sin(\theta^-)sin(\theta^+)} \tag{12.23}$$

Thus, multiplying and dividing this expression by $cos(\theta^-)cos(\theta^+)$ gives:

$$tan(\theta^- - \theta^+) = \frac{tan(\theta^-) - tan(\theta^+)}{1 + tan(\theta^-)tan(\theta^+)} \tag{12.24}$$

Since $\theta^{\pm} = tan^{-1}(\frac{y}{x - \pm a})$ substituting this into (12.13) we obtain:

$$\theta^- - \theta^+ = tan^{-1}(\frac{2ay}{x^2 + y^2 - a^2}) \tag{12.25}$$

and the equation for the body shape is :

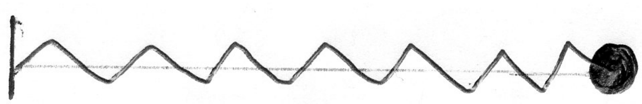
$$\psi = Uy - \frac{Q}{2\pi} tan^{-1}(\frac{2ay}{x^2 + y^2 - a^2}) = 0 \tag{12.26}$$

This relation is further simplified by introducing a constant $\gamma = \frac{2\pi U}{Q}$, so that

$$x = \pm\sqrt{1 + 2ycot(\frac{y}{\gamma}) - y^2} \tag{12.27}$$

Depending on the magnitude of γ, we find (see Fig.(..)) that the flow (...) is a flow past a family of ovals.

Potential flow: Oscillating cylinder. Here we consider application of potential flow theory to a problem of an oscillator in fluid. A point (zero radius) mass m (in our case cylinder) is connected to a support of an infinite mass M by a spring of stiffness κ. The spring is assumed to be very long, so that the influence of support on the motion of a mass m is negligibly weak. (See Fig.xx).



The equilibrium length of a spring is L and if the spring is deformed, so that its length becomes $L + x$, the attraction ($x > 0$) or repulsion ($x < 0$) forces are given by the Hooke's law:

$$f = -\kappa x$$

The Newton law in this case is:

$$m\frac{d^2x}{dt^2} = -\kappa x$$

or

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

where $\omega_0^2 = \kappa/m$. Solution to this equation is:

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

The constants A and B are found from initial conditions for x and $\partial_t x$ at $t = 0$.

In a fluid the force acting on an accelerating body (cylinder), calculated from potential flow theory includes added mass m_{ad} and the Newton equation is:

$$(m + m_{ad})\frac{d^2x}{dt^2} = -\kappa x$$

and

$$\frac{d^2x}{dt^2} + \frac{\kappa}{m + m_{ad}} x = 0$$

and the solution:

$$x = A\cos(\omega_1 t) + B\sin(\omega_1 t)$$

with

$$\omega_1^2 = \frac{\kappa}{m + m_{ad}}$$

We see that the added mass leads to the frequency shift of an oscillator. If a body of mass μ is adsorbed on a surface of an oscillating body, then the total mass becomes equal to $m_{tot} = m + m_{ad} + \mu$ and the resonance frequency shifts to the value $\omega^2 \approx \frac{\kappa}{m + m_{ad} + \mu}$ and if $\mu \ll m$, the final result is:

$$\omega \approx \omega_1 (1 - \frac{\mu}{2(m + m_{ad})})$$

Often, the frequency shift can be measured with high accuracy, thus enabling precise determination of the adsorbed mass μ. This fact has found various applications in nanotechnology and biosensing. This will be discussed on detail later in this course.

13 Flow past cylinder and potential vortex

From the viewpoint of aerodynamic applications, this is one of the most important cases. Consider, for example a curved ball moving with velocity $\mathbf{U}$ and spinning with angular velocity Ω. In real life, due to the action of the friction force, air (or any other fluid) in a close proximity to the ball starts spinning with the same angular velocity Ω, thus generating circular motion of the fluid. In the potential flow approximation, viscosity is equal to zero and this effect is absent. We, however, can study the basic physics of this system by considering a familiar flow past a cylinder with superimposed potential vortex discussed in a previous Section. **This way we, using a mathematically simple potential flow approximation, can mimic the viscous effects described by a much more complex Navier-Stokes equations. Until recently, this remarkably elegant approach, generalized for the bodies of an arbitrary shape, has been at the foundation of modern aerodynamics.**

Potential and streamfunctions are written readily:

$$\phi = Ucos(\theta)(r + \frac{R^2}{r}) + \frac{\Gamma}{2\pi}\theta \tag{13.28}$$

and

$$\psi = Usin(\theta)(r - \frac{R^2}{r}) - \frac{\Gamma}{2\pi}ln(r) \Rightarrow U\sin(\theta)(r - \frac{R^2}{r}) - \frac{\Gamma}{2\pi}ln(\frac{r}{R}) \tag{13.29}$$

Adding a constant term $\frac{\Gamma}{2\pi}ln(R)$ to the expression for the streamfunctions ensures $r = R = const$ as a solution to the equation $\psi = 0$. This means that again we are dealing with the flow over circular cylinder. In this case, however, the velocity field is:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = Ucos(\theta)(1 - \frac{R^2}{r^2}); \quad u_\theta = -\frac{\partial\psi}{\partial r} = -Usin(\theta)(1 + \frac{R^2}{r^2}) + \frac{\Gamma}{2\pi r} \tag{13.30}$$

The force calculation is performed as before:

$$\mathbf{F} = -\int_S (p - p_\infty)\mathbf{n}dS = -\int_S (p - p_\infty)(\mathbf{i}cos(\theta) + \mathbf{j}sin(\theta))dS \tag{13.31}$$

where the integral is evaluated over a surface $r = R$. The pressure, found from Bernoulli's equation is:

$$p - p_\infty = \frac{1}{2}\rho(U^2 - u^2) = \rho\frac{U^2}{2}(1 - (2sin(\theta) - \frac{\Gamma}{2\pi RU})^2) \tag{13.32}$$

The integral (12.41)-(12.42) can be evaluated very simply, especially using the above results, obtained for $\Gamma = 0$. Sum of all contributions to (12.41)-(12.42) which are not proportional to Γ or Γ^2, describing the flow past cylinder without potential vortex, computed above, is equal to zero. The term proportional to Γ^2 is also equal to zero since $\int_0^{2\pi} sin(\theta)d\theta = \int_0^{2\pi} cos(\theta)d\theta = 0$. Thus:

$$\mathbf{F} = -\frac{1}{2}U^2 \int_0^{2\pi} \frac{2\Gamma}{\pi R U} sin(\theta)(\mathbf{i}cos(\theta) + \mathbf{j}sin(\theta))Rd\theta \tag{13.33}$$

and

$$\mathbf{F} = -\rho U \Gamma \mathbf{j} \tag{13.34}$$

This force is called Magnus fource.

As we see, in this case the non-zero generated force is directed perpendicular to both the incoming flow and **vorticity** vector parallel to the cylinder axis. That is why we can write

$$\mathbf{F} = \rho \Gamma \times \mathbf{U} \tag{13.35}$$

This statement, valid for a body of an arbitrary shape, is called Kutta-Joukowskii theorem.
To understand the origin of the force, let us consider the stagnation points. First, we will be looking for the stagnation points at the cylinder surface $r = R$ where $u_r = 0$. The circumferential velocity

$$u_\theta = -2Usin(\theta) + \frac{\Gamma}{2\pi R} = 0 \tag{13.36}$$

and $\theta = sin^{-1}(\frac{\Gamma}{4\pi UR})$. This solution is possible only if $\Gamma \leq 4\pi UR$. We see that in this "subcritical" case, two stagnation points are above or below the x=axis, thus generating the net force in the y-direction. In the supercritical case $\Gamma\pi > UR$, the stagnation point is outside the cylinder surface. To show this, let us consider the zero of the velocity field corresponding to $\theta = \pm\pi/2$ and $r \neq R$. The radial component $u_r = 0$ on the entire vertical axis while the solution to the equation $u_\theta(r, \frac{\pi}{2}) = 0$ is

$$r = \frac{\Gamma}{4\pi U}[1 + \sqrt{1 - (\frac{4\pi RU}{\Gamma})^2}] \tag{13.37}$$

First consider $\theta = \pi/2$ and $\Gamma > 0$. In this case the stagnation point $r = \frac{\Gamma}{4\pi U} + \sqrt{(\frac{\Gamma}{4\pi U})^2 - R^2} > R$ is outside the cylinder surface on the $y > 0$ half-plane. If $\Gamma < 0$, then the stagnation point is on the $y < 0$ half-plane. The stagnation points $r < R$, which are inside the solid cylinder are non-physical and must be neglected.

In all these cases the stagnation point(s) break original symmetry of the problem, thus allowing the non-zero magnitude of the lifting force. This result is the basis for application of potential flow theory to aerodynamic calculations. A few dividing streamlines corresponding to subcritical, critical and supercritical regimes, which are the solutions to equation

$$\psi = Usin(\theta)(r - \frac{R^2}{r}) - \frac{\Gamma}{2\pi}ln(r) = 0 \tag{13.38}$$

are plotted on Fig..

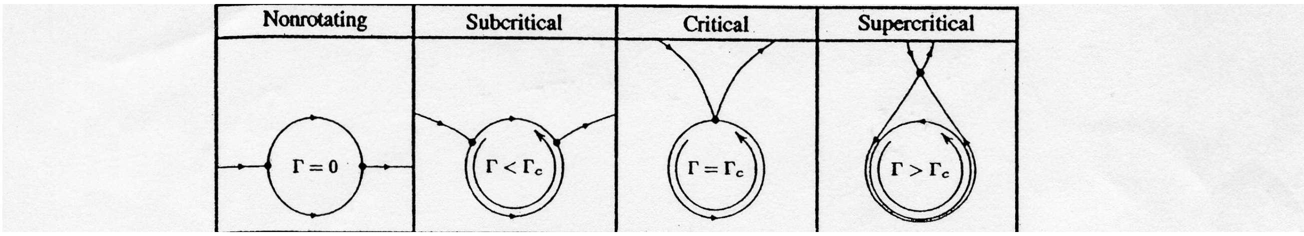


Figure 20: Dividing streamlines for flow past rotating cylinder. Critical $\Gamma_c = 4\pi RU$.

13.1 Flettner rotor ships.

The main idea of Flettner who was a Professor of the famous *Göttingen* University, patented in 1922, was that due to the Magnus effect, small motors powering a ship via rotating cylinders could propel it more efficiently than if they were driven by a conventional propeller. Assisted among others by Albert Benz and Ludvig Prandtl, Flettner constructed an experimental rotor vessel in 1924. It had two cylinders of hight $h = 15m$ and diameter $D = 3m$ driven by electric propulsion system of $50hp$ (30kW) power. In May 1926, made a cross Atlantic trip from Germany to New York. The system, though promising, was not efficient and the project was abandoned. At the present time new ships are being developed in Germany.

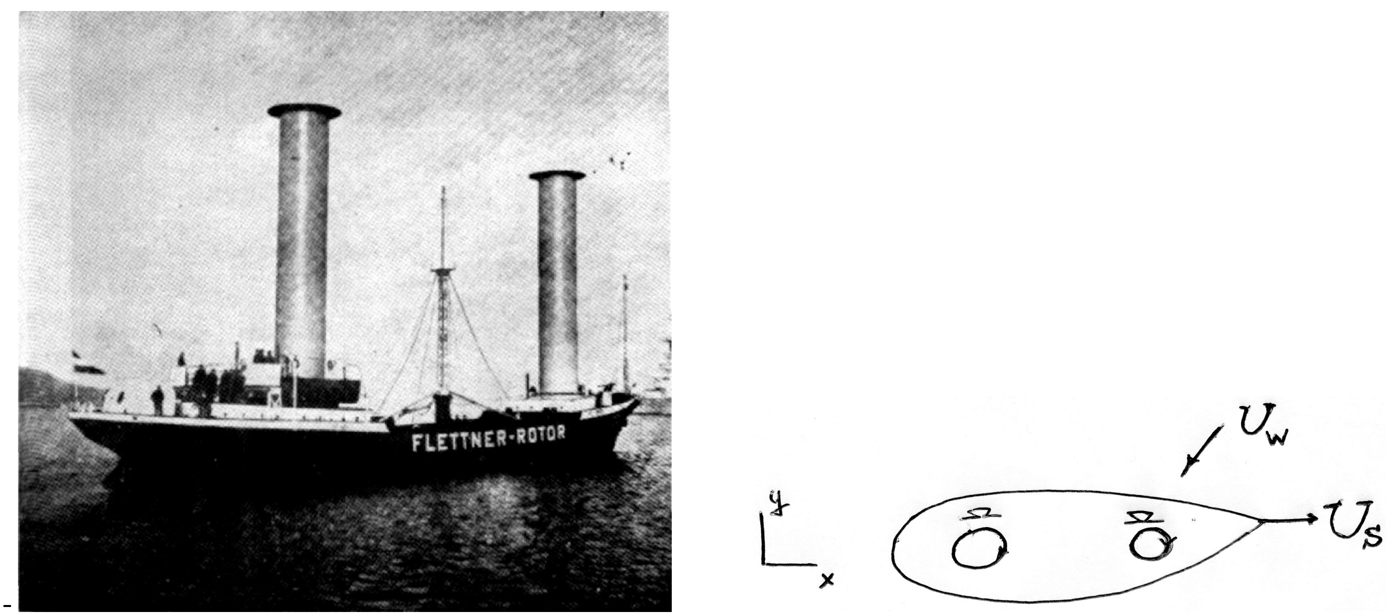


Figure 21: Flettner rotor ship.

Problem. Consider The Flettner-rotor ship. The height of the cylinders is H, their diameter D and angular velocity Ω. The ship sails with velocity $\mathbf{U_s} = U_s\mathbf{i}$ and the wind velocity is $\mathbf{U_w} = -V(\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha$.

a. Using potential flow theory, stimate the force on the ship as a function of all parameter and density ρ. b Find trust if the side force vanishes. Evaluate thrust if $\Omega = 750rpm$, $H = 15m$ and $D = 3m$ and

$U_s = 3knots$ and $V = 12knots$. ($\rho \approx 1.2kg/m^3$).

Solution. According to Kutta-Joukovskii theorem the force acting on each cylinder is:$\mathbf{F} = \rho \mathbf{u} \times \mathbf{\Gamma} H$. The theorem was derived for the **flow past a body**. Thus, it is convenient to operate in the frame of reference moving with the ship velocity:

$$\mathbf{u} = -\mathbf{U_s} + \mathbf{U_w} = (-U_s + V\cos\alpha)\mathbf{i} - V\sin\alpha\mathbf{j}$$

$$\mathbf{F} = -2\rho H[(-U_s + V\cos\alpha)\mathbf{i} - V\sin\alpha\mathbf{j}] \times \Gamma$$

$$\Gamma = (\Omega\frac{D}{2})(\pi D)(-\mathbf{k}) = -\frac{\pi}{2}\Omega D^2\mathbf{k}$$

$$\mathbf{F} = \pi\rho\Omega D^2 H[V\sin\alpha\mathbf{i} - (U + V\cos\alpha)\mathbf{j}]$$

The side F_y force vanishes when $U = -V\cos\alpha$. The thrust is equal to

$$F_x = \pi\rho\Omega D^2 HV\sin\alpha = \pi\rho\Omega D^2 HV\sqrt{1 - \frac{U^2}{V^2}}$$

Problem. A rotating cylinder of mass M, length H and radius R is pitched at $t = 0$ with velocity $\mathbf{U} = U\mathbf{i}$. Find angular velocity Ω needed to sustain horizontal flight. No viscous effects.

Problem. A rotating cylinder of mass M , length H, radius R and angular velocity Ω is pitched in the vertical direction with initial velocity $\mathbf{U} = U\mathbf{j}$. The density of air is ρ_f. Find maximum height if gravity $\mathbf{g} = -g\mathbf{j}$.

Problem. A rotating cylinder of mass M , length H and radius R and angular velocity Ω is pitched with initial velocity $\mathbf{U} = U(\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha)$. Density of the air is ρ_f. Find trajectory of this cylinder if gravity effects are neglected. a. Consider $\alpha = \pi/4$. b. Solve for an arbitrary α.

a.

$$M + m_{add} = m^*$$

$$\frac{du}{dt} = -\frac{2\pi\rho\Omega R^2}{m^*}v \equiv -\omega_0 v$$
$$\frac{dv}{dt} = -\frac{2\pi\rho\Omega R^2}{m^*}u \equiv \omega_0 u$$

The conservation of kinetic energy: multiply first equation by v and second by u. This gives

$$\frac{du^2}{dt} = -\frac{4\pi\rho\Omega R^2}{m^*}uv$$

$$\frac{dv^2}{dt} = -\frac{4\pi\rho\Omega R^2}{m^*}uv$$

and summing up these equations gives $\frac{d}{dt}(u^2 + v^2) = 0$ and $u^2 + v^2 = U^2 = const$. Differentiating these equations over time gives after simple manipulations:

$$\frac{d^2u}{dt^2} = -\omega_0^2 u; \qquad \frac{d^2v}{dt^2} = -\omega_0^2 v$$

$$u = \frac{U}{\sqrt{2}}\cos\omega_0 t + B_x \sin\omega_0 t; \qquad v = \frac{U}{\sqrt{2}}\cos\omega_0 t + B_y \sin\omega_0 t$$

The energy conservation is satisfied if $B_x^2 + B_y^2 = U^2$ and $B_x = -B_y$, thus:

$$B_{x,y} = \pm\frac{U}{\sqrt{2}}$$

$$u_{x,y} = \frac{U}{\sqrt{2}}(\cos\omega_0 t \pm \sin\omega_0 t)$$

$$(x, y) = \frac{U}{\sqrt{2}\omega_0}(\sin\omega_0 t \mp \cos\omega_0 t) + C_{x,y}$$

$$(x - C_x)^2 + (y - C_y)^2 = \frac{U^2}{2\omega_0^2}$$

The trajectory is a circle. Solve for an arbitrary α.

14 Airfoils.

The physics of the flow over cylinder with potential vortex, considered above, is a basis for applications of potential flow concepts to aerodynamics. The expression (13.35) derived for this case is general: if the circulation Γ evaluated along a closed contour along the body boundary is not equal to zero, then the lift force is:

$$\mathbf{L} = -\rho\Gamma \times U \tag{14.1}$$

This statement is called Kutta-Joukowsky theorem. The expression (14.1) can be derived as follows. Let us represent the velocity field as a sum of analytic and singular components, $\mathbf{v}_{reg} + \mathbf{v}_{sing}$, respectively. In general, on a two-dimensional plane, the singular term can be written as:

$$\mathbf{v}_{sing} = \mathbf{e}\frac{f(z)}{2\pi(z-a)} \tag{14.2}$$

where a is a position of singularity (simple pole) and $\mathbf{e}$ is a unit vector in the direction of the singular component to the velocity field. Then, the circulation evaluated as an integral over a contour containing this singularity is:

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = f(a)\mathbf{e} \tag{14.3}$$

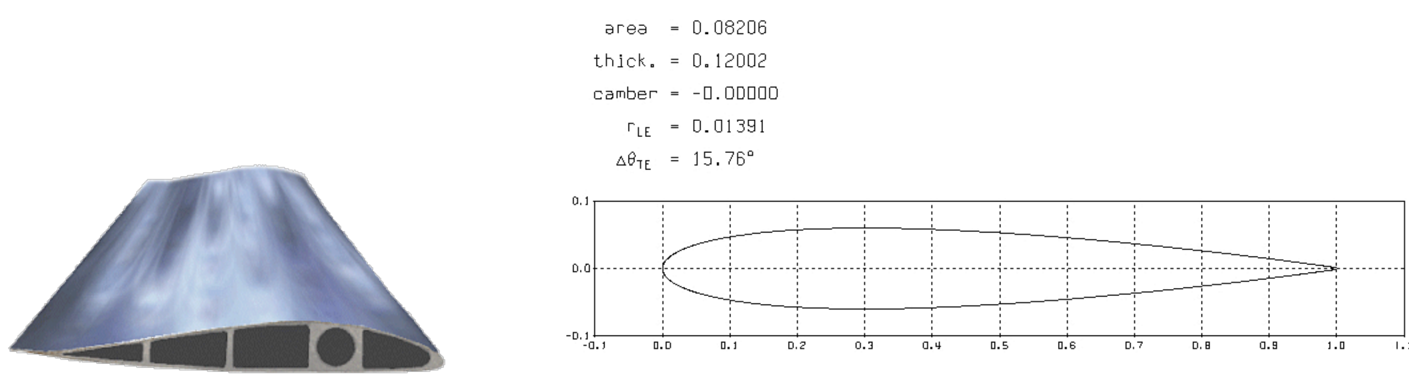


Figure 22: Left: NACA0012 turbine blade. Left: NACA0012 wingsection.

14.1 Nomenclature

A typical NACA airfoil is shown on the Figure.

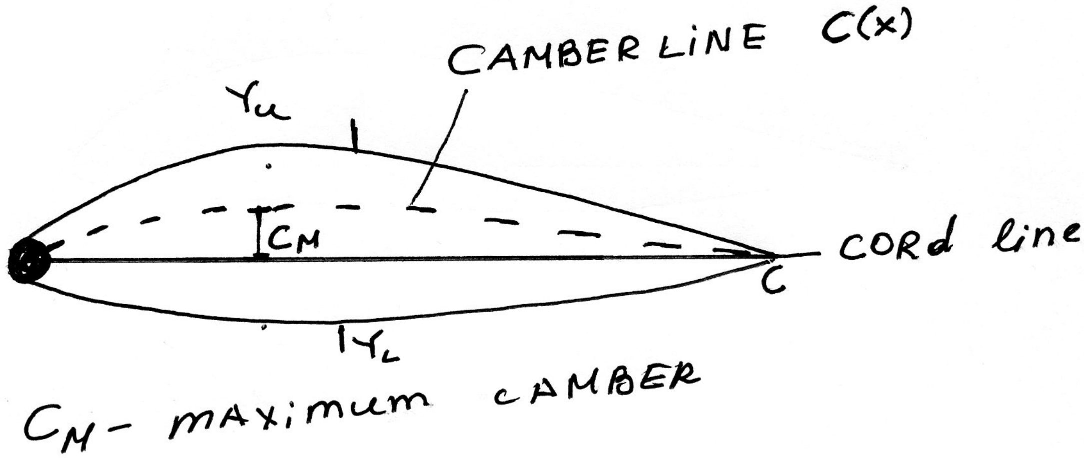


Figure 23: Illustration to nomenclature of NACA airfoils.

Introducing wo function $Y_u(x)$ and $Y_l(x)$ representing the shapes of the foils at $y > 0$ and $y < 0$, respectively, we define the mean camber line

$$C(x) = \frac{1}{2}(Y_u(x) + Y_l(x))$$

We can see that for symmetric airfoils $C(x) = 0$. The airfoil thickness is given by

$$T(x) = \frac{1}{2}(Y_u(x) - Y_l(x))$$

The 4-digit foils, for example, NACA 2412, are defined as follows: The first digit gives maximum camber in percentage of the cord length, i.e. in this case 2 means that $c_{max} = 0.02c$. Next one shows position of c_{max} in tenths of cord: 4 stands for $x(c_{max} = 0.4c$ and the last to give maximum thickness or $T_{max} = 0.12c$. The 5-digit NACA airfoil ....

In this section we are interested in applications of potential flow theory to flows past airfoils. The problem is formulated as follows: consider a wing of a mass M and span length (perpendicular to the page) W. For this wing to cruise with a speed U and angle of attack α, the lift $L = M$. It is clear that if the circulation Γ evaluated over the contour coinciding with a dividing streamline is equal to zero, no lift can be generated. *The main idea behind application of potential flow theory to aerodynamics is construction of a potential flow past a body (airfoil, plane etc) superimposed with a set of vortices distributed over the body surface. This way the nonzero lift force acting on a body can be generated, making the potential flow theory so useful. We have stated above that due to the no-slip condition on a solid boundary in a viscous flow, the vorticity is* *generated on a solid surface. Thus, the vortices placed on a body surface* **mimic** *the viscous effects absent in a potential flow.*

The number of possible combinations of potential vortices leading to a correct magnitude of the lifting force is infinite. The main question is how to choose the vortex distribution representing the physically plausible situation corresponding to a given lift and geometric characteristics of a body. Discussing the flow past a cylinder and a vortex, we saw that positions of stagnation points crucially depended upon magnitude of the circulation Γ and for $\Gamma \neq \Gamma_{cr}$ the singular "kinky" streamlines, ending on a body surface, were formed.

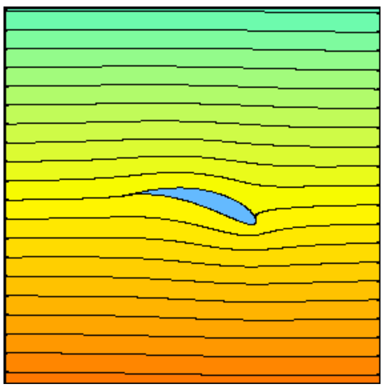


Figure 24: The streamline in a flow (from right to left) past airfoil. The rear stagnation point is exactly at the trailing edge.

Experimental data disagree with this effect: the observed streamlines always smoothly leave the trailing edge of a streamlined body. This fact can easily be understood. Consider a "kink" on a stream function $\Psi(x, y)$ at a point x_0, y_0. In the vicinity of this point $\Psi(x, y)$ is a continous function. However, the derivative $u = \frac{\partial \Psi(x_0)}{\partial x}$ has a finite discontinuity $\delta\Psi(x_0) = \delta u = O(1)$. Then , the second derivative $\frac{\partial^2 \Psi(x_0, y_0)}{\partial x^2} = \lim_{\Delta \to 0} \frac{\delta u}{\Delta} \to \infty$. This means that in the vicinity of the kink, the viscous dissipation tends to infinity thus effectively eliminating the kink and smoothing both the streamfunction and velocity field. This has indeed been observed in all flows. *This empirical fact can serve as a condition (Kutta condition) for the physically allowed circulation distribution on a body surface required for the forcing generation.* (For additional illustration, see next page)

14.2 Vortex sheets.

We illustrate the concept on a very important and simple example. Consider a flat plate with a cross-section presented on the Fig. We call the interval $0 \leq x \leq c$ a cord line with $x = c$ standing for cord length. We cover this plate with potential vortices with the circulation density $\gamma(x) = \frac{d\Gamma}{dx} = u_u(x) - u_l(x)$, so that circulation corresponding to the vortex on an infinitesimal element dx given by the differential circulation $d\Gamma = \gamma(x)dx$.

From the above figure, we see that

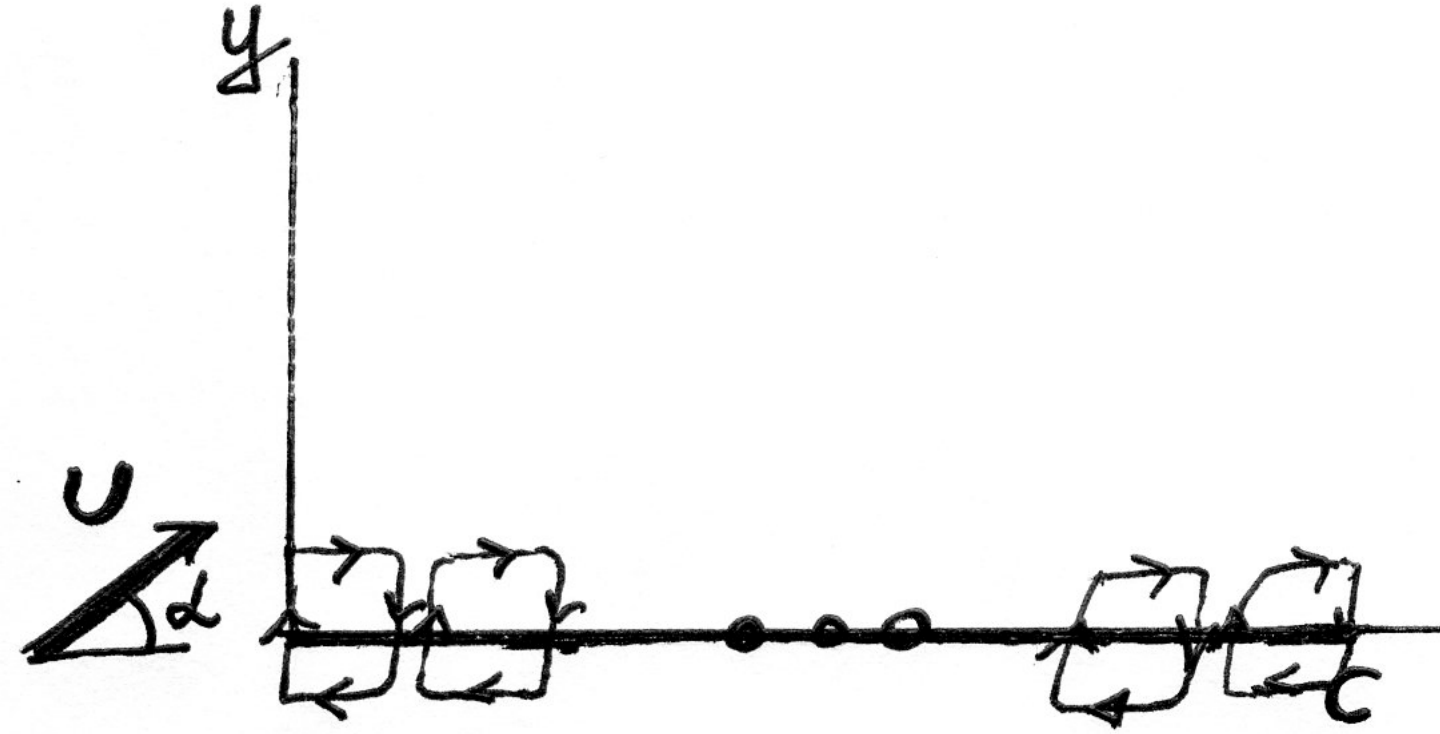


Figure 25: .

$$\Gamma = \oint u(x)dx = \sum_i (u_u(l_i) - u_l(l_i))\Delta = \int_0^c \gamma(x)dx = \int_0^c (u^+ - u^-)dx \quad (14.4)$$

This result becomes clear if we notice that $dl^i l_u = -dl_d^i$ and the vertical components of the velocity field from the neibouring vortices, canceling each other, do not contribute to the integral. Thus, the vortex placed on a point l of a contour is responsible for velocity discontinuity on this contour and for violation of the Kelvin theorem. This effect enables generation of the net force. The potential induced by the vortex sheet at a point $P = (x, y)$ (see Fig) is:

$$\phi(x,y) = -\int_0^c \frac{\gamma(\xi)}{2\pi}\theta(x-\xi, y)d\xi = -\frac{1}{2\pi}\int_0^c \gamma(\xi)tan^{-1}\frac{y}{x-\xi}d\xi \quad (14.5)$$

with the induced velocity field:

$$u(x,y) = -\frac{1}{2\pi}\int_0^c \gamma(\xi)\frac{\partial\theta}{\partial x}d\xi = \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)y}{(x-\xi)^2+y^2}d\xi \quad (14.6)$$

and

$$v(x,y) = -\frac{1}{2\pi}\int_0^c \gamma(\xi)\frac{\partial\theta}{\partial y}d\xi = -\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)(x-\xi)}{(x-\xi)^2+y^2}d\xi \quad (14.7)$$

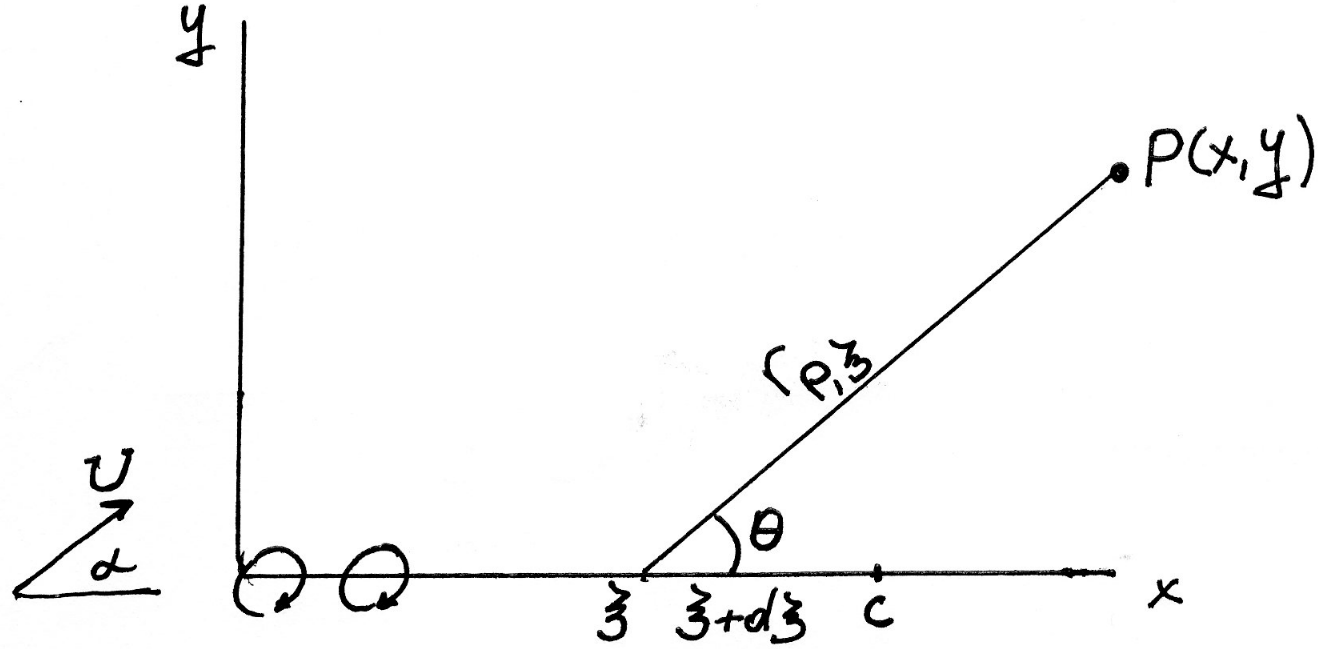


Figure 26: .

Now we prove a relation which will be useful below:

$$\lim_{y \to 0} \frac{y}{(x-\xi)^2 + y^2} = \pi\delta(x-\xi)$$

Indeed, when $x \neq \xi$, the limit is equal to zero and when $x = \xi$, to infinity. In addition,

$$\int_{-\infty}^{\infty} dy \frac{y}{(x-\xi)^2 + y^2} = \pi$$

independently upon magnitude of difference $x - \xi$. Thus, we have proved that the expression is indeed one of the representations of δ-function. Let us calculate the circulation on a contour around the plate in the limit $y \to 0$. It follows from this expressions that the vertical component v is symmetric under transformation $y \to -y$, $v_u = v_l$ and as a result v is a continuous function across the plate. On the other hand, we see that $\Delta u = u_u(x) - u_l(x) = 2u = \gamma(x)$. Now we take an arbitrary point $(0 \leq x \leq c, y \to 0)$ and conclude that the velocity component normal to the plane induced on the plate by the entire circulation distribution is

$$v = -\int \frac{\gamma(\xi) dl}{2\pi(x-\xi)} \quad ; \quad u(x, 0^{\pm}) = \pm\gamma(x)/2 \tag{14.8}$$

where in the limit $y \to 0$, we call $u_{u,l}(x,0) = u(,x0^{\pm})$, respectively.

Now we consider a homogeneous flow past the plate at $0 \le x \le c$ subject to boundary condition: as $r \to -\infty$, $\mathbf{U} \to U(\mathbf{i}\cos\alpha + \mathbf{j}\ sin\alpha)$. Since a surface of a solid body is a streamline corresponding the streamfunction $\Psi(x, y) = 0$, the normal component v_n of the total (freestream and induced by the vortex sheet) of velocity field must be equal to zero, we derive:

$$\mathbf{v}(\mathbf{x}, \mathbf{y} = \mathbf{0}) = \int \frac{\gamma(\xi) dl}{2\pi(x - \xi)} = U \sin\alpha \tag{14.9}$$

This is an integral equation for the function of interest $\gamma(x)$. The solution to this equation is:

$$\gamma(\xi) = 2U\sqrt{\frac{c - \xi}{\xi}} \sin\alpha \tag{14.10}$$

This remarkable expression automatically satisfies the Kutta condition $\gamma(c) = 0$ giving the smooth velocity distribution at the trailing edge.

Thus, the circulation

$$\Gamma = \int_0^c \gamma(\xi) d\xi = 2U \sin\alpha \int_0^c \sqrt{\frac{c - \xi}{\xi}} d\xi$$

and, in accord with the Kutta -Joukovsky theorem, lift lift per unit span is:

$$L = 2\rho U^2 \sin\alpha \int_0^c \sqrt{\frac{c - \xi}{\xi}} d\xi = 2\rho U^2 \sin\alpha\pi c/2 \approx \pi\rho U^2 c\alpha \tag{14.11}$$

With the lift coefficient

$$c_l = \frac{L}{\frac{\rho}{2}U^2 c} = 2\pi \sin\alpha \approx 2\pi\alpha \tag{14.12}$$

This is an exact analytic solution of the problem of a potential flow over flat plate.

Evaluation of the integral (14.11). Introduce a new variable $z = \xi/c$ and $q = z^2$. This gives:

$$\int_0^c \sqrt{\frac{c - \xi}{\xi}} d\xi = c \int_0^1 \sqrt{\frac{1 - z}{z}} dz = 2c \int_0^1 \sqrt{1 - q^2} dq \tag{14.13}$$

With $q = \cos\theta$, this integral is:

$$2c \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = c\pi/2$$

leading to (14.12).

Alternative derivation of the integral. Take (14.13) and introduce new variables:

$$z = \frac{1}{2}(1 - \cos\theta); \qquad dz = \frac{1}{2}\sin\theta$$

so that

$$c\int_0^1 \sqrt{\frac{1-z}{z}}dz = \frac{c}{2}\int_0^\pi \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\sin\theta d\theta = \frac{c}{2}\int_0^\pi \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}\sin\theta d\theta = \frac{c}{2}\int_0^\pi (1+\cos\theta)d\theta = \frac{c\pi}{2}$$

In these variables, the solution (14.10) is written as:

$$\gamma(\theta) = 2U\alpha\frac{1+\cos\theta}{\sin\theta}$$

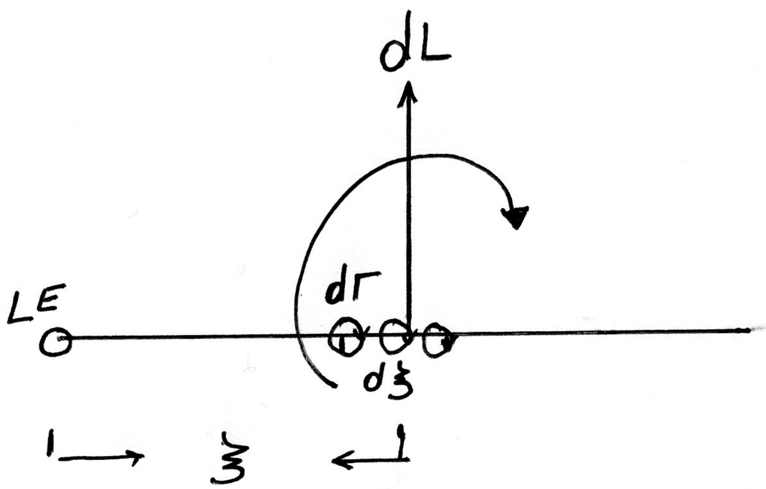


Figure 27: Evaluation of the moment about leading edge. Schematic.

These representation will be useful for evaluation of the moment about leading edge per unit span:

$$M'_{LE} = -\int_0^c \xi dL = -\rho U\int_0^c \xi\gamma(\xi)d\xi = -2\alpha U^2\frac{c^2}{4}\int_0^\pi \frac{1-\cos\theta)(1+\cos\theta)}{\sin\theta}\sin\theta d\theta \tag{14.14}$$

and

$$M'_{LE} = -\pi\rho U^2 c^2\alpha/4 \tag{14.15}$$

with the leading edge moment coefficient

$$c_{m,LE} = \frac{M'_{LE}}{\frac{\rho}{2}U^2c^2} = \frac{\pi}{2}\alpha = -\frac{c_l}{4} \tag{14.16}$$

It is useful to calculate the moment about a point $\xi = c/4$. It is clear that the procedure is identical to the one outlined above but with

$$dM'_{c/4} = (\xi - \frac{c}{4})dL$$

with the result:

$$M'_{c/4} = 0; \qquad\qquad c_{m,\frac{c}{4}} = 0 \tag{14.17}$$

Therefore, the point (line) $x = c/4$ is an **aerodynamic center"** of a flat plate. The results derived above are valid only for the small angle of attacks. In this limit the flow is attached over the entire plate and the vortex-producing flow-separation phenomenon does not happen. In this limit in all expressions derived above we can set $\sin\alpha \approx \alpha$.

The expression (14.12) derived here for a simple case of a potential flow past flat plate can be applied a flow past "thin airfoil", i.e. body of the type shown on Fig.22, provided its maximum thickness T (dimension perpendicular to the cord line) is small, i.e. $T/c \ll 1$. This is demonstrated on a figure below.

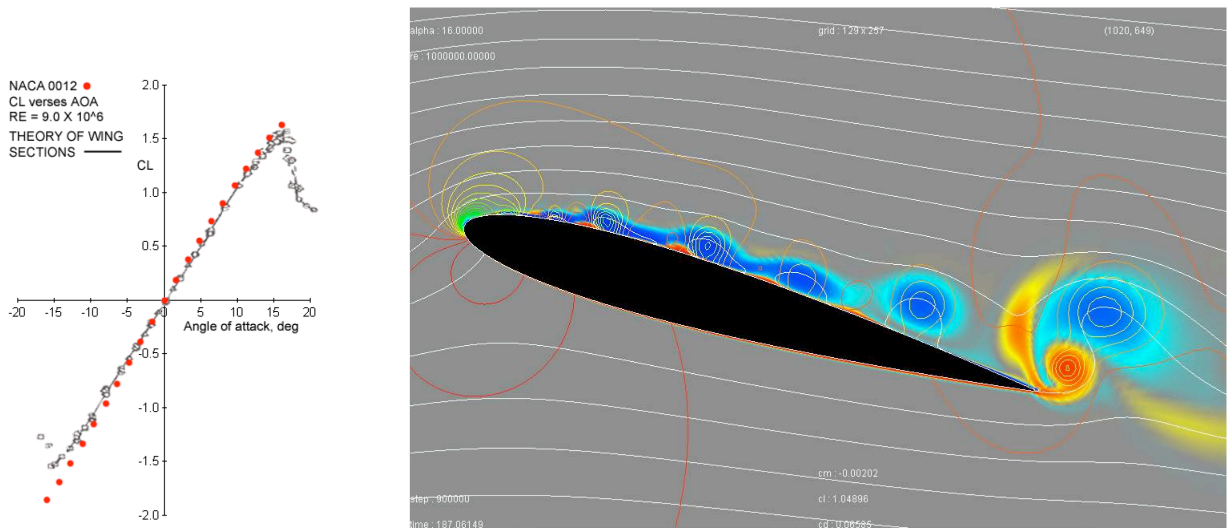


Figure 28: NACA0012. Left: Lift coefficient as a function of angle of attack α. Dots: expression (4.12). Line:experimental data. In the range $-15^o \leq \alpha \leq 15^o$, the agreement is very good. Right: Large angle of attack. Stall.

As we see, the data on an airfoil agree well with a simple relation (14.12) obtained for a flow past a plate. At the large angles of attack the streamlines are separated forming strong vortices. In this regime the potential flow approximation breaks down and a flow can be described only using full Navier-Stokes equations of viscous fluid.

Problem. Consider an airfoil of mass $M = 20000kg$ and wing span $W = 50m$. Find an angle of attack α if the low- altitude ($\rho \approx 1.2kg/m^3$) cruising velocity $U = 150m/sec$. Is the thin airfoil theory applicable?

Solution:

$$Mg = 2\pi\alpha W \frac{\rho U^2}{2} c$$

This gives $\alpha \approx 0.027rad \approx 1.54^o < 15^o$. For these parameters, expression (14.12) is accurate.

14.3 Thin airfoil theory

.

Now we generalize the approach developed above to the case of a two-dimensional airfoil which is basically a curved closed surface (see Fig on page 42.). As in the case of flat plate we will call the segment of the x-axis $o \leq x \leq c$ passing from the leading edge to the trailing edge the cord line and the upper and lower surfaces given by two curves $Y_u(x)$ and $Y_l(x)$ respectively. The half thickness $T(x)$ and camber line $C(x)$ are defined as;

$$C(x) = \frac{1}{2}(Y_u(x) + Y_l(x)); \quad T(x) = \frac{1}{2}(Y_u(x) - Y_l(x)) \tag{14.18}$$

We see that for symmetric airfoil $C(x) = const$ and $\partial_x C(x) = 0$.

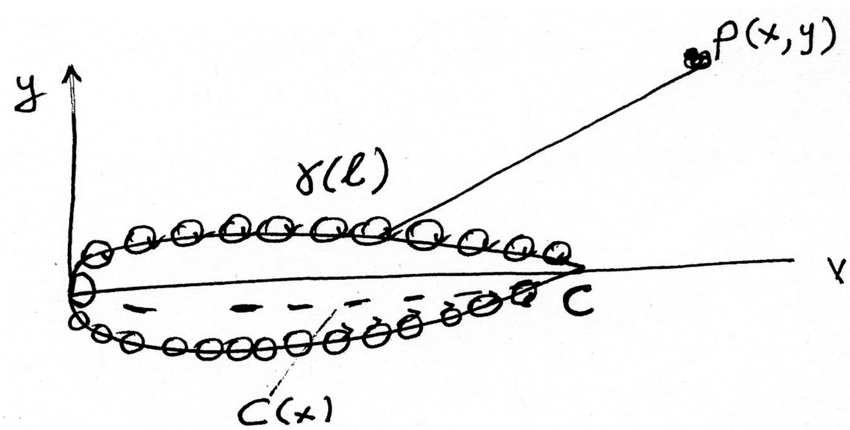


Figure 29: Airfoil surface as a vortex sheet.

The potential at a point P is evaluated according to the general rules of potential theory:

$$\phi(x, y) = \phi(P) = -\frac{1}{2\pi} \int_a^b \theta(l)\gamma(l)dl \tag{14.19}$$

In this section we are interested in a thin airfoil $T/c \ll 1$. In this case, the vortices on the upper and lowe surfaces act in opposite directions and with a good accuracy we can substitute the foil with a vortex sheet placed on a **mean camber line**.

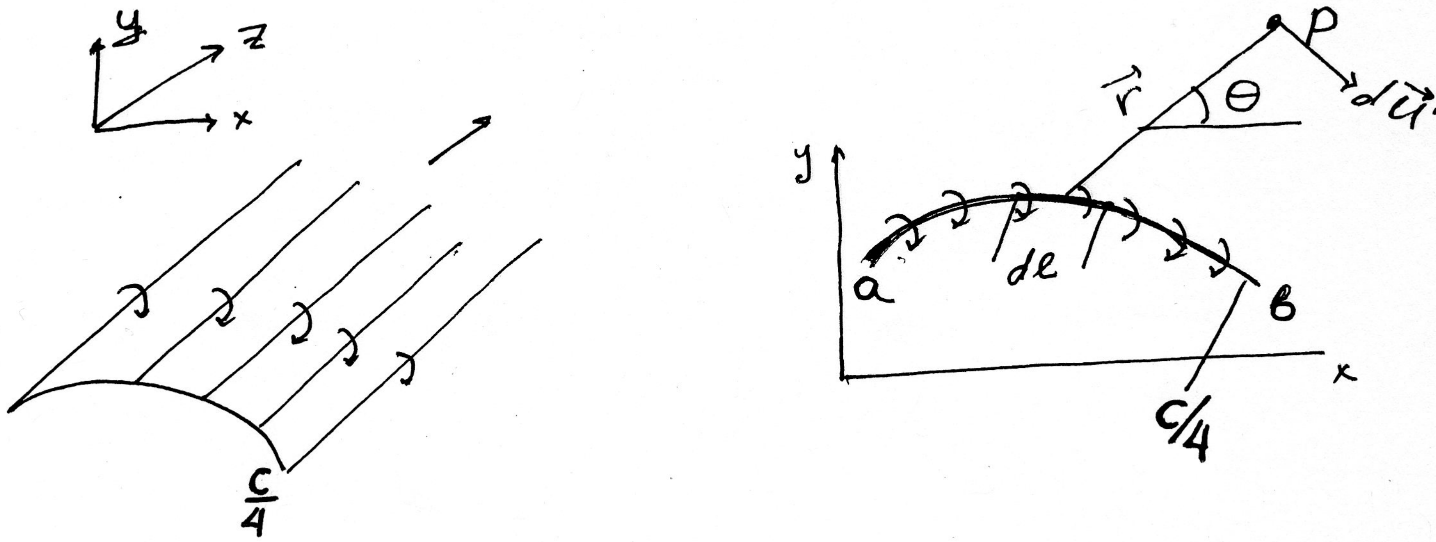


Figure 30: Simplified representation of entire airfoil of Fig. XX. as a vortex sheet covering camber line

As in the case of flat plate we consider a homogeneous flow with velocity $\mathbf{U} = U(cos\ \alpha\mathbf{i} + \sin\ \alpha\mathbf{j})$ where $U = const$ and α is the angle with the x-axis. The velocity field is a sum of two components

$$\mathbf{u} = \mathbf{U} + \mathbf{u}' \tag{14.20}$$

where $\mathbf{u}'$ is the velocity field induced by the vortex sheet. The main difference between this case and the one considered above (flat plate) is modification of the local value of the angle of attack due to the curvature of the mean camber line $C(x)$, so that $\alpha \to \alpha + \theta$. The it is clear from the Figure (xxx) that $\tan\theta = -\frac{dC}{dx}$ and if the airfoil is this, the angle is small and $\theta \approx -\frac{dC}{dx}$. In another approximation, valid in case of this airfoil, we place the vortices not on a camber but the cord line and repeat all considerations which led us to the theory of flat plate. The result is: instead of (14.9) we have now:

Again, the surface of the airfoils is a streamline and the normal component of total velocity on a surface velocity must be equal to zero. This leads to:

$$U(\alpha - \frac{dC}{dx}) - \int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)} = 0 \tag{14.21}$$

This is an integral equation for the circulation distribution $\gamma(\xi)$ along the airfoil. It is to be solved subject to boundary condition $\gamma(c) = 0$. (Kutta condition.)

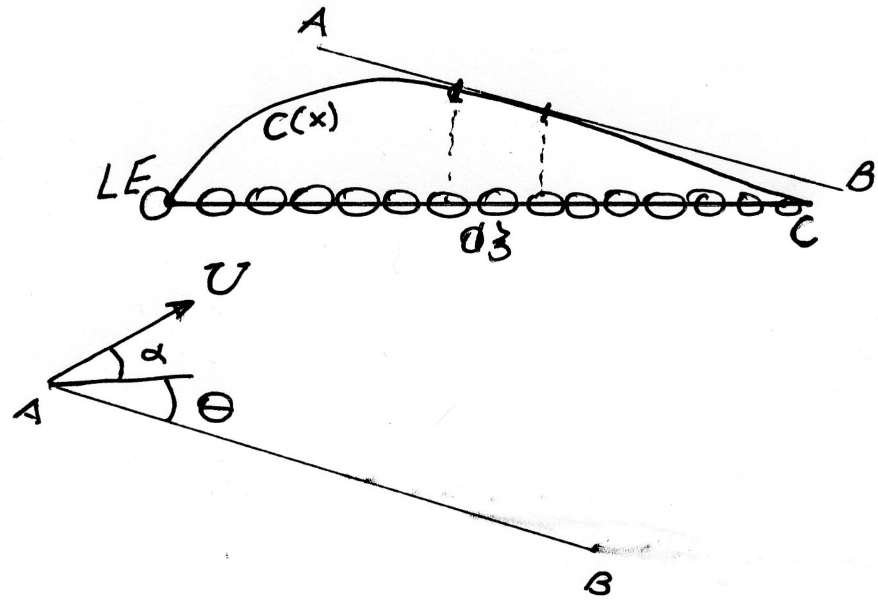


Figure 31: Modification of a local magnitude of the angle of attack due to curvature of the mean camber line.

14.4 Symmetric Airfoil.

We are interested in calculating the lift force on a thin airfoil. Before we proceed further, let us pause and try to figure out what the expected solution should be. First of all, due to the total symmetry of a problem when the angle of attack is equal to zero, we expect the lift on the airfoil to be equal to zero. It is clear that the lift is not zero when $\alpha \neq 0$ and it changes sign when the angle α changes sign, i.e. as $\alpha \rightarrow -\alpha$, the lift $L \rightarrow -L$. Thus, the lift must be proportional to the odd powers of α: $L \propto \alpha + a_3\alpha^3 + ...$ In the limit of small angle of attack , the high powers of α can be neglected and the first non-vanishing contribution to the expansion gives: $L \propto \alpha$. The proportionality coefficient is also estimated readily from the dimensional considerations. We have learned in the previous sections (Bernoulli's equation) that the pressure on a body is proportional to ρU^2 and therefore, the force acting on the airfoil of the area S must be proportional to $pS = \rho U^2 cW$ where W is the width of the airfoil (perpendicular to the page). As a result we expect the lift on the unit width of the airfoil:

$$L \propto \rho U^2 c\alpha$$

.

The proportionality coefficient will be evaluated below. It is clear that this is exactly the result obtained above for the flat plate.

For symmetric airfoil the cumber is zero, so that $\frac{dC}{dx} = 0$ and the problem becomes identical to the one for flat plate.:

$$U\alpha = \frac{1}{2\pi}\int_o^c \frac{\gamma(\xi)d\xi}{x-\xi} \tag{14.22}$$

Let us introduce a set of coordinates

$$\xi = \frac{c}{2}(1 - cos\ \theta); \quad x = \frac{c}{2}(1 - cos\ \theta_0); \quad d\xi = \frac{c}{2}sin\ \theta d\theta \tag{14.23}$$

The integral (14.22) is:

$$U\alpha = \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)sin\ \theta d\theta}{cos\ \theta - cos\ \theta_0} \tag{14.24}$$

Since, in general

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0}d\theta = \frac{\pi \sin n\theta_0}{\sin\theta_0}$$

$$\int_0^\pi \frac{1 + cos\ \theta}{cos\ \theta - cos\ \theta_0} = \pi$$

,

the solution to the integral equation (14.24) is:

$$\gamma(\theta) = 2\frac{1 + cos\ \theta}{sin\ \theta}U\alpha \tag{14.25}$$

As a result:

$$\Gamma = \int_0^c \gamma(\xi)d\xi = \frac{c}{2}\int_0^\pi \gamma(\theta)sin\ \theta d\theta = \alpha cU\int_0^\pi (1 + cos\ \theta)d\theta = \pi\alpha cU \tag{14.26}$$

and the lift force acting on an airfoil of a unit width is: fluid is:

$$\mathbf{F} = \mathbf{L} = -\rho U\Gamma = \pi\alpha c\rho U^2 \tag{14.27}$$

This result is of course in accord with the qualitative considerations presented in the beginning of this section. The lift coefficient is:

$$c_l = \frac{L}{\rho\frac{U^2}{2}c} = 2\pi\alpha \tag{14.28}$$

This simple conclusion is in a remarkable agreement with experimental data on the real life airfoils in the range of the angle of attack $|\alpha| \leq 12 - 15$ degrees. (see Fig.)

Knowing the force $\mathbf{F}$, the moment about leading edge is computed readily. The differential lift originating from an element in the vicinity of the point ξ is: $dL = \rho U d\Gamma = \rho U\gamma(\xi)d\xi$. Since the distance to the leading edge is ξ, the differential moment coming from this element is $dM_{LE} = -\xi dL = -\rho U\xi\gamma(\xi)d\xi$ and the moment is given by an integral:

$$M_{LE} = \int_0^c \xi dL = \xi\gamma(\xi)d\xi \tag{14.29}$$

Using the coordinate transformations leading to (14.24) and the solution (164.25) for $\gamma(\xi)$ gives:

$$M = -\rho\alpha\frac{U^2c^2}{2}\int_0^\pi (1-cos\ \theta)(1+cos\ \theta)d\theta = -\rho\alpha\frac{U^2c^2}{2}\int_0^\pi (1-cos^2\ \theta)d\theta = -\frac{\rho c^2\pi U^2}{4}\alpha \tag{14.30}$$

and the leading edge moment coefficient is:

$$c_{m,LE} = -\pi\alpha/2 = -c_l/4 \tag{14.31}$$

Let us calculate the moment about a point $x = c/4$:

$$M_{c/4} = \int_0^c (x-\frac{c}{4})\gamma(\xi)d\xi = M_{LE} + cL/4 = 0 \tag{14.32}$$

Thus, $x = c/4$ is the center of momentum point.

14.5 Cambered Airfoils.

In this case the term $\frac{\partial C}{\partial x}$ in (14.21) cannot be neglected. Still, qualitative considerations developed for the constant -camber case $C = 0$ can easily be generalized. According to the Figure, the relevant , symmetry-breaking angle is: $\alpha + \frac{\partial C}{\partial x}$. If this angle is small, we expect the lift L be a sum of two independent contributions $L = L_1 + L_2$, where $L_1 = \pi\alpha c U^2$, evaluated above and $L_2 \propto \overline{\frac{\partial C}{\partial x}}$ where $\overline{\frac{\partial C}{\partial x}}$ is a weighted value of $\frac{\partial C}{\partial x}$, i.e.

$$L_2 \propto \int_0^\pi \frac{\partial C(\theta)}{\partial x}\phi(\theta)d\theta$$

To evaluate L_2, we have to determine the weighting function $\phi(\theta)$. In the coordinates (14.23), the equation (14.21) reads:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)sin\ \theta d\theta}{\cos\theta - \cos\theta_0} = U(\alpha - \frac{\partial C}{\partial x}) \tag{14.33}$$

Our goal is to express $\gamma(\theta)$ in terms of given dynamic (U) and geometric (α, $\frac{\partial C}{\partial x}$) parameters of the problem. Since we are dealing with linear equations, It is natural to seek solution of a problem as a superposition of the solution (14.25) and a correction due to the non-zero camber:

$$\gamma(\theta) = 2U[A_0 \frac{1+\cos\theta}{\sin\ \theta} + \sum_{n=1}^{\infty} A_n \sin n\theta] \tag{14.34}$$

Now, we have to express all constant coefficient A_n in terms of parameters of the problem. Substituting (14.34) into (14.33) gives:

$$U(\alpha - \frac{dC}{dx}) = \frac{2U}{2\pi} \int_0^{\pi} \frac{\sin\theta d\theta}{\cos\theta - \cos\theta_0} [A_0 \frac{1+\cos\theta}{\sin\ \theta} + \sum_{n=1}^{\infty} A_n \sin n\theta] \tag{14.35}$$

and :

$$\alpha - \frac{\partial C}{\partial x} = A_0 + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{\pi} A_n \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_0} d\theta \tag{14.36}$$

This expression is simplified using the integral:

$$-\frac{1}{\pi} \int_0^{\pi} \frac{\sin n\theta d \cos\theta}{\cos\theta - \cos\theta_0} = -\cos n\theta_0 \tag{14.37}$$

leading to:

$$\frac{\partial C}{\partial x} = \alpha - A_0 + \sum_{n=1}^{\infty} A_n cos\ n\theta_0 \tag{14.38}$$

The coefficient A_0 is found by integrating (14.38) in the interval $0 \leq \theta_0 \leq \pi$ and taking into account that on this interval all integrals involving $\cos n\theta_0$ are equal to zero:

$$\alpha - \frac{1}{\pi} \int_0^{\pi} \frac{\partial C}{\partial x} d\theta = A_0 \tag{14.39}$$

To determine the Fourier coefficients A_n, let us multiply (14.38) by $\cos\theta_0$ and since all integrals $\int_0^{\pi} \cos\theta_0 \cos n\theta_0 d\theta_0 = 0$ for $n \neq 1$, we derive readily:

$$\pi A_1/2 = \int_0^{\pi} \frac{\partial C}{\partial x} \cos\theta d\theta \tag{14.40}$$

Substituting this into (14.34) gives the expression for $\gamma(\theta)$:

$$\gamma(\theta) = 2U[A_0 \frac{1+\cos\theta}{\sin\ \theta} + \int_0^{\pi} \frac{\partial C}{\partial x} \cos\theta d\theta] \tag{14.41}$$

and finally:

$$\Gamma = \int_0^{\pi} \gamma(\theta) sin\ \theta d\theta = \pi c U(\alpha + \frac{1}{\pi} \int_0^{\pi} \frac{\partial C}{\partial x} (cos\ \theta - 1) d\theta) \tag{14.42}$$

Indeed, this expression is in agreement with $\Gamma = \Gamma_1 + \Gamma_2$ obtained above for the case of a thin airfoil and small camber on a qualitative grounds. The lift, evaluated readily from the familiar Kutta-Joukovskii formula is:

$$L = \rho U \Gamma = \pi c \rho U^2 (\alpha + \frac{1}{\pi}\int_0^{\pi} \frac{\partial C}{\partial x}(\cos\theta - 1) d\theta) \tag{14.43}$$

Sometimes one is interested in the angle of attack $\alpha_{L=0}$, so that $L_(\alpha_{L=0})$ which is evaluated from the relation:

$$\alpha_{L=0} = -\frac{1}{\pi}\int_0^{\pi} \frac{\partial C}{\partial x}(\cos\theta - 1) d\theta \tag{14.44}$$

The lift coefficient is thus:

$$c_l = 2\pi[\alpha + \frac{1}{\pi}\int_0^{\pi} \frac{\partial C(x)}{\partial x}(\cos\theta - 1) d\theta] = 2\pi(\alpha - \alpha_{L=0}) = \frac{dc_l}{d\alpha}(\alpha - \alpha_{L=0}) \tag{14.45}$$

The moment per span about the leading edge is:

$$M'_{LE} = \int_0^c \xi dL(\xi) = -\rho U \int_0^c \xi\gamma(\xi) d\xi = -\pi\frac{c^2\rho U^2}{4}[A_0 + A_1 - A_2/2] \tag{14.46}$$

The integral is evaluated as follows:

$$M_{LE} = \int_0^c \xi dL(\xi) = \rho U \int_0^c \xi\gamma(\xi) d\xi =$$
$$-\frac{\rho U^2 c^2}{2}\int_0^{\pi}[(1-\cos\theta)\sin\theta d\theta(A_0\frac{1+\cos\theta}{\sin\theta} + \sum A_n \sin n\theta)] \tag{14.47}$$

The ntegrals are:

$$I_0 = A_0 \int_0^{\pi}(1-\cos^2\theta) d\theta = A_0\pi/2$$

The second integral is calculated readily

$$I_1 = A_1 \int_0^{\pi}(1-\cos\theta)\sin^2\theta d\theta = A_1\pi/2$$

and the third one is calculated using the relation $\sin 2\theta = 2\sin\theta\cos\theta$ and substitution $\kappa = 2\theta$:

$$I_2 = A_2 \int_0^{\pi}(1-\cos\theta)\sin\theta\sin 2\theta = -\frac{A_2}{4}\int_0^{2\pi}\sin^2\kappa d\kappa = -A_2\pi/4$$

Combining I_0, I_1 and I_2 gives (14.46).

The moment coefficients are:

$$c_{m,LE} = -\frac{\pi}{2}(A_0 + A_1 - A_2/2) = -\frac{\pi}{2}(c_l/4 + \pi(A_1 - A_2)/4) \tag{14.48}$$

and

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1) \tag{14.49}$$

14.6 Panel Method for Lifting Bodies.

The described in previous sections thin airfoil theory is an approximation giving quite good results when the effects of finite thickness are negligibly small. Panel method, introduced below, is an approximate construction for a rapid and accurate numerical solution of a problem of potential flow past a body of arbitrary geometry. The beauty of thin airfoil theory is that it enables one to find an analytic solution to the general equation (14.33) and accurately describe the foil, provided $T/c \ll 1$. This feat is impossible for a body of an arbitrary shape.

Panel method for lifting bodies is an approximate numerical method leading to fast and accurate results. Instead of covering an airfoil with infinite number of infinitesimal vortices, we first represent a body surface as a set of strait panel $1 \leq n \leq N$ and attach to control points at centers of each panel a vortex of circulation γ_i. As $N \to \infty$, the representation becomes exact. As before, our goal is to find such a set γ_i, so that the total generated velocity field be tangential to the airfoil surface. In addition, this field is to satisfy Kutta condition of smoothness of the flow leaving trailing edge: $\gamma(TE) = 0$. The setup is shown on a Figure (xx). In accord with potential theory, the potential generated by a j^{th} vortex at an arbitrary point $P = (x, y)$ is:

$$\Delta\phi_j = -\frac{1}{2\pi} \int_j \theta_{p,j} \gamma_j ds_j \tag{14.50}$$

where the integration is carried over j^{th} panel only. We can see from Fig. (xx,b) that:

$$\theta_{pj} = tan^{-1} \frac{y - y_j}{x - x_j} \tag{14.51}$$

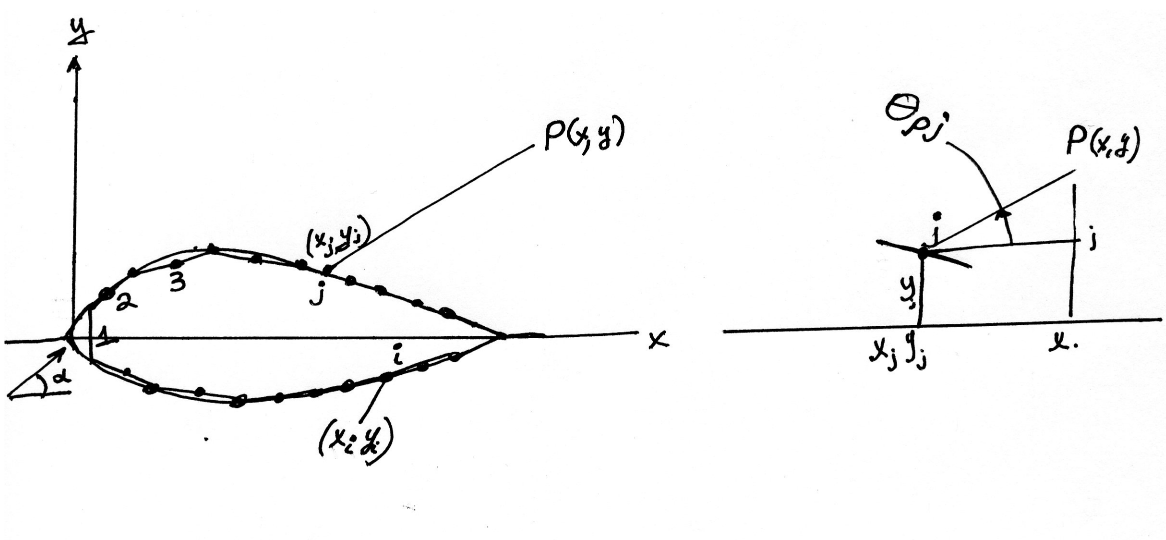


Figure 32: a. Choice of panels b. Geometric parameters.

Thus, the potential from all panels is:

$$\phi(P) = \sum_{j=1}^{N} \Delta\phi_j = -\sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \int_j \theta_{Pj} ds_j \tag{14.52}$$

This expression is valid for each point P, including those on a surface of each panel so that

$$\phi(x_i y_i) = -\sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \int_j \theta_{ij} ds_j \tag{14.53}$$

$$\theta_{ij} = tan^{-1} \frac{y_i - y_j}{x_i - x_j} \tag{14.54}$$

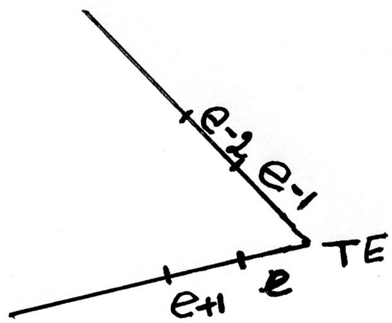
Since geometry of the body is given, all coordinates $x_i y_i$ are known and, in principle, the system (14.53), (14.54) can be solved subject to condition on the normal velocity on each panel:

$$u_{n,i} + U \cos\beta_i = 0 \tag{14.55}$$

where the local angle of attack on each panel is β_i and the induced velocity is expressed in terms of potential:

$$u_{n,i} = \frac{\partial}{\partial n_i}(\phi(x_i, y_i)) = -\sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i} ds_j \tag{14.56}$$

The set (14.54)-(14.56) is the basis of a panel method. We are dealing with N equations for N un knowns γ_i with well-known geometric parameters. In principle it can be solved using modern computers and, if N is large enough, accurately describe the airfoil. There is a caveat, though. The Kutta condition $\gamma(TE) = 0$ is a must and this make the system overdefined: we are to solve $N + 1$ for N unknowns, there is an infinite number of solutions and it is not a trivial task to choose a single correct one. A popular prescription is the following: just drop one of the equations, say for a panel adjacent to the trailing edge (see Fig.(xx) solve for the remaining ones and impose $\gamma(e) = -\gamma(e-1)$. This way we automatically obtain on the trailing edge $\gamma(e) + \gamma(e-1) = 0$.



Example.

Consider an NACA 23012 airfoil. The mean camber line for this airfoil is given by

$\frac{dz}{dx} \equiv \frac{dc}{dx}$

$$\frac{z}{c} = 2.6595\left[\left(\frac{x}{c}\right)^3 - 0.6075\left(\frac{x}{c}\right)^2 + 0.1147\left(\frac{x}{c}\right)\right] \quad \text{for } 0 \le \frac{x}{c} \le 0.2025$$

and
$$\frac{z}{c} = 0.02208\left(1 - \frac{x}{c}\right) \quad \text{for } 0.2025 \le \frac{x}{c} \le 1.0$$

Calculate (*a*) the angle of attack at zero lift, (*b*) the lift coefficient when $\alpha = 4°$, (*c*) the moment coefficient about the quarter chord, and (*d*) the location of the center of pressure in terms of x_{cp}/c, when $\alpha = 4°$. Compare the results with experimental data.

Solution

We will need dz/dx. From the given shape of the mean camber line, this is

$$\frac{dz}{dx} = 2.6595\left[3\left(\frac{x}{c}\right)^2 - 1.215\left(\frac{x}{c}\right) + 0.1147\right] \quad \text{for } 0 \le \frac{x}{c} \le 0.2025$$

and
$$\frac{dz}{dx} = -0.02208 \quad \text{for } 0.2025 \le \frac{x}{c} \le 1.0$$

Transforming from x to θ, where $x = (c/2)(1 - \cos\theta)$, we have

$$\frac{dz}{dx} = 2.6595\left[\frac{3}{4}(1 - 2\cos\theta + \cos^2\theta) - 0.6075(1 - \cos\theta) + 0.1147\right]$$

or
$$= 0.6840 - 2.3736\cos\theta + 1.995\cos^2\theta \quad \text{for } 0 \le \theta \le 0.9335 \text{ rad}$$

and
$$= -0.02208 \quad \text{for } 0.9335 \le \theta \le \pi$$

(*a*) From Equation (4.61),

$$\alpha_{L=0} = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta - 1)\, d\theta$$

(*Note:* For simplicity, we have dropped the subscript zero from θ; in Equation (4.61), θ_0 is the variable of integration—it can just as well be symbolized as θ for the variable of integration.)

54

The thin airfoil approximation enables one to apply potential flow theory to calculate the **two-dimensional** airfoil performance parameters in terms of the foil geometry, angle of attack and speed. The success of the method is based on the theoretically unjustified but experimentally observed Kutta condition on the trailing edge:

$$\gamma(TE) \equiv \gamma(c) = 0$$

ensuring smoothness of velocity field at the trailing edge. The potential flow theory is easily implemented in the numerical panel method for calculation of the airfoil features beyond the limits of applicability of the thin airfoil theory. The panel method is described in the class notes or Anderson's book). Here are the tasks

Example continued

Substituting the equation for dz/dx into Equation (4.61), we have

$$\alpha_{L=0} = -\frac{1}{\pi}\int_0^{0.9335} (-0.6840 + 3.0576\cos\theta - 4.3686\cos^2\theta + 1.995\cos^3\theta)\,d\theta \qquad \textbf{[E.1]}$$

$$-\frac{1}{\pi}\int_{0.9335}^{\pi} (0.02208 - 0.02208\cos\theta)\,d\theta$$

From a table of integrals, we see that

$$\int \cos\theta\,d\theta = \sin\theta$$

$$\int \cos^2\theta\,d\theta = \tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta$$

$$\int \cos^3\theta\,d\theta = \tfrac{1}{3}\sin\theta(\cos^2\theta + 2)$$

Hence, Equation (E.1) becomes

$$\alpha_{L=0} = -\frac{1}{\pi}[-2.8683\theta + 3.0576\sin\theta - 2.1843\sin\theta\cos\theta + 0.665\sin\theta(\cos^2\theta + 2)]_0^{0.9335} - \frac{1}{\pi}[0.02208\theta - 0.02208\sin\theta]_{0.9335}^{\pi}$$

Hence, $\alpha_{L=0} = -\frac{1}{\pi}(-0.0065 + 0.0665) = -0.0191$ rad

or $\boxed{\alpha_{L=0} = -1.09^\circ}$

(*b*) $\alpha = 4^\circ = 0.0698$ rad
From Equation (4.60),

$$c_l = 2\pi(\alpha - \alpha_{L=0}) = 2\pi(0.0698 + 0.0191) = \boxed{0.559}$$

(*c*) The value of $c_{m,c/4}$ is obtained from Equation (4.64). For this, we need the two Fourier coefficients A_1 and A_2. From Equation (4.51),

$$A_1 = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos\theta\,d\theta$$

$$A_1 = \frac{2}{\pi}\int_0^{0.9335} (0.6840\cos\theta - 2.3736\cos^2\theta + 1.995\cos^3\theta)\,d\theta + \frac{2}{\pi}\int_{0.9335}^{\pi} (-0.02208\cos\theta)\,d\theta$$

$$= \frac{2}{\pi}[0.6840\sin\theta - 1.1868\sin\theta\cos\theta - 1.1868\theta + 0.665\sin\theta(\cos^2\theta + 2)]_0^{0.9335} + \frac{2}{\pi}[-0.02208\sin\theta]_{0.09335}^{\pi}$$

$$= \frac{2}{\pi}(0.1322 + 0.0177) = 0.0954$$

55

Example continued

From Equation (4.51),

$$A_2 = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos 2\theta\,d\theta = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}(2\cos^2\theta - 1)\,d\theta$$

$$= \frac{2}{\pi}\int_0^{0.9335} (-0.6840 + 2.3736\cos\theta - 0.627\cos^2\theta - 4.747\cos^3\theta + 3.99\cos^4\theta)\,d\theta + \frac{2}{\pi}\int_{0.9335}^{\pi} (0.02208 - 0.0446\cos^2\theta)\,d\theta$$

Note:

$$\int \cos^4\theta\,d\theta = \tfrac{1}{4}\cos^3\theta\sin\theta + \tfrac{3}{8}(\sin\theta\cos\theta + \theta)$$

Thus,

$$A_2 = \frac{2}{\pi}\left\{-0.6840\theta + 2.3736\sin\theta - 0.628\left(\frac{1}{2}\right)(\sin\theta\cos\theta + \theta) - 4.747\left(\frac{1}{3}\right)\sin\theta(\cos^2\theta + 2) + 3.99\left[\frac{1}{4}\cos^3\sin\theta + \frac{3}{8}(\sin\theta\cos\theta + \theta)\right]\right\}_0^{0.9335} + \frac{2}{\pi}\left[0.02208\theta - 0.0446\left(\frac{1}{2}\right)(\sin\theta\cos\theta + \theta)\right]_{0.9335}^{\pi}$$

$$= \frac{2}{\pi}(0.11384 + 0.01056) = 0.0792$$

From Equation (4.64)

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1) = \frac{\pi}{4}(0.0792 - 0.0954)$$

$$\boxed{c_{m,c/4} = -0.0127}$$

(*d*) From Equation (4.66)

$$x_{cp} = \frac{c}{4}\left[1 + \frac{\pi}{c_l}(A_1 - A_2)\right]$$

Hence, $\frac{x_{cp}}{c} = \frac{1}{4}\left[1 + \frac{\pi}{0.559}(0.0954 - 0.0792)\right] = 0.273$

Comparison with Experimental Data The data for the NACA 23012 airfoil are shown in Figure 4.22. From this, we make the following tabulation:

	Calculated	Experiment
$\alpha_{L=0}$	-1.09°	-1.1°
c_l (at $\alpha = 4^\circ$)	0.559	0.55
$c_{m,c/4}$	-0.0127	-0.01

56

for the numerical X-FOIL project.

Problem 1. Consider the airfoil NACA2412 (Figure. (xxx)). If $U = 15m/sec$, $c = 60cm$,

1. determine the magnitude of integral $\int_0^{\pi} \frac{dG}{dx}(\cos\theta - 1)d\theta$;
2. find lift per unit spane for angle of attack $\alpha = 4^o$;
3. Find moment about leading edge: M'_{LE}.

Problem 2. For the same airfoil, find angle of attack if cruising speed is $U = 50m/sec$, $c = 2m$, $L' = 1353N$ (per span) and $\rho_{air} = 1kg/m^3$.

Problem 3.. Consider a vortex sheet with strength:

$$\gamma(\xi) = \gamma_0 \sqrt{\frac{\xi}{c}(1 - \frac{\xi}{c})}$$

If this is a solution in the thin airfoil approximation, find the shape of a mean camber line $C(x)$.

Solution. The equation for $\alpha = 0$ is:

$$\frac{dC}{dx} = -\frac{1}{2\pi U} \int_0^c \frac{\gamma(\xi)}{x - \xi}) d\xi$$

Using the variables (14.23), this equation is:

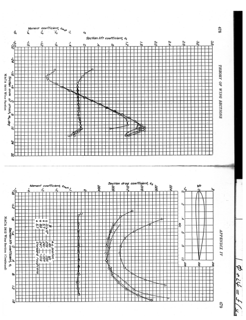
$$\frac{dC}{dx} = -\frac{\gamma_0 c}{4\pi U} \int_0^\pi \frac{\sqrt{(1 - \cos^2\theta)}}{\cos\theta - \cos\theta_0} \sin\theta d\theta$$

$$\frac{dC}{dx} = -\frac{\gamma_0 c}{8\pi U} \int_0^\pi \frac{1 - \cos 2\theta}{\cos\theta - \cos\theta_0} d\theta = \frac{\gamma_0 c}{8\pi U} \pi \frac{\sin 2\theta_0}{\sin\theta_0} = \frac{\gamma_0 c \cos\theta_0}{8U} = -\frac{\gamma_0}{8U}(1 - 2x/c)$$

and

$$C(x) = \frac{\gamma_0 c}{8U}(x - x^2/c)$$

$C(0) - C(c) = 0.$



15 Finite Wings.

Airfoils considered in previous sections are in fact wing sections of a mathematically convenient infinite wing, which does not exist in real life. This construction enabled us, using potential flow theory, calculate lift on airfoils for small angle of attack. This model, however, misses a few important effects resulting from three-dimensionality of a real thing. The first one is a tip vortex. It is clear that the mean pressure at the bottom of a lifting airfoil is larger than the one at the top. This leads to a flow around the wing tips, schematically shown on Fig. A new geometric characteristic of a finite wing, reflecting variation of cord lines along the wing, is a taper ratio c_t/c_r. It is clear that for a rectangular wing this ratio is equal to 1.

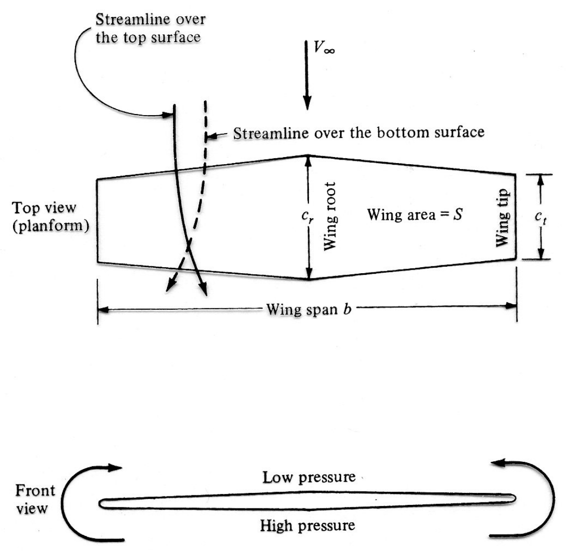
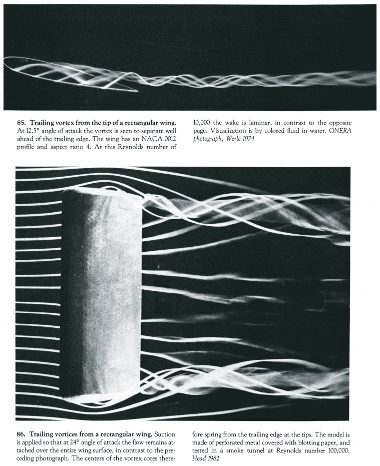


Figure 5.1 Finite wing. In this figure, the curvature of the streamlines over the top and bottom of the wing is exaggerated for clarity.

Since the wing on the Fig. xx moves into the page, the generated flow, called "tip vortex", looks like a vortex tube. The effect is quite dramatic and, smaller planes, caught in the field of tip vortices generated by the large ones, can can lead loose control. This is the main reason the airports impose strict regulation of the intervals between plane take-offs.

The formation of tip vortices can be visualized by seeding the air flux in the wind tunnel by the dust particles and shining the laser light on it. In this case, tip vortices are seen as smoke l filaments In (See Fig.)



In important effect caused by tip vortex is generation of the downward velocity field $\mathbf{w} = -w\mathbf{k}$, so that the total angle of attack, "felt" by an arbitrary crossection of the wing decreases by a magnitude:

$$-\frac{w}{U} = \tan \alpha_i \approx \alpha \tag{15.1}$$

, The phenomenon, called "downwash" is schematically shown on the Figure xx, where the effective angle of attack is denoted as: $\alpha_{eff} \equiv \alpha_R = \alpha - \alpha_i$.

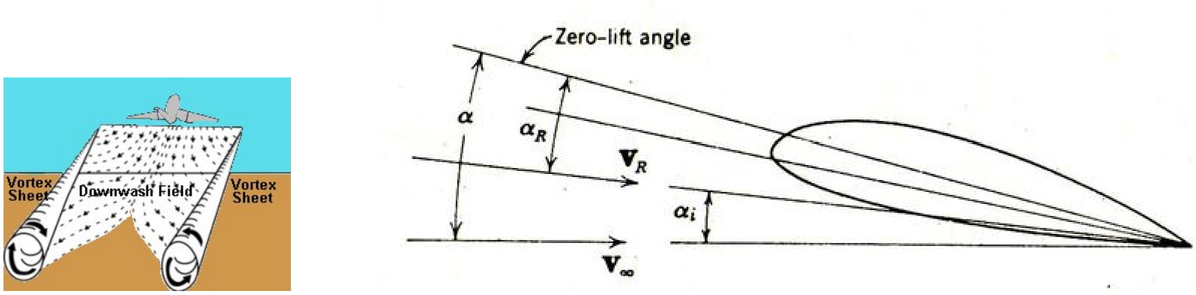


Figure 33: *Left panel: velocity field of the tip vortex. Right panel: mechanism of generation of induced angle of attack*

The effect can be summarized as follows: Consider a plane cruising with velocity $U\mathbf{i}$ and the angle of attack (between $\mathbf{U}$ and the cord line) α. The downwash tends to modify the relative velocity so that $\mathbf{U}$ becomes **somewhat "more parallel" to the cord line.** This leads to the effective decrease of the angle of attack and that of lift.

Another important effect of the downwash is formation of induced drag. The mechanism is schematically shown on the Fig. xxx. As we see, according to Joukovsky theorem the lift is $\mathbf{L} = \rho(\mathbf{U} + \mathbf{w}) \times \Gamma$. Since vector Γ points into the page, in this case lift is perpendicular to the relative velocity $\mathbf{U} + \mathbf{w}$. This leads to generation of the velocity component parallel to the wing cruising velocity $\mathbf{U}$ which, by definition is called

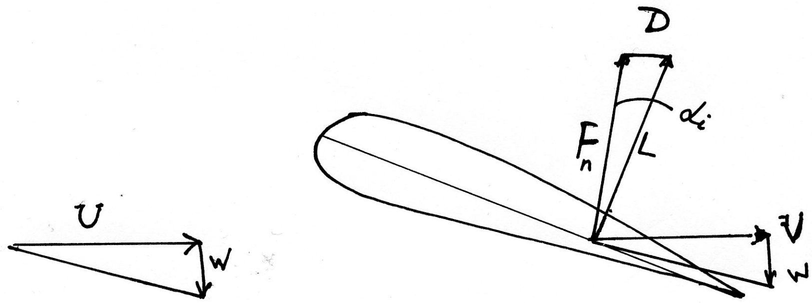


Figure 34: Induced drag.

drag. It is clear that this effect, absent in the 2D potential flow theory, is caused by a three-dimensionality of the wing.

The theory of the finite wings was developed by Prandtl (of course). Considering 2D potential vortex with the center at the origin, we derived for the velocity field $v_\theta = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta$. Here, r is the distance to the origin. Prandtl introduced a concept of vortex filament which is to be percieved as a vortex tube depicted on the Fig. 14

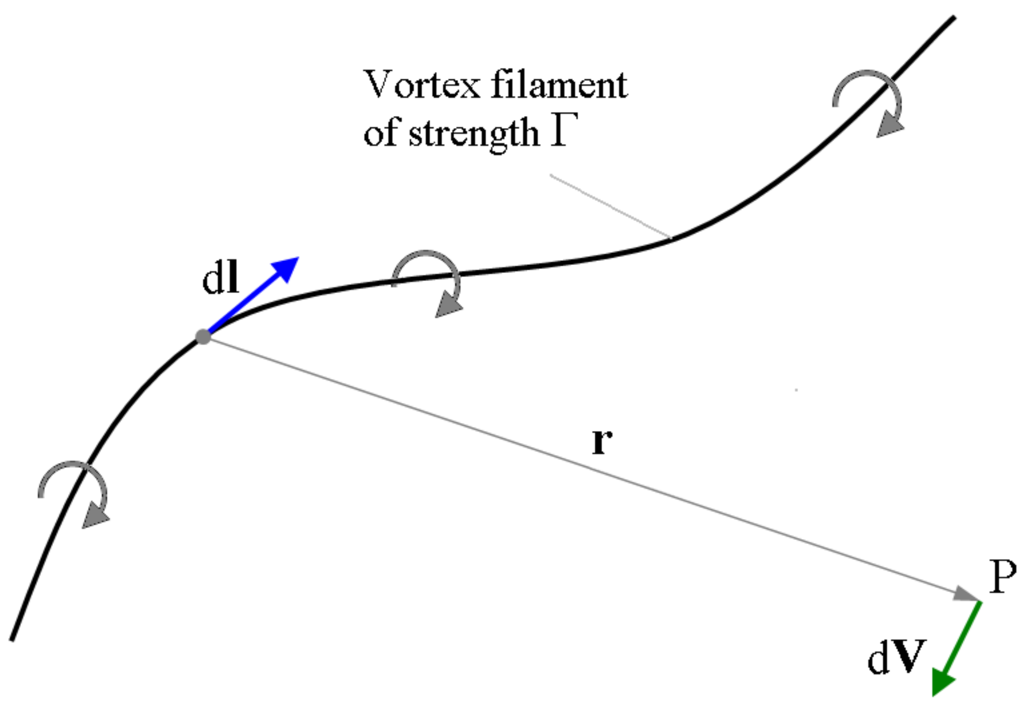


Figure 35: Vortex filament of strength Γ. Biot-Savart law

He considered a vortex filament surrounding a streamline. Since, in accord with Kelvin theorem, in the inviscid flow the circulation is constant on an evolving streamline, the strength of the filament $\Gamma = const.$

(In aerodynamics, Kelvin's theorem is often called Helmholtz theorem.) We also know that in potential flows, streamlines cannot form closed loops and thus, they must start and end at infinity. According to Bio-Savart law, the velocity generated at each point P by an element $d\mathbf{l}$ of the filament is:

$$d\mathbf{V} = \frac{\Gamma}{4\pi}\frac{d\mathbf{l} \times \mathbf{r}}{r^3} \tag{15.2}$$

The velocity field $\mathbf{V}$ induced by the filament is

$$\mathbf{V} = \int \frac{\Gamma}{4\pi}\frac{d\mathbf{l} \times \mathbf{r}}{r^3} \tag{15.3}$$

In the simplest case of a strait vortex filament a detailed calculation gives:

$$\mathbf{V} = \frac{\Gamma}{2\pi h} \tag{15.4}$$

where h is the shortest distance distance to the filament. (See Fig.15).

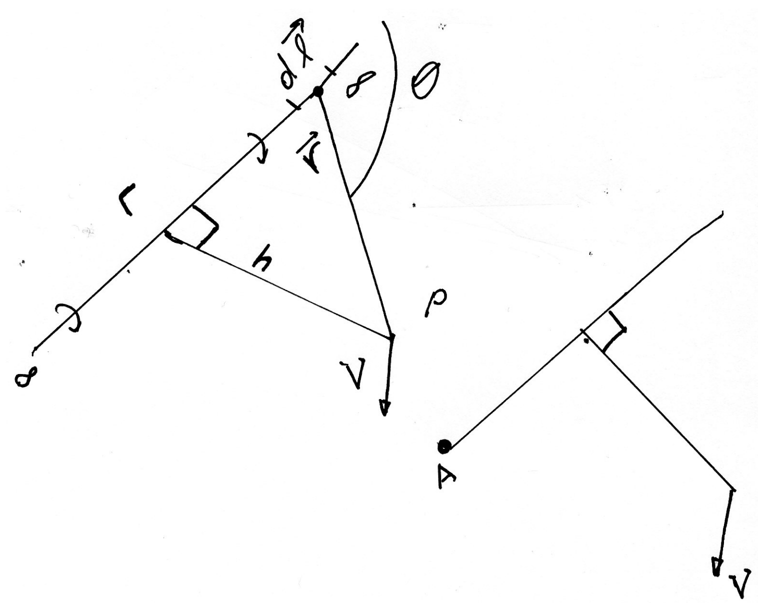


Figure 36: Strait vortex filament of strength Γ. Biot-Savart law

The integral (2.3), leading to (2.4) can be easily calculated. However, the result can be understood without detailed mathematics. All points on the strait filament shown on Fig. 15 right and left from h are equivalent and their contributions to the velocity $\mathbf{V}$ cancel each other. The only one, which is not cancelled is point h contributing to the finial answer which is exactly the one we derived in potential flow approximation.
The velocity generated at point P by a semi-infinite filament (see Fig. 15) between point A and infinity, is given by:

$$V = \frac{\Gamma}{4\pi r} \tag{15.5}$$

Horse-shoe vortex is a ubiquitous flow feature observed in flows past three-dimensional bodies.

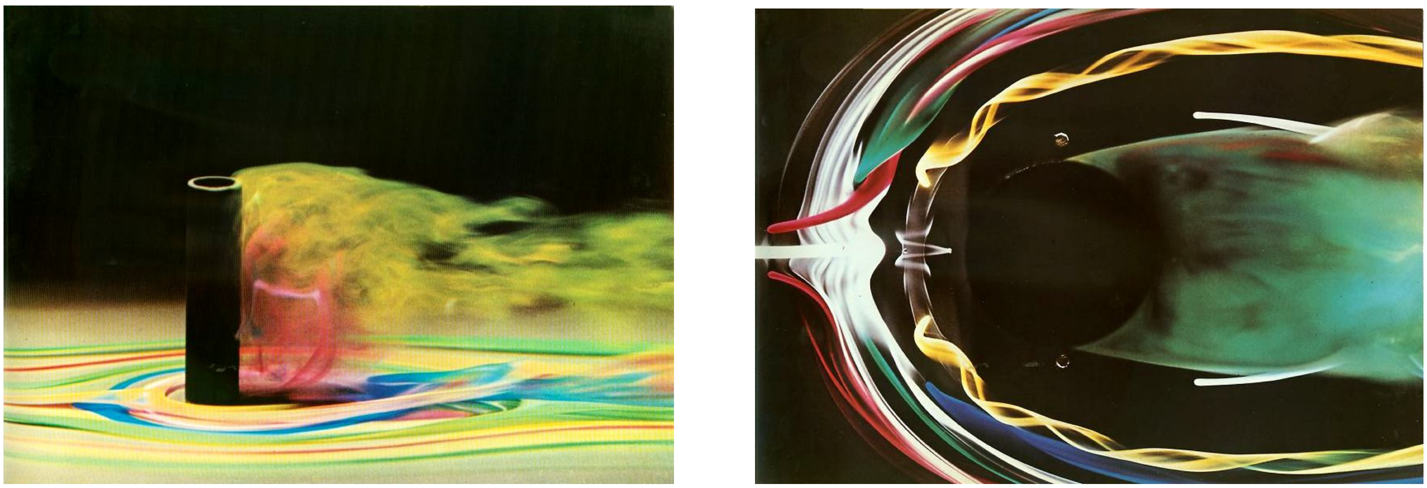


Figure 37: *Left panel: velocity field of the tip vortex. Right panel: mechanism of generation of induced angle of attack*

A typical example is shown on Fig. 16 where the flows past cylinder on a floor of a wind tunnel is visualized. We can see stagnation point and a flow separating as two **trailing vortices** on both sides of a cylinder.
Prandtl considered a finite wing with its stagnation point and tips and substituted it by a horseshoe vortex filament (see Fig. 17). This construction does not contradict theorems of potential flow theory: the filament starts and ends at infinity and is characterized by a constant strength (circulation) $\Gamma = const$. We are interested in the velocity field generated by the this vortex in the interval $-\frac{b}{2} \leq y \leq \frac{b}{2}$ where $d\mathbf{l} \times \mathbf{r} = 0$. It is clear that the only non-zero contributions come from the trailing vortices, which are to be considered as semi-infinite filaments. The result for downwash $\mathbf{w}$ is given by (2.5):

$$\mathbf{w} = -\frac{\Gamma}{4\pi}[\frac{1}{\frac{b}{2}+y} + \frac{1}{\frac{b}{2}-y}] = -\frac{\Gamma}{4\pi}\frac{1}{(\frac{b}{2})^2 - y^2} \tag{15.6}$$

As wee, this construction breaks down at the tips $y = \pm b/2$. The reason for this failure is the constant value of the vortex strength $\Gamma = const$ along the filament. This simplified theory lacked something important. It took many years of hard work to develop these ideas to fruition.
Consider a set of horseshoe vortices shown on Figure 18.
Again, this set does not contradict basic feature of potential flow theory: in each vortex number i, starting and ending at infinity, the corresponding strength $\Gamma_i = const$ etc. It can also be seen from the Fig. 18 that this way one can assemble a system with an arbitrary distribution $\Gamma = \Gamma(y)$ of a bound vortex. Introducing

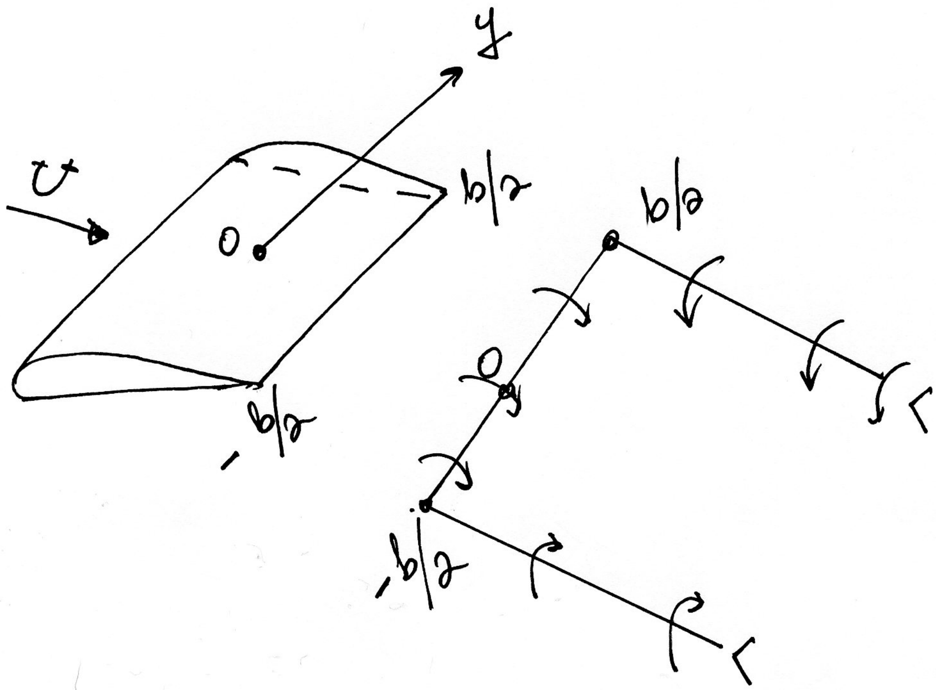


Figure 38: *Left panel: velocity field of the tip vortex. Right panel: mechanism of generation of induced angle of attack*

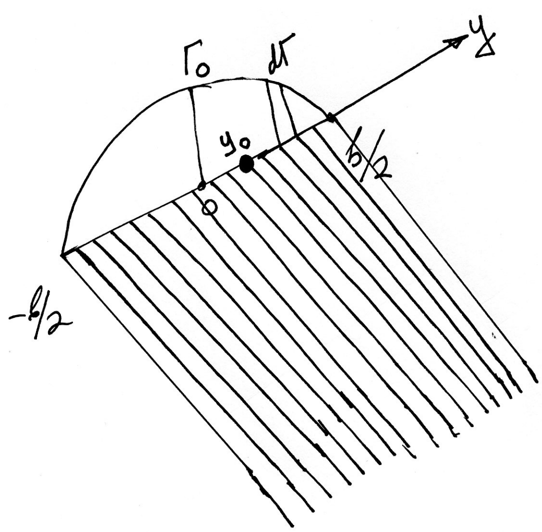


Figure 39: *Set of horseshoe vortices expliang Prandtl construction.*

the density of vortex strength $\frac{\partial \Gamma}{\partial y}$, we can write a general expression for the downwash velocity at any point of a bound vortex y_0:

$$w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{\frac{\partial\Gamma(y)}{\partial y}dy}{y_0 - y} \tag{15.7}$$

and induced angle $\alpha_i(y_0)$ at each section (airfoil) at $y = y_0$:

$$\alpha_i(y_0) \approx -\frac{w(y_0)}{U} = \frac{1}{4\pi U}\int_{-b/2}^{b/2}\frac{\frac{\partial\Gamma(y)}{\partial y}dy}{y_0 - y} \tag{15.8}$$

With the effective angle of attack (per section) $\alpha_{eff} = \alpha - \alpha_i$, the lift coefficient per section can be expressed in terms of this airfoil theory: $c_l = 2\pi(\alpha_{eff} - \alpha_{L=0})$, the lift coefficient for the section of the cord $c(y_0)$ is given by Joukovsky theorem:

$$L' = \frac{1}{2}\rho U^2 c(y_0)c_l = \rho U^2\Gamma(y_0) \tag{15.9}$$

so that:

$$c_l = \frac{2\Gamma(y_0)}{Uc(y_0)} = 2\pi(\alpha - \alpha_{L=0}) \tag{15.10}$$

and

$$\alpha_{eff} = \alpha - \alpha_i = \frac{\Gamma(y_0)}{\pi U c(y_0)} + \alpha_{L=0} = \alpha(y_0) - \frac{1}{4\pi U}\int_{-b/2}^{b/2}\frac{\frac{\partial\Gamma(y)}{\partial y}dy}{y_0 - y} \tag{15.11}$$

and

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi U c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi U}\int_{-b/2}^{b/2}\frac{\frac{\partial\Gamma(y)}{\partial y}dy}{y_0 - y} \tag{15.12}$$

This is fundamental Prandtl's lifting wing theory. It states that geometric angle of attack is equal to the effective angle (see (2.11)) plus induced one, This is the integro-differential equation for $\Gamma(y)$. Given this parameter, we can readily have:

1. $L'(y_0) = \rho U\Gamma(y_0)$;
2. $L = \int_{-b/2}^{b/2} L'(y)dy = \rho U\int_{-b/2}^{b/2}\Gamma(y)dy$;
3. $C_L = \frac{L}{\frac{1}{2}\rho U^2 S} = \frac{2}{US}\int_{-b/2}^{b/2}\Gamma(y)dy$
4. $D_i' = L_i'\sin\alpha_i \approx L_i'\alpha_i$
5. $D = \int_{-b/2}^{b/2} L'(y)\alpha_i dy = \rho U\int_{-b/2}^{b/2}\Gamma(y)\alpha_i(y)dy$
6. $C_{D,i} = \frac{D}{\frac{1}{2}\rho U^2 S} = \frac{2}{US}\int_{-b/2}^{b/2}\Gamma(y)\alpha_i(y)dy$

Elliptic lift.

Consider a given distribution:

$$\Gamma(y) = \Gamma_0\sqrt{1 - \frac{4y^2}{b^2}} \tag{15.13}$$

satisfying the Kutta condition $\Gamma(\pm b/2) = 0$. Lift per section $y = y_0$ is:

$$L'(y_0) = \rho U \Gamma \sqrt{1 - \frac{4y^2}{b^2}} \tag{15.14}$$

Let us calculate down wash:

$$\frac{d\Gamma}{dy} = -\frac{4\Gamma_0}{b^2} \frac{y}{\sqrt{1 - \frac{4y^2}{b^2}}} \tag{15.15}$$

and

$$w(y_0) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{\frac{d\Gamma}{dy}}{y_0 - y} dy = \frac{\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y dy}{\sqrt{1 - \frac{4y^2}{b^2}}(y_0 - y)} \tag{15.16}$$

Using variables: $y = \frac{b}{2}\cos\theta$, $y_0 = \frac{b}{2}\cos\theta_0$ gives:

$$w(\theta_0) = -\frac{\Gamma_0}{2\pi b} \int_{\pi}^{0} \frac{\cos\theta}{\cos\theta_0 - \cos\theta} d\theta = -\frac{\Gamma_0}{2b} \tag{15.17}$$

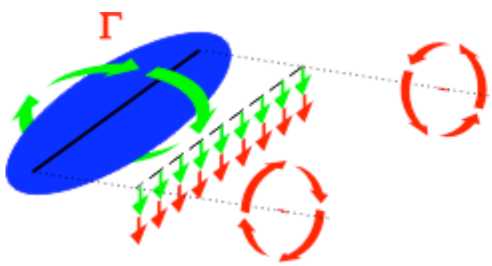


Figure 40: *Elliptical wing. Constant downwash..*

We see that for this distribution, downwash is independent upon y. Then, for the induced angle of attack:

$$\alpha_i = -\frac{w}{U} = \frac{\Gamma_0}{2bU} \tag{15.18}$$

or

$$L = \rho U \Gamma = \rho U \int_{-b/2}^{b/2} \Gamma(y) dy = \rho U \Gamma_0 \int_{-b/2}^{b/2} \sqrt{1 - \frac{4y^2}{b^2}} dy \tag{15.19}$$

The integral is evaluated using the same transformation of integration variables $y = \frac{b}{2}\cos\theta$, with the result:

$$L = \rho U \Gamma_0 \frac{b}{4}\pi \tag{15.20}$$

so that

$$\Gamma_0 = \frac{4L}{\rho U b \pi} = \frac{4 \times \frac{1}{2}\rho U^2 S C_L}{\rho U b \pi} = \frac{2USC_L}{b\pi} \tag{15.21}$$

We can rewrite (2.18):

$$\alpha_i = \frac{2USC_L}{b\pi}\frac{1}{2bU} = \frac{SC_L}{\pi b^2} \tag{15.22}$$

Finally, introducing the aspect ratio $AR = b^2/S$ gives:

$$\alpha_i = \frac{C_L}{\pi AR} \tag{15.23}$$

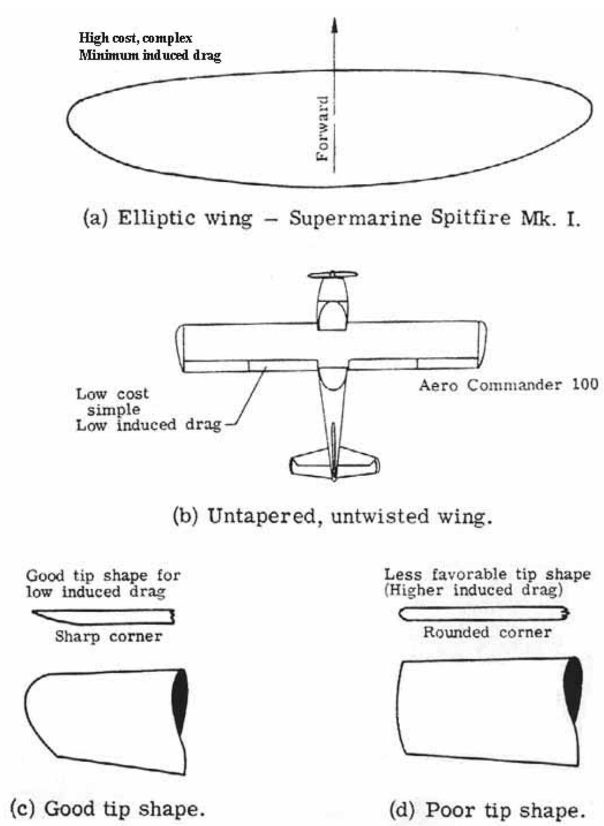
The induced drag:

$$C_{D,i} = \frac{2\alpha_i}{US}\int_{-b/2}^{b/2}\Gamma(y)dy = \frac{\pi\alpha_i\Gamma_0 b}{2US} \tag{15.24}$$

or

$$C_{D,i} = (\frac{\pi b}{2US}) \times (\frac{C_L}{\pi AR}) \times (\frac{2USC_L}{\pi b}) = \frac{C_L^2}{\pi AR} \tag{15.25}$$

We see that induced drag is not small but reaches some 25% of the total drag. The elliptic wings were widely used.



(a) Elliptic wing – Supermarine Spitfire Mk. I.
(b) Untapered, untwisted wing.
(c) Good tip shape.
(d) Poor tip shape.

wing.jpg

15.1 General Lift Distribution.

The results of a previous section were obtained by postulating distribution $\Gamma(y)$ given by (2.13). This was Prandtl's stroke of genius which enables qualitative understanding of main features of finite wings. However, since (2.13) was not obtained from the basic equation (2.13), we cannot be sure that the parabolic lift theory is correct. Thus, to obtain a general relations for lift and induced drag, we have to solve integro-differential equations (2.13).

Let us introduce a new variable:

$$y = -\frac{b}{2}\cos\theta \tag{15.26}$$

In the elliptic case:

$$\Gamma(y) = \Gamma_0\sqrt{1-\frac{4y^2}{b^2}} = \Gamma_0\sin\theta = \Gamma(\theta)$$

Thus, we conclude that parabolic distribution may be a first term of the Taylor series. Based on (2.18), we seek solution as:

$$\Gamma(\theta) = 2bU\sum_1^N A_n\sin n\theta \tag{15.27}$$

where the coefficients A_n are to be found from the equation (2.13). Using

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy} = 2bU\sum_1^N A_n n\cos n\theta\theta\frac{d\theta}{dy}$$

the equation (2.13) reads:

$$\alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)}\sum_1^N A_n\sin n\theta_0 + \alpha_{L=0}(\theta_0) + \frac{1}{\pi}\int_0^\pi \frac{\sum_1^N nA_n\cos n\theta}{\cos\theta-\cos\theta_0} \tag{15.28}$$

This integral was evaluated in (1.24). Using the result, we rewrite (2.28) as:

$$\alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)}\sum_1^N A_n\sin n\theta_0 + \alpha_{L=0}(\theta_0) + \sum_1^N nA_n\frac{\sin n\theta_0}{\sin\theta_0} \tag{15.29}$$

This equation is defined on each and every spanwise location θ_0. From the known geometry and this airfoil theory, we know the cord length $c(\theta_0)$ and the span b and $\alpha_{L=0}(\theta_0)$. The only unknown are the A_n's which can be obtained readily (numerically) from N equations written for N locations θ_0. Thus, in principle, the distribution $\Gamma(\theta)$ is known as a solution to a set of algebraic relations (2.29).

By Joukovsky's theorem, the lift coefficient is:

$$C_L = \frac{\rho U\Gamma}{\frac{1}{2}\rho U^2 S} = \frac{2}{US}\int_{-b/2}^{b/2}\Gamma(y)dy = \frac{2b^2}{S}\sum_1^N A_n\int_0^\pi \sin n\theta\sin\theta d\theta \tag{15.30}$$

The only integral contributing to the sum in the one with $n = 1$. This gives:

$$C_L = A_1 \pi \frac{b^2}{S} = A_1 \pi AR \tag{15.31}$$

As follows from relation (6) on page 28, the indiced drag coefficient

$$C_{D,i} = \frac{2}{US} \int_{-b/2}^{b/2} \Gamma(y) \alpha_i(y) dy \tag{15.32}$$

Skipping mathematical details, one derives:

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta) \equiv \frac{C_L^2}{\pi eAR} \tag{15.33}$$

where

$$\delta = \sum_2^N n(\frac{A_n}{A_1})^2 > 0$$

The parameter $e = (1+\delta)^{-1}$ is called a span efficiency factor. It was shown above, that Prandtl's parabolic distribution is equivalent to (2.27) when only the first terms with $n = 1$ is accounted for. We see that the correction $\delta > 0$ comes from A_n's with $n > 2$ not appearing in the simplified, parabolic, version of the theory. Moreover, it follows from (2.33) that parabolic lifting wing ($\delta = 0$) corresponds to the minimum of induced drag.

For a general aspect and taper ratios, evaluation of δ is not a simple task. We can use Prandtl's curves presented on the Figure.

First, one chooses a curve corresponding to a given magnitude of AR. Then, the parameter δ for a given magnitude of the taper ratio c_t/c_r is found from the curve. For example, if taper ratio $c_t/c_r = 0.8$ and aspect ration $AR = 6$, the parameter $\delta \approx 0.5$.

Thus, the total drag of a finite wing is:

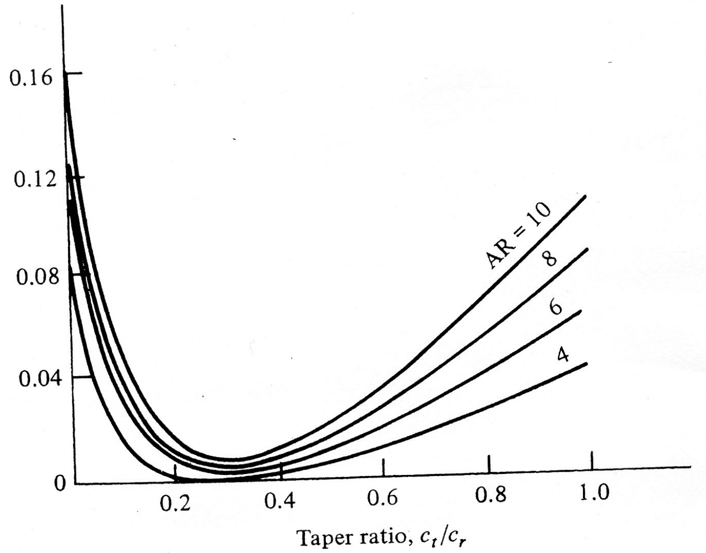
$$C_D = c_d + \frac{C_L^2}{\pi eAR} \tag{15.34}$$

where c_d can be obtained from the airfoil calculations. The second term is this equation accounts for induced drag which can reach 25% of total.

Considering airfoil we found that $c_L = a_0(\alpha - \alpha_{L=0})$, where in the simplest case of a**thin** airfoil $a_0 = 2\pi$. In this expresion, α is the angle between cord line and relative velocity. In case of finite wing this relative velocity is corrected for the downwash, so that

$$\frac{dC_L}{d\alpha_{eff}} = \frac{dC_L}{d(\alpha - \alpha_i)} = a_0 \tag{15.35}$$

and



Induced drag factor δ as a function of taper ratio. (*Source:* McCormick, B. W., *Aerodynamics, Aeronautics, and Flight Mechanics*, John Wiley & Sons, New York, 1979.)

$$C_L = a_0(\alpha - \alpha_i) + const = a_0(\alpha - \alpha_{L=0} - \frac{C_L}{\pi AR}(1+\tau)) \tag{15.36}$$

where, as follows from (2.33) $\tau \approx \delta$. From this one derives:

$$C_L = \frac{a_0(\alpha - \alpha_{L=0})}{1 + \frac{a_0}{\pi AR}(1+\tau)}$$

which leads to

$$\frac{dC_L}{d\alpha} = a \tag{15.37}$$

where

$$a = \frac{a_0}{1 + \frac{a_0}{\pi AR}(1+\tau)} \tag{15.38}$$

Problem. Consider a finite wing $AR = 8$ and taper ratio $\frac{c_t}{c_r} = 0.8$. The airfoil section is thin and symmetric. The angle of attack is 5^o. Calculate the lift and induced drag coefficients. Assume $\tau = \delta$.

Solution. From the Figure: $\delta = 0.055$. The airfoil is thin, so that $a_0 = 2\pi$. Thus,

$$a = \frac{a_0}{1 + \frac{a_0}{\pi AR}(1+\tau)} = 4.97 rad^{-1} = 0.0867 degree^{-1}$$

The foil is symmetric, so that $\alpha_{L=0} = 0$. Thus,

$$C_L = a\alpha = 0.0867\ degree^{-1} \times 5\ degrees = 0.4335$$

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta) = 0.00789$$

Problem. Consider a rectangular wing ($c_t/c_r = 1$) of $AR = 6$, induced drag factor (from the Figure) $\delta = 0.55$ and $\alpha_{L=0} = -2^o$. At the angle of attack $\alpha = 3.4^o$, the induced drag coefficient for this wing is $C_{D,i} = 0.01$. Calculate the induced drag coefficient for a similar wing (rectangular wing with the sa

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta) = 0.00789$$

Problem. Consider a rectangular wing ($c_t/c_r = 1$) of $AR = 6$, induced drag factor (from the Figure) $\delta = 0.55$ and $\alpha_{L=0} = -2^o$. At the angle of attack $\alpha = 3.4^o$, the induced drag coefficient for this wing is $C_{D,i} = 0.01$. Calculate the induced drag coefficient for a similar wing (rectangular wing with the same airfoil section but with $AR_1 = 10$..

Solution. In order to solve this problem, first we have to find the coefficient a_0 which is the same for similar wings, i.e. is independent upon AR. From the Figure, in this case $\delta = \tau \approx 0.105$. First, calculate the lift coefficient for $AR = 6$.:

$$C_L^2 = \frac{\pi AR C_{D,i}}{1+\delta} = \frac{\pi \times 6\ \times 0.01}{1+0.055} = 0.1787. \qquad C_L = 0.423.$$

Therefore

$$\frac{dC_L}{d\alpha} = \frac{0.423}{3.4^o - (-2^o)} = 0.078\ degree^{-1} = 4.485\ rad^{-1}$$

$$\frac{dC_L}{d\alpha} = a = \frac{a_0}{1 + \frac{a_0}{\pi AR}(1+\tau)} = 4.485\ rad^{-1} = \frac{a_0}{1+0.056a_0} \tag{15.39}$$

This gives $a_0 = 5.989\ rad^{-1}$.

NOW WE CAN SOLVE THE CASE OF $AR_1 = 10$. The lift slop of this wing is:

$$a = \frac{a_0}{1 + \frac{a_0}{\pi AR}(1+\tau)} = \frac{5.989}{1 + \frac{5.989 \times 1.105}{(\pi \times 10)}} = 4.95\ rad^{-1} = 0.086\ degree^{-1}$$

and

$$C_L = a(\alpha - \alpha_{L=0}) = 0.086(3.4^o - (-2^o)) = 0.464$$

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta) = 0.0076$$

Problem. Consider a rectangular wing of a span $b = 20m$ and cord line $c = 2m$. The section of this wing are NACA0012. The speed $U = 100m/sec$, angle of attack $\alpha = 2^o$. Find:

1. Lift.
2. Total drag.
3. Repeat the calculation for NACA2412.
4. Compare with experimental data.

Assume: the foil is thin, $\tau - = \delta$ and evaluating friction drag neglect thickness of the wing.

Solution.

$$Re_b = 100 \times 2/(0.15 \times 10^{-4}) = 1.33 \times 10^7$$

$$C_D = c_d + C_{D,i} = c_d + \frac{C_L^2}{\pi AR}(1+\delta)$$

$$C_L = \frac{2\pi(\alpha - \alpha_{L=0})}{1 + \frac{2\pi}{\pi AR}(1+\tau)} = 0.18$$

$$F_f = 2b\frac{\rho U^2}{2}\int_0^c 0.0576 Re_x^{-0.02} dx = 0.219\rho U^2/2 = \frac{\rho U^2}{2} S \equiv c_d \frac{\rho U^2}{2} S$$

$$c_d = 0.0027$$

$$C_D = 0.0027 + 0.0011 = 0.0038$$

16 Potential Flows: Sound and Gravity Waves.

Waves in fluids is a huge field of fluid mechanics covering sound propagation, surface and internal waves, gravity-induced waves etc. Often, waves control environmental flows, mixing and many other important phenomena. The thorough study of waves in fluids in much beyond the scope of these notes. In this Section, we would like to present an elementary demonstration how the equations of ideal flows (Euler equation), even in the simplest possible potential flow approximation introduced above, can lead to the equations describing various types of waves.

Sound. Now, we analyze a compressible flow of density $\rho \neq const$. For simplicity, consider a gas filling the entire space. The density of nonperturbed gas is $\rho = \rho_0 = const$ ($p = p_0 = const$). If at initial instant $t = 0$ a compression ($\rho > \rho_0$) is created in a small volume $\approx \Delta^3$ positioned in the interval $r_0 - \Delta \leq r \leq r_0 + \Delta$, then, due to the mass conservation, the volumes of lower density $\rho < \rho_0$ must be formed in the adjacent fluid/gas patches $r > r_0 + \Delta$ and $r < r_0 < \Delta$. This compression leads to the net, noncompensated, surface forces acting on the surrounding fluid elements. This surface force leads to compression of the neighboring fluid elements accompanied by a pressure loss in the original volume. This way, compression and rarefaction are propagated across fluid. We would like to stress that no transfer of substantial amount of material is generated by this mechanism. The pressure p is a scalar and, as a result, the compressed fluid volume equally "pushes" the adjacent fluid element in all directions. Only normal forces are involved and no vorticity is created. As a result, the potential flow theory is valid. This picture of compressed and rarified regions in gas, solid or liquid) will propagate forming the *sound wave*.

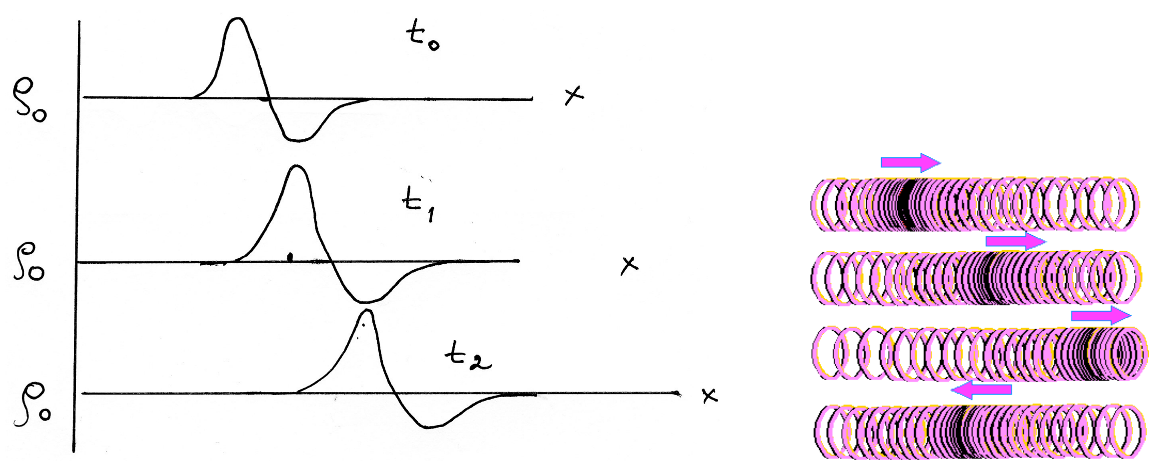


Figure 41: Propagation of the density perturbation.

We assume that the density and pressure perturbations are very small and write

$$p = p_0 + p' \qquad \rho = \rho_0 + \rho' \tag{16.1}$$

and $\rho' \ll \rho_0$ and $p' \ll p_0$. The Euler equation can be written as:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla(p_0 + p')}{\rho_0 + \rho'} \tag{16.2}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p'}{\rho_0}(1 - \frac{\rho'}{\rho_0}) \tag{16.3}$$

The velocity of an unperturbed gas $\mathbf{u}(\rho_0) = 0$ and we conclude that the non-zero velocity, caused by the density variation, is $O(\rho')$. Since the pressure/density fluctuations are very small, the non-linear $\mathbf{v} \cdot \nabla \mathbf{v} = O(\nabla(\rho')^2)$ - contribution to the Euler equation is very small, compared with $\frac{1}{\rho_0}p'$ and can be neglected together with the $O(p'\rho')$-contributions. This gives:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla p'}{\rho_0} \tag{16.4}$$

In the same approximation, the continuity equation:

$$\frac{\partial(\rho_0 + \rho')}{\partial t} + \nabla \cdot (\rho_0 + \rho')\mathbf{u} \approx \partial \rho' \partial t + \rho_0 \nabla \cdot \mathbf{u} = 0 \tag{16.5}$$

In deriving this equation the contributions $\rho' \nabla \cdot \mathbf{u}$ was neglected because both ρ' and u are small. According to the equation of state pressure $p = p(\rho(\mathbf{x}))$ is a function of density and depends upon spatial coordinates only due the density variation: $p = p(\rho_0 + \rho'(\mathbf{x})) = p_0 + p'(\mathbf{x}) = p(\rho_0) + (\frac{\partial p(\rho_0)}{\partial \rho})_s \rho'$ and

$$p' = (\frac{\partial p(\rho_0)}{\partial \rho})_s \rho' \tag{16.6}$$

Differentiating this over t and using (15.5) gives:

$$\frac{\partial p'}{\partial t} + \rho_0 (\frac{\partial p(\rho_0)}{\partial \rho})_s \nabla \cdot \mathbf{u} = 0 \tag{16.7}$$

That is where potential flow approximation $\mathbf{u} = \nabla \phi$, becomes handy: substituting this into equation (15.4) we derive readily:

$$p' = -\rho_0 \frac{\partial \phi}{\partial t} \tag{16.8}$$

and finally the expression (15.7) reads

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \tag{16.9}$$

where the speed of sound $c^2 = \frac{\partial p}{\partial \rho}_s$. This equation describing propagation of the density waves in fluids, is one of canonical equations of mathematical and theoretical physics. The derivation, presented here, demonstrates the broad range of applications described by the Euler equations in a potential flow approximation.

If all properties of a flow depend only on one coordinate x (*plane wave*), then (15.9) is:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{16.10}$$

To solve this equation, we introduce new variables $\xi = x - ct$ and $\eta = x + ct$. It these variables, the equation (15.9) becomes:

$$\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0 \tag{16.11}$$

and gives $\partial_\eta \phi = F(\eta)$ where F is an arbitrary function. Second integration gives $\phi = \int F(\eta) d\eta + F_1(\xi) \equiv f_1(\xi) + f_2(\eta)$ where f_1 and f_2 are arbitrary functions. Thus,

$$\phi(x,t) = f_1(x - ct) + f_2(x + ct) \tag{16.12}$$

describing two propagating or traveling plane waves.

The equation (15.10) can be solved using the Fourier transform. We seek solution is a form $\phi = \mathcal{R}e^{i\omega t - ikx}$. Substituting this into (15.10) gives:

$$\omega = \pm ck \tag{16.13}$$

and

$$\phi = A\cos(ckt + kx) + B\sin(ckt - kx) \tag{16.14}$$

The velocty field is then $u_x = \partial_x \phi$:

$$u_x = -Ak\sin(ckt + kx) + Bk\cos(ckt - kx) \equiv a\sin(ckt) + b\cos(ckt - kx) \tag{16.15}$$

and the constant a and b are found from initial conditions. We see that if at any instant of time and at any point in space $u = u_0$, then after time t the same value will be found at a distance ct from this point. Thus any pattern in a fluid propagates with the sound velocity c. Since only one velocity component $u_x = \partial_x \phi$ in a plane wave is not equal to zero , we conclude that the sound waves in fluids are *longitudinal.*

Consider a travelling way propagating in the x direction. The x-component of velocity can also is written as $u_x = \partial_x \phi(x - ct) = \phi'$ The Bernoulli equation, neglecting the $O(u^2)$ contributions as small is:

$$\frac{\partial \phi}{\partial t} + p'/\rho = \frac{\phi(x - ct)}{\partial t} + p'/\rho = c\phi' + p'/\rho = 0 \tag{16.16}$$

Combining two expression gives:

$$u_x = \frac{p'}{\rho c} \tag{16.17}$$

Since $p' = c^2\rho'$, we have

$$u_x = c\rho'/\rho \approx c\rho'/\rho_0 \tag{16.18}$$

The root -mean-square of the wave velocity averaged over one cycle is

$$u_{x,rms} = a/\sqrt{2} = c\rho'_{rms}/\rho_0 > 0 \tag{16.19}$$

Introducing the Mach number $Ma = u_x/c$ we see that the approximations used for derivation of the above expressions are valid if :

$$Ma = u_{x,rms}/c = \rho'_{rms}/\rho \ll 1 \tag{16.20}$$

Based on these relations, we can calculate the energy of a unit volume of a plane travelling wave:

$$\overline{E} = \rho_0 u_{x,rms}^2 = \rho_0 c^2 Ma^2 \tag{16.21}$$

and the total energy is simply equal to $\overline{E}V$.
Accoding to thermodynamics $(\frac{\partial p}{\partial \rho})_s = (\frac{c_p}{c_v})(\frac{\partial p}{\partial \rho})_T \equiv \gamma(\frac{\partial p}{\partial \rho})_T$ and if $pV = p/\rho = RT/m$, where m is the molecular weght, we have:

$$c = \sqrt{\gamma RT/m} \tag{16.22}$$

For air at room temperature and normal (atmospheric) pressure, $\gamma = 1.4$. For other substances, see the books on thermodynamics.
In general, the expression describing a sound wave propagating forward in the x-direction, can be written as:

$$\phi = a\cos(\frac{\omega x}{c} - \omega t + \alpha) \tag{16.23}$$

where a is called an amplitude and, in general, the wave -vector is defined as $\mathbf{k} = \frac{\omega}{c}\mathbf{n}$, where $\mathbf{n}$ is a unite vector in direction of propagation. The wave characterized by a single frequency and a single wave-vector , described above, is called monochromatic wave. Any more complex wave, can be represented as a superposition of these waves, which is basically an expansion in Fourier series.

Sound in a moving medium. Sound from moving sources. The dispersion relation $\omega = ck$ derived above for a monochromatic (single frequency) wave propagating in an **non-moving** infinite homogeneous medium. Now, imagine gas moving as a whole with velocity **V**, relative to an observer (us) staying in a fixed frame of reference K, where the coordinates of each point are defined by three numbers (x,y,z). We are interested in properties of sound detected by a non-moving observer (us).

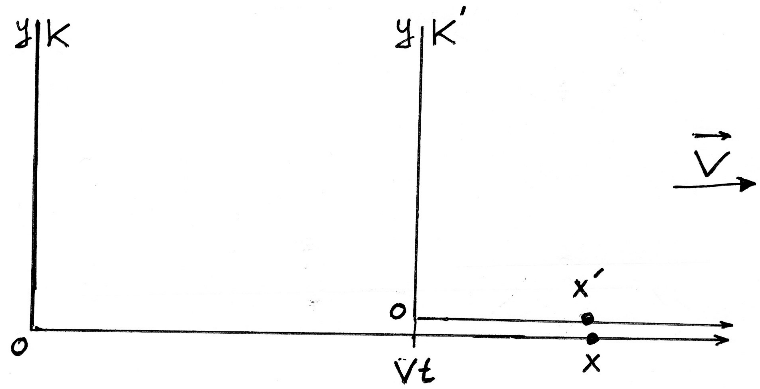


Figure 42: The coordinate frame K' moves with velocity V relative to the non-moving frame K. At time t $x = x' + Vt$. At $t = 0$, $x = x'$.

In the frame of reference K' moving with velocity **V**, the fluid is at rest and the solution to the wave equation $\phi = a\mathcal{R}e^{i(\mathbf{k}\cdot\mathbf{x}'-ikct))}$ is valid. It is clear from the Figure 28, $\mathbf{x}' = \mathbf{x} - \mathbf{V}t$ and the solution for a wave propagating in a **moving medium** (frame K') detected by a steady observer in a frame K is:

$$\phi = a\mathcal{R}e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{V}t)-ckt} = a\mathcal{R}e^{i(\mathbf{k}\cdot\mathbf{x}-(ck+\mathbf{V}\cdot\mathbf{k}t))} \equiv a\mathcal{R}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

with a frequency:

$$\omega = ck + \mathbf{k}\cdot\mathbf{V} \tag{16.24}$$

Thus, depending on the sign of velocity **V**, the frequency detected by a non-moving observer depends upon the angle between the wave-vector **k** and **V**. The effect is illustrated on Fig.29. We see that the left observer facing the "wind" detects the high frequency wave ($\omega > \omega_0 = ck$ ("blue shift") while the right one feels $\omega < \omega_0$ ("red shift").

Based on the above results, we can investigate the Doppler effect: the frequency of sound, as received by an observer moving relative to the source at rest with velocity **V** is not equal to the one generated by the source. In the frame K, where observer is steady, the velocity of the medium is $-\mathbf{V}$. The frequency generated by a source relative to the steady medium (now this is frame K') is $\omega_0 = ck$. Thus, the frequency received by a moving observer is

$$\omega = \omega_0 - \mathbf{k}\cdot\mathbf{V} = \omega_0(1 - \frac{V}{c}\cos\theta) \tag{16.25}$$

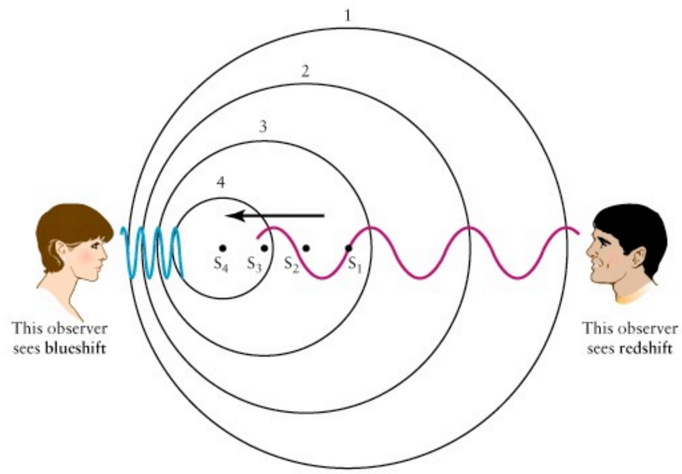


Figure 43: Doppler effect. Two "observers" talking with each other. The arrow indicates the velocity of the medium (wind) $\mathbf{V}$. Both receive distorted sound waves of the frequencies not equal to the frequencies of their voices in a closed room

Now, consider a medium at rest with a sound wave emitted from source moving with velocity $\mathbf{V}$. The frequency, generated in the moving frame where source is at rest is equal to $\omega = \omega_0 = ck$. The frequency in the frame K is thus: $\omega_0 = \omega(1 - \frac{V}{c}\cos\theta)$ and the frequency received by a non-moving observer is

$$\omega = \frac{\omega_0}{1 - \frac{V}{c}\cos\theta} \tag{16.26}$$

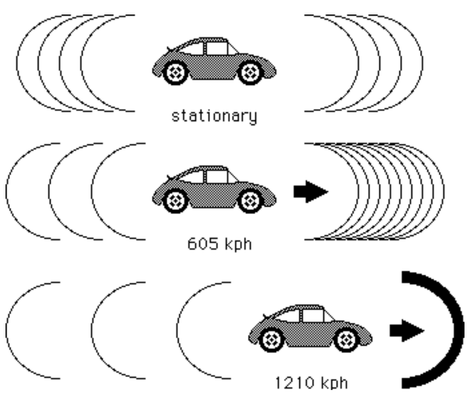


Figure 44: Doppler effect. Propagation of sound generated by a moving car.

This effect is illustrated on Fig. 30. If an observer is behind the car moving away so that $\mathbf{V}\cdot\mathbf{k} < 0$, the received frequency $\omega < \omega_0$. In the opposite case $\mathbf{k}\cdot\mathbf{V} > 0$ and the frequency $\omega > \omega_0$. The case $Ma = V/c > 1$ is singular corresponding to hypersonic flow regime.

Red shift in astronomy.

Gravity waves.

Another important example of potential flow is gravity waves generated by perturbation of a free fluid surface. In previous sections we saw how external forces or imposed pressure gradients, lead to the flow generation and, as a result, fluid mass transfer over large distances. The wave is a response to external perturbation which propagates, sometimes over huge distances, without any substantial mass transfer. Moreover, often, its speed is not related to the wave -generating flow velocity. It was Leonardo da Vinchi, who in the fifteenth century noticed that the wave on the water surface can move much faster than the water mass itself. A remarkable and widely observed example of the waves , not related to substantial "particle" transfer, are the wind-generated waves on a surface of the grain field. Waves which occur in electrodynamics, acoustics, solid state physics, are one of the most important effects in nature. Below, we briefly discuss examples of waves resulting from potential flow theory.

Consider a fluid layer $-H \leq z \leq 0$ and $-\infty < x, y < \infty$. It follows from equations of hydrostatics, the unperturbed free surface at $z = 0$ is plane.

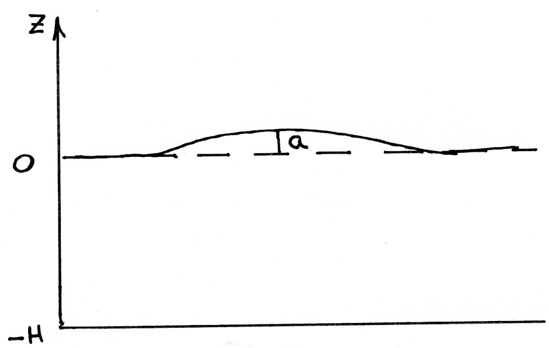


Figure 45: Perturbation of the surface of a fluid layer.

The Euler equations in a gravitational field are:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{v} + g\mathbf{k} \tag{16.27}$$

If the surface is perturbed from its equilibrium position $z = 0$, the access of pressure $\Delta p \approx \rho g a$, where a is the amplitude of perturbation, is created which tends to restore the fluid layer to its unperturbed shape. However, the pressure, being a scalar tends to push the surrounding perturbation fluid elements will propagate along the surface as a wave with the wave-length λ and amplitude a.

Let characteristic time of the oscillation be τ, so that the typical velocity can be estimated as $v \approx a/\tau$ and

$$\frac{\partial v}{\partial t} \approx \frac{a}{\tau^2}$$

The spatial derivative in the x-direction is then $\frac{\partial v}{\partial x} \approx v/\lambda$ and $v\nabla v \approx v^2/\lambda \approx \frac{1}{\lambda}\frac{a^2}{\tau^2}$. Interested in the long-wave propagation $a \ll \lambda$, we see that :

$$v\nabla v \approx \frac{v^2}{\lambda} \approx \frac{a^2}{\tau^2 \lambda} << \frac{\partial v}{\partial t} \approx \frac{a}{\tau^2}$$

meaning that, if $a/\lambda \ll 1$, the non-linear term can be neglected. Since the viscous contribution can be estimated as: $\nu \nabla^2 v \approx \nu \frac{v}{\lambda^2} \approx \nu \frac{a}{\tau \lambda^2}$, we see that it is small in comparison with time-derivative if: $\frac{\nu v}{\lambda^2} \ll v/\tau 1$, so that $\frac{\nu \tau}{\lambda^2} \ll 1$.

The remaining equation is:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\nabla p}{\rho} - g\mathbf{k} \tag{16.28}$$

It follows from this equation that

$$\frac{\partial \omega}{\partial t} = const = 0 \tag{16.29}$$

Thus, under these assumptions, the considered flow is *potential.* Taking into account that $\mathbf{v} = \nabla \phi$ and $g\mathbf{k} = \nabla g z$, the Bernoulli equation, corresponding to (15.28) (see (9.4)) can be written readily

$$\rho \frac{\partial \phi}{\partial t} + p + \rho g z = f(t) \tag{16.30}$$

Let us introduce the vertical displacement of the surface ζ and denoting the atmospheric pressure $p = p_0 = const$ on a surface, gives:

$$p_0 = -\rho g \zeta - \rho \frac{\partial \phi}{\partial t} + f(t) \tag{16.31}$$

Since $\mathbf{v} = \nabla \phi = \nabla(\phi + f(t))$, we can add an arbitrary function $f(t)$ to the potential ϕ, including $f(t) = \frac{p_0}{\rho} t$. Sustituting this into (15.31) gives on a surface $z = \zeta$:

$$g\zeta + \frac{\partial \phi}{\partial t}|_{z=\zeta} = 0 \tag{16.32}$$

Since ζ is a surface displacement in the z-direction, by definition:

$$v_z = \frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} \tag{16.33}$$

and, differentiating (15.32) over time t leads to:

$$g v_z + \frac{\partial^2 \phi}{\partial t^2}|_{z=\zeta} = (g\frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial t^2})_{z=\zeta} = 0 \tag{16.34}$$

or, taking into account that $\zeta \ll \lambda$, this equation can be simplified:

$$(g\frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial t^2})_{z=0} = 0 \tag{16.35}$$

and, as always in potential flows,:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{16.36}$$

The equations (15.35)-(15.36) describe the so called gravity waves.

Deep water waves.

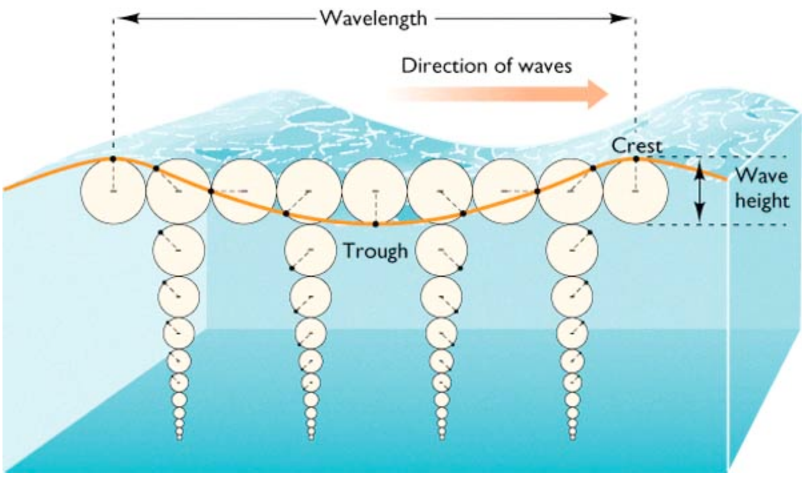


Figure 46: Deep water waves. The depth of the layer is much larger than the wave amplitude.

Considering first the "deep water waves propagating in the x-direction", we seek solution to this equation as:

$$\phi = A(z)\cos(\omega t - kx) \tag{16.37}$$

where ω is called "circular frequency" related to the period of oscillations as $T = 2\pi/\omega$ and the wave number $k = 2\pi/\lambda$. Substituting this solution into the equation (15.36) gives:

$$\frac{d^2 A}{dz^2} - k^2 A = 0 \tag{16.38}$$

This equation has two solutions: Be^{kz} and $B_1 e^{-kz}$. Since at the bottom $z \to -\infty$ ("deep water"), the velocity $v = 0$, we have:

$$\phi = Be^{kz}\cos(kx - \omega t) \tag{16.39}$$

From the boundary condition at $z = 0$, given by (15.35), we derive relation between ω and k which is called the "dispersion relation". Indeed, substituting the solution (15.39) into(15.35) gives:

$$\omega^2 = kg \tag{16.40}$$

The velocity field is found from the definition $v_x = \frac{\partial \phi}{\partial x}$ and $v_y = \frac{\partial \phi}{\partial y}$. The result is:

$$v_x = -Bke^{kz} \sin(kx - \omega t); \qquad v_z = Bke^{kz} \cos(kx - \omega t) \tag{16.41}$$

We see that velocity exponentially decreases outside the surface layer. This kind of waves is called "surface waves". The path of particles in the wave is found by the rules introduced above: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. Integrating the velocity field over time gives:

$$x - x_0 = -B\frac{k}{\omega}e^{kz_0} \cos(kx_0 - \omega t); \qquad y - y_0 = -B\frac{k}{\omega}e^{kz_0} \cos(kx_0 - \omega t); \tag{16.42}$$

These expressions give:

$$(x - x_0)^2 + (y - y_0)^2 = B^2(\frac{k}{\omega})^2 e^{2kz_0} \tag{16.43}$$

meaning that the fluid particle trajectories are circles with the radii $B\frac{k}{\omega}e^{kz_0}$, decreasing away from the surface into the "water" (see Fig.29). The speed of propagation is given by a general theory of waves:

$$U = \frac{\partial \omega(k)}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}} \tag{16.44}$$

Problem. Consider the same problem but in a layer of a finite depth H.

Shallow water waves. (Long gravity waves.)

Now we consider the case of shallow water waves, i.e. $\lambda >> H$ dominating the coastal area of oceans, seas, rivers etc. (see Fig. 30). Again, we neglect the nonlinear term $\mathbf{v} \cdot \nabla \mathbf{v}$ and assume that the wave propagates in the x-direction. In this case $v_x >> v_y, v_z$. The Euler equation reads ($v_x \equiv v$):

$$\rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0 \tag{16.45}$$

and

$$\frac{\partial p}{\partial z} = \rho g \tag{16.46}$$

Integrating this equation and taking into account that on the surface $z = \zeta$, pressure $p = p_0$, gives:

$$p = p_0 + \rho g(\zeta - z) \tag{16.47}$$

Combining this with (15.45), we have:

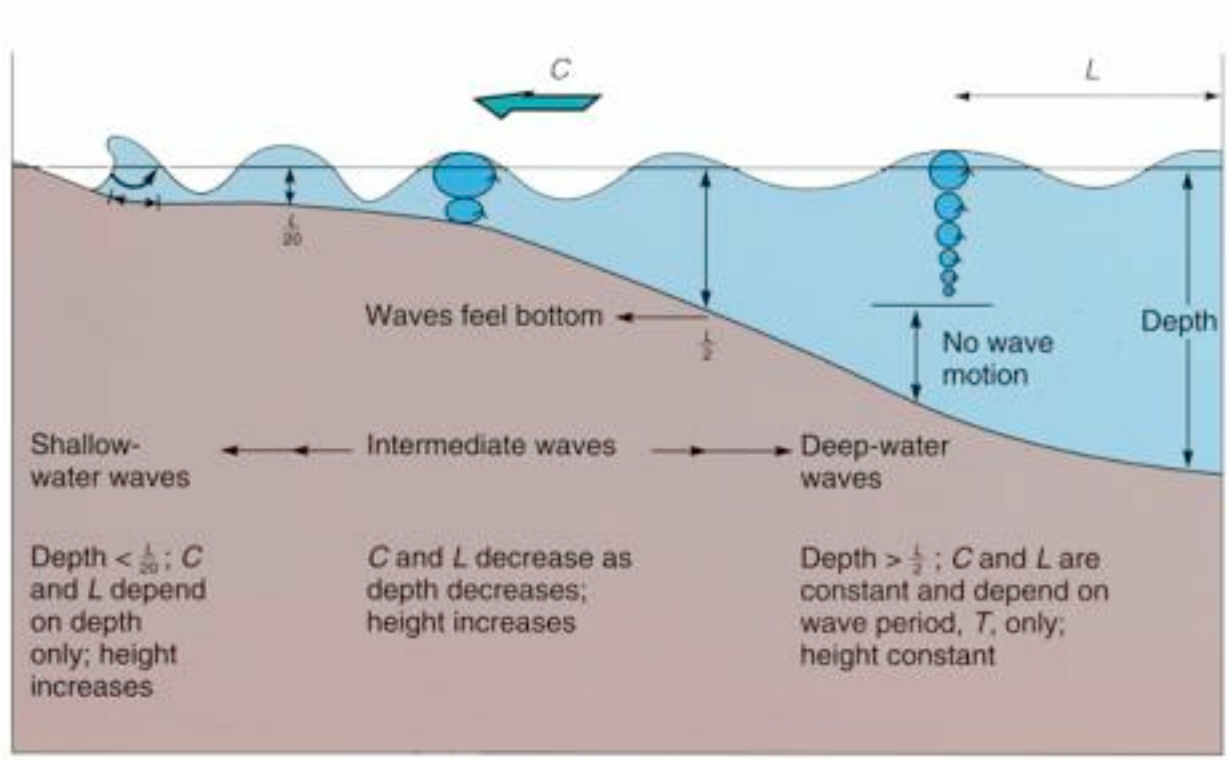


Figure 47: Deep and shallow water waves. Shallow ($\lambda \gg H$) waves appear near the coast.

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x} \tag{16.48}$$

Another important relation is derived directly from the continuity equation (mass conservation law). In this case the density $\rho = const$ but, due to the surface variation, the crossection of the flow varies in both time and space. Consider two crossections at points $x + dx$ and x. If width of the layer is $W = const$, the mass of the fluid contained in the volume $M(x,t) = \rho W S(x,t)dx$. The change of mass in this volume is then: $\frac{\partial M(,t)}{\partial t} = \rho W \frac{\partial S(x,t)}{\partial t}$. By the mass conservfation law, the variation of the fluid mass in the chosen volume is equal to total mass flux through two crossections $S(x + dx, t)$ and $S(x,t)$:

$$\rho W[(Sv)_{x+dx} - (Sv)_{dx}] = \rho W \frac{\partial Sv}{\partial x} dx$$

The balance is:

$$\frac{\partial S}{\partial t} + \frac{\partial Sv}{\partial x} = 0 \tag{16.49}$$

If the width of the channel is W, the area of a crossection is $S = (H + \zeta)W = S_0 + \zeta W$, where $S_0 = const$ is a width of a plane surface in the absence of perturbation, i.e. $\zeta = 0$. Since, ζ and v are small, we neglect

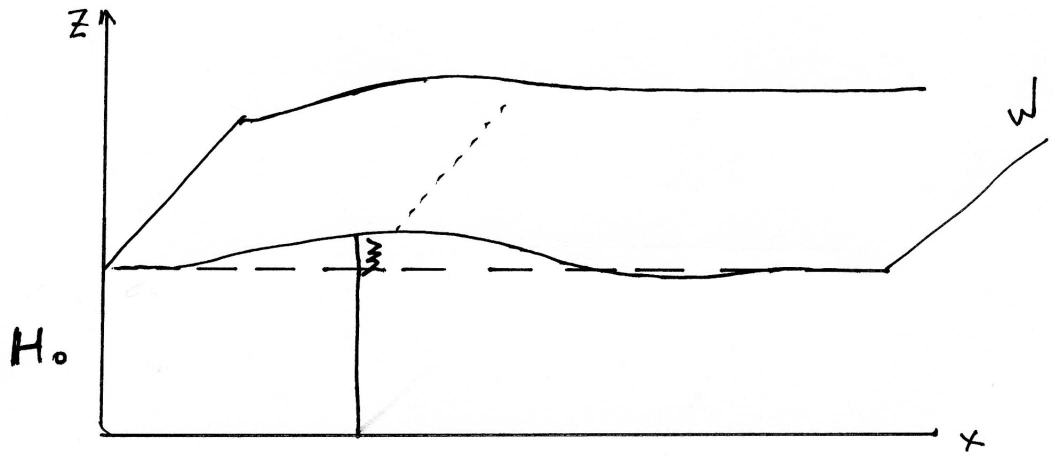


Figure 48: Deep water waves. The depth of the layer is much larger than the wave amplitude.

the $O(v\zeta)$ contribution, i.e. approximate $Sv = W(H+\zeta)v \approx S_0 v$, and obtain:

$$W\frac{\partial \zeta}{\partial t} + \frac{\partial S_0 v}{\partial x} = 0 \tag{16.50}$$

Differentiating this equation over t and taking into account

$$\frac{\partial v}{\partial t} = -g\frac{\partial \zeta}{\partial x}$$

we derive:

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{gS_0}{W}\frac{\partial^2 \zeta}{\partial x^2} = 0 \tag{16.51}$$

which is a familiar "wave equation"-one of the canonical equations of mathematical physics. Since $H = S_0/W$, we, looking for solution in the form $\zeta \propto \cos(\omega t - kx)$, derive dispersion relation for the shallow water waves:

$$\omega = \sqrt{gH}k \tag{16.52}$$

and the wave velocity:

$$c = \sqrt{gH} \tag{16.53}$$

If initial perturbation $\zeta(x,0) = \zeta_0$, the general solution to the equations is:

$$\zeta(x,t) = \zeta_0(x-ct) \tag{16.54}$$

meaning that this perturbation propagated along the surface of the layer without major distortions. This effect is very important in many enviromental flows. It is clear that if initial perturbation is plane, then it propagates as a whole with velocity c. In this case it not called a wave but a *bore*. If it is a point (stone dropped into a pond), then resulting wave will be a propagating circular pattern with the right edge moving with velocity c and the left (rear) one with velocity $-c$.

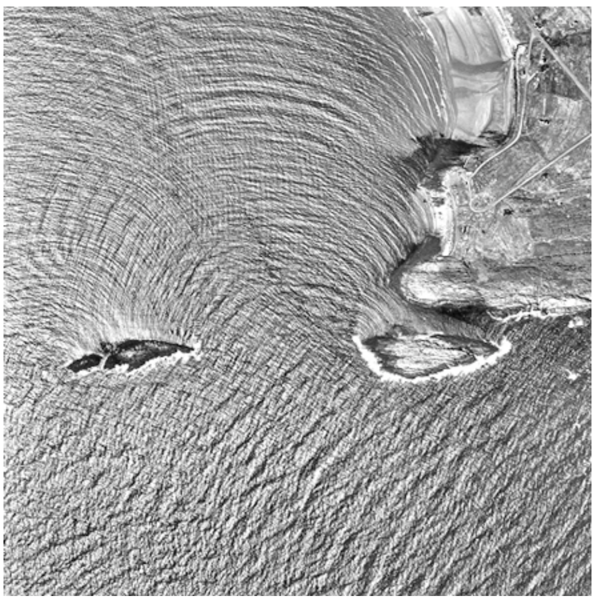


Figure 49: Shallow water waves. The waves originating from many different sources form a diffraction pattern.

A simple derivation of expression (15.53).

The above results were derived from the **first principles**, i.e. using the basic equations of hydrodynamics in a well-defined limit $\lambda >> H$ and $\zeta \ll \lambda$. The structure of the flow was not assumed but appeared as a solution to the wave equation (15.45)-(15.46).

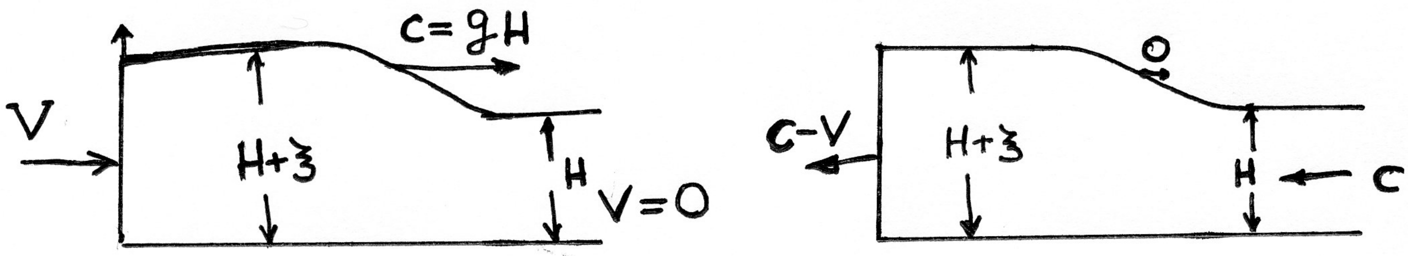


Figure 50: Flow structure in steady (left) and moving (right) frames.

Here we demonstrate how the expression (15.53) can be obtained by **assuming** the relation (15.54) stating that the initial perturbation of the surface propagates without substantial deformation. We consider the initial perturbation (see Fig.33) $\zeta(x) = \zeta_0(x) = const$ defined on a moving domain $x < ct$, propagating from left to right with an unknown velocity c. The velocity of the fluid at $x > ct$ is equal to $V = 0$, meaning that the fluid, which is on the right from a moving wave front is immobile. However, behind perturbation's front, the bulk of the flow moves with velocity $V \neq 0$. Since the amplitude of the wave is small, $V \ll c$.

As we saw above, in the case of a steady flow, Bernoulli's equation is very simple. Considering a problem of propagating perturbation (see Fig.33) in a frame of reference moving with velocity c, the time-independent Bernoulli's equation for a moving front can be used. In this frame, the flow behind and before "wave" front has velocities $c - V$ and and c, respectively. On a surface, where pressure $p = p_0 = const$, Bernoulli's and continuity equations are:

$$(c - V)(H + \zeta) = cH \tag{16.55}$$

and

$$\frac{1}{2}(c - V)^2 + g(H + \zeta) = gH + \frac{1}{2}c^2 \tag{16.56}$$

Since $c \ll V$, we neglect the $O(V^2)$ contributions to obtain:

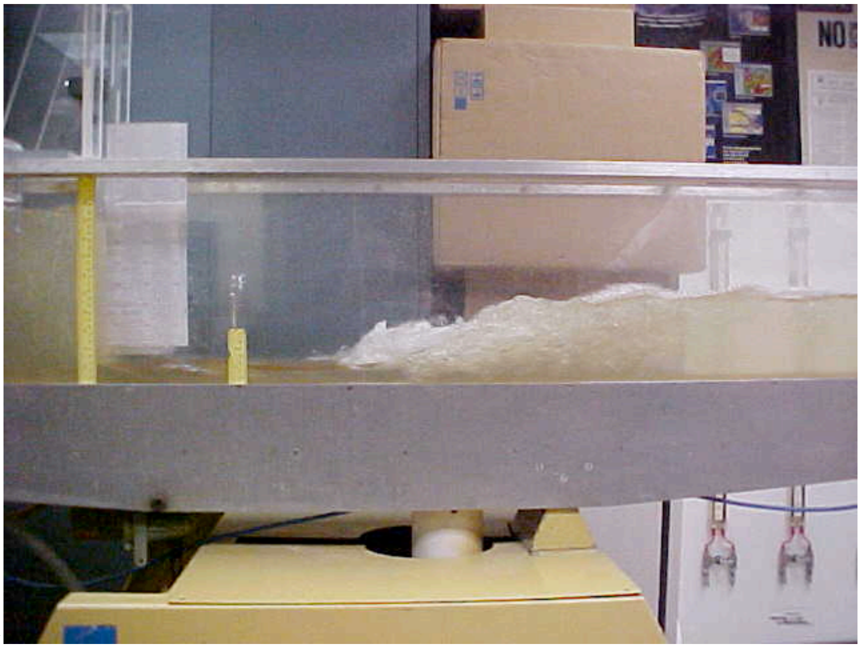
$$VH = \zeta c; \qquad g\zeta = cV \tag{16.57}$$

giving the expression (15.53) and justifying the assumption $\frac{V}{c} = \frac{\zeta}{H} \ll 1$ used in the derivation. We would like to stress that this "derivation" is based strong assumptions of a stable surface perturbation of a particular shape moving without any appreciable distortions. In the previous, much more superior, derivation these assumptions were not needed.

Shallow water waves: Tsunamis.

Perturbations of water surface can be generated by wind, impact of falling bodies like meteorites, tiny earth tremors etc. Generated far from the coastal areas, typically, these perturbations give rise to the deep water waves ($\lambda << H$) propagating with speed given by the relation (15.44). Approaching the shores where $\lambda \gg H$, these waves turns into shallow water waves which safely decay due to the bottom friction.

Tsunami is a different story. Tsunamis can be generated when the sea floor abruptly deforms and vertically displaces the overlying water. Tectonic earthquakes are a particular kind of earthquake that are associated with the earth's crustal deformation; when these earthquakes occur beneath the sea, the water above the deformed area is displaced from its equilibrium position. Waves are formed as the displaced water mass, which acts under the influence of gravity, attempts to regain its equilibrium. When large areas of the sea floor elevate or subside, a tsunami can be created. Tsunamis are unlike wind-generated waves, which



Jump.pdf

Figure 51: Hydraulic jump. The surface perturbations generated by a paddle in the left side of a channel reach the shallow water leading to the " jump".

many of us may have observed on a local lake or at a coastal beach, in that they are characterized as shallow-water waves, with long periods and wave lengths. The wind-generated swell one sees at a California beach, for example, spawned by a storm out in the Pacific and rhythmically rolling in, one wave after another, might have a period of about 10 seconds and a wave length of 150 m. A tsunami, on the other hand, can have a wavelength in excess of 100 km and period on the order of one hour.

As a result of their long wave lengths, tsunamis behave as shallow-water waves. A wave becomes a shallow-water wave when the ratio between the water depth and its wave length gets very small. As was derived in this Section, shallow-water waves move at a speed $c = \sqrt{gH}$. In the Pacific Ocean, where the typical water depth is about $H \approx 4000m$, a tsunami travels at about $c = \sqrt{gH} \approx 200m/s \approx 700km/hr$. Because the rate at which a wave loses its energy is inversely related to its wave length, tsunamis not only propagate at high speeds, they can also travel great, transoceanic distances with limited energy losses.

Large vertical movements of the earth's crust can occur at plate boundaries. Plates interact along these boundaries called faults. Around the margins of the Pacific Ocean, for example, denser oceanic plates slip under continental plates in a process known as subduction. Subduction earthquakes are particularly effective in generating tsunamis.

As a tsunami leaves the deep water of the open ocean and travels into the shallower water near the coast, it transforms. As the water depth decreases, since $c = \sqrt{gH}$, the tsunami slows. The tsunami's energy flux, which is dependent on both its wave speed and wave height, remains nearly constant. Consequently, as the tsunami's speed diminishes as it travels into shallower water, its height grows. Because of this shoaling effect, a tsunami, imperceptible at sea, may grow to be several meters or more in height near the coast. When it finally reaches the coast, a tsunami may appear as a rapidly rising or falling tide, a series of breaking waves, or even a bore. Capable of inundating, or flooding, hundreds of meters inland past the typical high-water level, the fast-moving water associated with the inundating tsunami can crush homes and other coastal structures. Tsunamis may reach a maximum vertical height onshore above sea level, often called a runup height, of 10, 20, and even 30 meters.

Froude number. All above considerations have been developed for the waves propagating in a non-moving, steady, fluid layer. Now, let us consider a layer ("river", brook) moving in the x-direction with velocity U together with source of the surface perturbation ("paddle"). In a frame of reference moving with velocity U, the wave pattern will be exactly the same as in the case of non-moving fluid. We are interested in the wave-shape, formed in a moving layer, "seen" by a stationary observer. Relative to this observer, the upstream (rear) part of the pattern will propagate with velocity $c-U$, while the downstream part will move with the speed $c+U$.

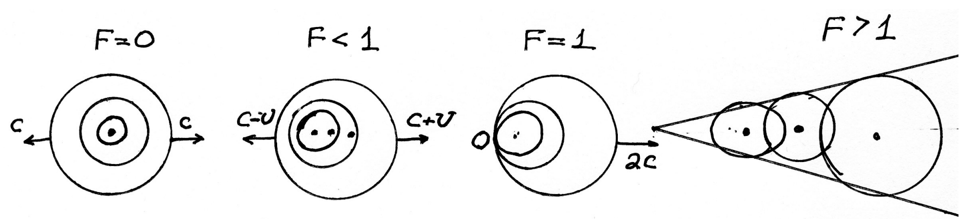


Figure 52: The surface perturbations propagating in a fluid layer moving with velocity U. The wave speed is $c = \sqrt{gH}$.

To characterize various situations, let us define the so called Froude number:

$$F = \frac{U}{c} = \frac{U}{\sqrt{gH}} \tag{16.58}$$

When $F = 0$ the wave has a circular shape. If $F < 1$, since upstream and downstream parts move with different velocities, it is somewhat deformed. At the critical $F = F_c = 1$, the rear part of the wave will not propagate, i.e. will stay frozen relative to the stationary observer. If $F > 1$ the entire pattern will be swept downstream with velocity $U > c$ and a wedge-shape form will be developed. If a boat moored upstream of

the wave origin. If $F < 1$ or $U < c$ eventually, the wave will reach and rock the boat. However, if $F \geq 1$, the fisherman will never know about the wave existence.
To elucidate the pattern formed in the case $F \geq 1$, consider Fig.39.

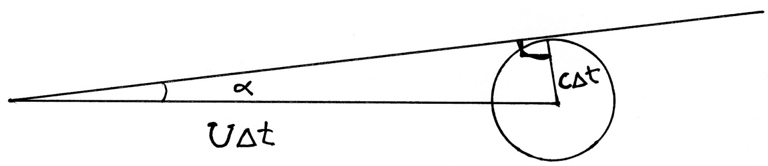


Figure 53:

Generated at time $t = 0$ at the origin $x = 0$, the local perturbation propagates in a moving water layer. After time-interval Δt, the center of the generated circular pattern will reach a point $x = U\Delta t$ and the radius of the circle measured from this center, determined by the wave propagation speed, is $c\Delta t$. We can see that the angle of the cone available to the wave is given by a relation:

$$\sin\alpha = \frac{c\delta t}{U\Delta t} = \frac{c}{U} = \frac{1}{F} \tag{16.59}$$

Similar patterns occur when the source of the sound wave moves with velocity which larger than the sound speed. In compressible fluids small density fluctuations are propagated by the sound waves moving with velocity $c \approx \sqrt{k_B T/m}$. If the source moves with a very large velocity $U \geq c$, the pile up of the density fronts "downstream" leads to a very large density and pressure jump, which is called shock wave. In this case the cone pattern can be readily detected using optical methods.

Hydraulic jump. Hydraulic jump is formed when a water from a faucet falls on a surface of a plate plate: first the water propagates away as a thin layer and then, suddenly, a substantial rise in the water level occurs. In an open channel the effect can be observed when membrane a separating two compartments filled with water of different height is suddenly lifted. To understand the physics governing this phenomenon, consider the jump shown on Fig. 40.

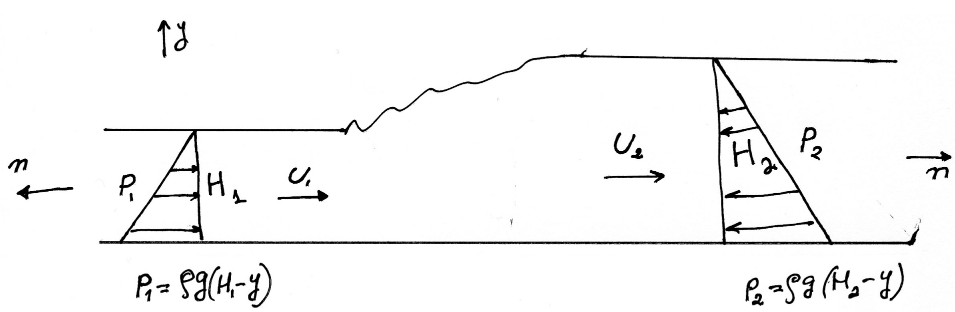


Figure 54: Hydraulic jump.

By continuity:

$$U_1 H_1 = U_2 H_2 \tag{16.60}$$

The equation for the x-component of momentum balancing the forces acting on crossections a control volume between crosssections 1 and 2 is:

$$-\mathbf{i} \int p\mathbf{n} dA = \mathbf{i} \int \rho(\mathbf{n} \cdot \mathbf{U})\mathbf{U} dA$$

or

$$\int_0^{H_1} \rho g(H_1 - y)W dy - \int_0^{H_2} \rho g(H_2 - y)W dy = -\rho U_1^2 W H_1 + \rho U_2^2 W H_2 \tag{16.61}$$

which gives

$$\frac{g}{2}(H_1^2 - H_2^2 = U_2^2 H_2 - U_1^2 H_1 \tag{16.62}$$

Combining this with (15.60) and defining upstream Froude number $F_1 = U_1/\sqrt{gH_1}$ we have:

$$(\frac{H_2}{H_1})^2 + \frac{H_2}{H_1} - 2F_1^2 = 0 \tag{16.63}$$

Since $H_2 \geq H_1$ this gives

$$\frac{H_2}{H_1} = \frac{1}{2}(\sqrt{1 + 8F_1^2} - 1) \tag{16.64}$$

This formula is called *hydraulic jump relationship.* This relation can be easily understood if we recall that for $F_1 > 1$, the speed of the flow is larger that of the surface perturbation propagation. Thus, if rise of a surface level occurs at a a point due to an obstacle, for example, the water upstream cannot react to it and readjust its motion. This leads to a formation of a steady hydraulic jump in the immediate vicinity of the obstacle. (Think about multiple car crash.)

(FIGURES.)