

## ENG ME 566 Advanced Engineering Mathematics

**Instructor:**

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**Prerequisites:**

Multivariate Calculus; Ordinary Differential Equations; or instructor permission. It is expected that you can already:

- solve simple first and second order linear ordinary differential equations
- differentiate and integrate elementary functions, including trigonometric, exponential and hyperbolic functions
- integrate by parts; evaluate simple surface and volume integrals
- use the binomial theorem and the series expansions of elementary functions (sine, cosine, exponential, logarithmic and hyperbolic functions)
- do all the problems in the **prerequisites self test** at the end of these notes.

**Course Outcomes:**

- consolidate understanding of vector calculus and applications to graduate level engineering
- introductory understanding of complex variable theory with applications to engineering problems
- ability to solve standard partial differential equations of engineering science using eigenfunction expansions, Fourier transforms and generalised functions
- become proficient in documenting calculations

**Textbook:**

Lectures are based on *Mathematical Methods for Mechanical Sciences* (M. S. Howe; 6th edition). It can be downloaded in pdf form from the ME 566 *BlackBoard* web site. You are expected to 'read around' the subject, and are recommended to consult other textbooks such as those listed on page 5.

**Course grading:**

- 4 take-home examinations (12.5% each)
- final **closed book** examination (50%)

**Homework:**

Four ungraded homework assignments provide practice in applying techniques taught in class – model answers will be posted on *BlackBoard*. In addition there are four take home exams each consisting of a short **essay** and 5 problems.

Each take-home examination problem is graded out of 10 as follows: up to 7 points for obtaining the correct answer and showing all your working; up to 3 points for setting-out the answer in a neat and readable manner. The essay is graded out of 25.

You have at least **ONE WEEK** for each take-home examination; exam scripts are due at the end of class on the date specified.

The following example illustrates how solutions should be set-out, and how you should include written explanations of your steps:

**Question:** Use the divergence theorem to evaluate  $\oint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F} = \frac{1}{3}(x^3, y^3, z^3)$  and S is the surface of the sphere  $|\mathbf{x}| \leq R$ .

**Solution:** To evaluate

$$I = \oint_S \mathbf{F} \cdot d\mathbf{S}$$

where  $\mathbf{F} = \frac{1}{3}(x^3, y^3, z^3)$  and S is the surface  $|\mathbf{x}| = R$ .

By the divergence theorem

$$I = \int_V \operatorname{div} \mathbf{F} dV,$$

where the integration is over the volume V of the sphere, and

$$\operatorname{div} \mathbf{F} = \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = \frac{1}{3}(3x^2 + 3y^2 + 3z^2) = r^2, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

Using spherical polar coordinates, and taking  $dV = 4\pi r^2 dr$  (because the integrand is radially symmetric):

$$I = \int_{r < R} r^2 dV = \int_0^R 4\pi r^4 dr = \left[ \frac{4\pi}{5} r^5 \right]_0^R$$

$$\therefore \quad I = \frac{4\pi}{5} R^5$$


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## SAMPLE ESSAY

**Essay:** Write a short essay (not more than two sides of paper) in which you give the definition of a vector field, define the *curl* of a vector field  $\mathbf{F}(\mathbf{x})$ , and state and prove Stokes' theorem.

A single valued vector function defined over a region of space is called a vector field. For example, the velocity vector in a fluid, and vector gravitational and electrical force distributions are vector fields.

The 'curl' of a vector field  $\mathbf{F}(\mathbf{x})$  is a new vector field  $\mathbf{curl} \mathbf{F}$  whose value at  $\mathbf{x}$  is determined by the following limiting procedure. For any fixed unit vector  $\hat{\mathbf{a}}$ , the component of  $\mathbf{curl} \mathbf{F}$  in the direction of  $\hat{\mathbf{a}}$  is

$$\hat{\mathbf{a}} \cdot \mathbf{curl} \mathbf{F} = \lim_{A \rightarrow 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r} \quad (1)$$

where  $C$  is a closed contour enclosing  $\mathbf{x}$  in the plane through  $\mathbf{x}$  with normal  $\hat{\mathbf{a}}$ .  $C$  encloses an area  $A$  of the plane, and is traversed in the positive sense with respect to the direction  $\hat{\mathbf{a}}$ .

For rectangular coordinates  $\mathbf{x} = (x, y, z)$ , with unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  respectively parallel to the  $x$ ,  $y$ , and  $z$  directions, this definition yields

$$\begin{aligned} \mathbf{curl} \mathbf{F} &\equiv \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}, \end{aligned}$$

where  $\mathbf{F} = (F_1, F_2, F_3)$ .

The definition (1) also leads to a direct proof of Stokes' theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S \mathbf{curl} \mathbf{F} \cdot d\mathbf{S} \equiv \int_S \mathbf{n} \cdot \mathbf{curl} \mathbf{F} dS,$$

where  $C$  is a closed contour,  $S$  an open, two-sided surface bounded by  $C$ , and the line integral is taken along  $C$  in the positive direction with respect to the unit normal  $\mathbf{n}$  on  $S$ .

We can write  $\oint_C \mathbf{F} \cdot d\mathbf{r} = \lim_{k \rightarrow \infty} \sum_k \oint_{C_k} \mathbf{F} \cdot d\mathbf{r}$ , where the surface  $S$  is partitioned into infinitesimal elements of area  $\delta S_k$  with unit normal  $\mathbf{n}_k$  and boundary  $C_k$ , because the integral along each section of  $C_k$  common to two adjacent elements enters twice, once for each element, but with opposite signs, and therefore makes no contribution to the sum. What remains is just the integral along those sections of the  $C_k$  that make up the contour  $C$ . The definition (1), with  $\hat{\mathbf{a}} = \mathbf{n}_k$ , then gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \lim_{k \rightarrow \infty} \sum_k \oint_{C_k} \mathbf{F} \cdot d\mathbf{r} = \lim_{k \rightarrow \infty} \sum_k \mathbf{n}_k \cdot \mathbf{curl} \mathbf{F} \delta S_k = \int_S \mathbf{curl} \mathbf{F} \cdot d\mathbf{S}. \quad \text{Q.E.D.}$$

## Schedule

Sep T6	}	Vector/Multivariate Calculus	
Sep R8			
Sep T13			
Sep R15			
Sep T20			
Sep R22			
	TH #1		
Sep T27	}	Complex Variable and Integration	
Sep R29			
Oct T4			
Oct R6			
Oct T11			
Oct R13			
Oct T18	TH #2		
Oct R20	}	PDEs and Fourier transforms	
Oct T25			
Oct R27			
Nov T1			
Nov R3			
Nov T8			
Nov R10	TH #3		
Nov T15	}		
Nov R17			
Nov T22			
	TH #3 due		
Nov T29	TH #4	Review	
Dec R1	}		
Dec T6			
Dec R8			
	TH #4 due		

## Syllabus

During the course of the semester you are expected to read and understand the material in those sections of **MMMS**, 6th edition, indicated after each heading in the following syllabus:

### 1. Vector/Multivariate Calculus

Sections 2.1 - 2.5; Section 2.7.

Assignment 1: Take-home examination 1; Essay 1.

### 2. Complex Variable and Integration

Sections 3.1 - 3.10.

Assignment 2: Take-home examination 2; Essay 2.

### 3/4. Partial differential equations; Fourier series and transforms

Sections 1.9, 1.10, 1.11;

Sections 4.1 - 4.3, 4.6, 4.9, 4.10.

Assignment 3: Take-home examination 3; Essay 3.

Assignment 4: Take-home examination 4; Essay 4.

**Alternative Texts:** (*not* recommended for purchase):

Lopez, R. J. *Advanced Engineering Mathematics*. Addison Wesley.

Kreyszig, E. *Advanced Engineering Mathematics*. Wiley.

Greenberg, M. E. *Advanced Engineering Mathematics*. Prentice Hall.

Zill, D. G. and Cullen, M. R. *Advanced Engineering Mathematics*. Jones and Bertlett.

## ASSIGNMENT 1

## Problems 2A

- Find the value of  $x$  given that  $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$  and  $\mathbf{b} = 4\mathbf{i} + x\mathbf{j}$  are perpendicular. [6].
- Solve for  $\mathbf{x}$  the vector equation  $\mathbf{x} + \mathbf{a}(\mathbf{b} \cdot \mathbf{x}) = \mathbf{c}$ , where  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are constant vectors. What happens when  $\mathbf{a} \cdot \mathbf{b} = -1$ ? [ $\mathbf{x} = \mathbf{c} - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})/(1 + \mathbf{a} \cdot \mathbf{b})$ ].
- Solve the simultaneous equations  $\mathbf{x} + \mathbf{y} \times \mathbf{p} = \mathbf{a}$ ,  $\mathbf{y} + \mathbf{x} \times \mathbf{p} = \mathbf{b}$ .  
[ $\mathbf{x} = \{(\mathbf{p} \cdot \mathbf{a})\mathbf{p} + \mathbf{a} - \mathbf{b} \times \mathbf{p}\}/(1 + p^2)$ ,  $\mathbf{y} = \{(\mathbf{p} \cdot \mathbf{b})\mathbf{p} + \mathbf{b} - \mathbf{a} \times \mathbf{p}\}/(1 + p^2)$ ].
- Show that  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) \times \mathbf{d} = (\mathbf{a} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})$ .

## Problems 2B

Calculate the gradients of

- $\varphi = x$  [i].
- $\varphi = r^n$ ,  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ . [ $nr^{n-2}\mathbf{r}$ ].
- $\varphi = \mathbf{r} \cdot \nabla(x + y + z)$ ,  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ . [ $\mathbf{i} + \mathbf{j} + \mathbf{k}$ ].

Find the directional derivatives in the direction of  $\mathbf{a}$  of

- $\varphi = e^x \cos y$ ,  $\mathbf{a} = (2, 3, 0)$ ,  $\mathbf{x} = (2, \pi, 0)$  [ $-2e^2/\sqrt{13}$ ].

Find the unit normal to the surfaces:

- $z = \sqrt{x^2 + y^2}$  at  $\mathbf{x} = (3, 4, 5)$  [ $(3, 4, -5)/5\sqrt{2}$ ].

## Problems 2C

Find the divergence of

- $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$  [0].
- $\mathbf{F} = (x, y^2, z^3)$  [ $1 + 2y + 3z^2$ ].
- $\mathbf{F} = xyz(\mathbf{i} + \mathbf{j} + \mathbf{k})$  [ $yz + xz + xy$ ].
- $\mathbf{F} = \mathbf{r}(\mathbf{r} \cdot \mathbf{a})$ ,  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ,  $\mathbf{a} = \text{constant}$ . [ $4\mathbf{r} \cdot \mathbf{a}$ ].

Prove that

- $\nabla^2(r^n) = n(n+1)r^{n-2}$ ,  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ .
- $\text{div}(f\nabla g) - \text{div}(g\nabla f) = f\nabla^2 g - g\nabla^2 f$ .
- $V = \frac{1}{6} \oint_S \mathbf{n} \cdot \nabla(r^2) dS$ , where  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and  $V$  is the volume enclosed by  $S$ .

Evaluate  $\oint_S \mathbf{n} \cdot \mathbf{F} dS$  when

- $\mathbf{F} = (x, x^2y, -x^2z)$  and  $S$  is the surface of the tetrahedron with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$  [ $\frac{1}{6}$ ].
- $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$  where  $a$ ,  $b$ ,  $c$  are constants, and  $S$  is the unit sphere  $|\mathbf{x}| = 1$  [ $\frac{4}{3}\pi(a + b + c)$ ].

**Problems 2E**

Evaluate the line integrals  $\int_C \mathbf{F} \cdot d\mathbf{r}$ :

1.  $\mathbf{F} = (3x^4, 3y^6, 0)$  where  $C$  is the curve:  $x^2 + y^2 = 4$ ,  $z = 0$  from  $(2, 0, 0)$  to  $(-2, 0, 0)$   $\left[-\frac{192}{5}\right]$ .

3.  $\mathbf{F} = (e^x, e^{4y/x}, e^{2z/y})$  where  $C$  is  $\mathbf{r} = (t, t^2, t^3)$ ,  $0 < t < 1$   $\left[\frac{3}{8}e^4 + \frac{3}{4}e^2 + e - \frac{13}{8}\right]$ .

Show that the following integral is path-independent and find the corresponding potential function:

5.  $\int_C [y \cos xy \, dx + x \cos xy \, dy - dz]$ ,  $[\varphi = \sin xy - z]$ .

7. Show that if  $\mathbf{r} = \mathbf{r}(t)$  on  $C$ , the *length of arc* between  $t = a$  and  $t = b$  is given by  $\ell = \int_a^b |\dot{\mathbf{r}}(t)| dt$ .

9.  $\mathbf{r} = (a \cos \theta, a \sin \theta, a \theta \tan \alpha)$  on a helix, where  $a, \alpha$  are constants. Show that the length of arc measured from  $\theta = 0$  is given by  $\ell = a \theta \sec \alpha$ .

**Problems 2F**

Evaluate  $\int_S \mathbf{F} \cdot d\mathbf{S}$ :

1.  $\mathbf{F} = (2x, 2y, 0)$  where  $S$  is the surface:  $z = 2x + 3y$ ,  $0 < x < 2$ ,  $-1 < y < 1$ ,  $[-16]$ .

3.  $\mathbf{F} = (1, x^2, xyz)$  where  $S$  is  $z = xy$ ,  $0 < x < y$ ,  $0 < y < 1$ ,  $\left[-\frac{59}{180}\right]$ .

5.  $\mathbf{F} = (2xy, x^2, 0)$  where  $S$  is  $\mathbf{r} = (\cosh u, \sinh u, v)$ ,  $0 < u < 2$ ,  $-3 < v < 3$ ,  $[2 \cosh^3 2 - 2]$ .

7.  $\mathbf{F} = (y^2, z^2, x^2 z)$  where  $S$  is the surface bounding the region  $x^2 + y^2 \leq 4$ ,  $x \geq 0$ ,  $y \geq 0$ ,  $|z| \leq 1$ ,  $[2\pi]$ .

Use Stokes's theorem to evaluate  $\oint_C \mathbf{F} \cdot d\mathbf{r}$ :

9.  $\mathbf{F} = x^2 y z \mathbf{j}$  where  $C$  is the quadrilateral with vertices  $(0, 1, 0)$ ,  $(1, 1, 0)$ ,  $(1, 0, 1)$ ,  $(0, 0, 1)$  traversed in this order.  $\left[-\frac{1}{6}\right]$ .

11.  $\mathbf{F} = xyz \mathbf{j}$  where  $C$  is the triangle with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ ,  $[0]$ .

13.  $\mathbf{F} = (-3y, 3x, z)$  where  $C$  is  $x^2 + y^2 = 4$ ,  $z = 1$ ,  $[24\pi]$ .

15.  $\mathbf{F} = 2z \mathbf{i} + 4x \mathbf{j} + 5y \mathbf{k}$  where  $C$  is  $x^2 + y^2 = 4$ ,  $z - x = 4$ , orientated anticlockwise when viewed from above.  $[-4\pi]$ .

## ASSIGNMENT 2

### Problems 3A

1. Express in the form  $a + ib$ :  
 (i)  $(2 + i)^2 + (2 - i)^2$ ; (ii)  $\frac{\alpha + \beta i}{\alpha - \beta i} - \frac{\alpha - \beta i}{\alpha + \beta i}$ .  $[6, 4i\alpha\beta/(\alpha^2 + \beta^2)]$ .
3. Find the modulus and principal value of the argument of:  $z = -1, i, 3 + 4i, -i - \sqrt{3}$ .  
 $[1, \pi; 1, \frac{\pi}{2}; 5, 0.927 \text{ radians}; 2, -\frac{5\pi}{6}]$ .
5. Express  $\sqrt{5 + 12i}, \sqrt{-5 + 12i}, \sqrt{i}$  in the form  $a + ib$ .  $[\pm(3 + 2i), \pm(2 + 3i), \pm\frac{1}{\sqrt{2}}(1 + i)]$ .
7. Find two real numbers  $a, b$  such that  $(1 + i)a + 2(1 - 2i)b - 3 = 0$ .  $[a = 2, b = \frac{1}{2}]$ .
9. If  $z_1, z_2, z_3$  are complex numbers, show that  
 (i)  $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$   
 (ii)  $|2z_1 - z_2 - z_3|^2 + |2z_2 - z_3 - z_1|^2 + |2z_3 - z_1 - z_2|^2 = 3\{|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2\}$
11. Show that

$$\frac{1 + \cos \theta + i \sin \theta}{1 - \cos \theta + i \sin \theta} = \cot \left( \frac{\theta}{2} \right) e^{i(\theta - \frac{\pi}{2})}.$$

Evaluate the roots:

13.  $\sqrt[8]{1}$   $[\pm 1, \pm i, \pm(1 \pm i)/\sqrt{2}]$
15.  $z^2 - (5 + i)z + 8 + i = 0$   $[z = 3 + 2i, 2 - i]$

### Problems 3D Evaluate:

1.  $\oint_{|z|=2} \frac{dz}{z+i}$   $[2\pi i]$ .
3.  $\oint_C \frac{dz}{z^2 - \frac{1}{4}}$  where  $C$  is the square with corners  $\pm(1 \pm i)$   $[0]$ .
5.  $\oint_{|z - \frac{\pi}{2}|=1} \frac{\sin z \, dz}{(z^2 - \frac{\pi^2}{4})}$   $[2i]$
7.  $\oint_{|z|=\frac{3}{2}|a|} \frac{dz}{z-a}$ .  $[2\pi i]$ .
9.  $\oint_{|z|=1} \frac{(2z-3)dz}{z^2-3z}$ .  $[2\pi i]$ .

### Problems 3E

Evaluate the residues of:

1.  $z \cosh(3/z)$ .  $[\frac{9}{2} \text{ at } z = 0]$ .
3.  $z^4/(z^2 + 1)$ .  $[-\frac{i}{2} \text{ at } z = i; \frac{i}{2} \text{ at } z = -i]$ .
5.  $\frac{\sin z}{z^4}$ .  $[-\frac{1}{6} \text{ at } z = 0]$ .
7.  $z^2 e^{\frac{1}{z}}$ .  $[\frac{1}{6} \text{ at } z = 0]$ .
9.  $(\cosh 2z)/z^5$ .  $[\frac{2}{3} \text{ at } z = 0]$ .



Evaluate by the residue theorem (contours are traversed in the anticlockwise sense):

$$11. \oint_{|z|=1} \frac{z^6+7}{z^2-2z} dz \quad [-7\pi i].$$

$$13. \oint_{|z|=1} \frac{\sinh z}{4z^2+1} dz \quad [\pi i \sin(\frac{1}{2})].$$

$$15. \oint_{|z-\frac{i}{2}|=1} \frac{z^4}{z^2+1} dz \quad [\pi].$$

$$17. \oint_{|z-\frac{i}{2}|=1} \frac{1}{z^3(z-1)^2} dz \quad [6\pi i].$$

$$19. \oint_{|z|=\frac{3}{2}} e^{\frac{1}{z}}/(z-1)^2 dz \quad [0].$$

### Problems 3F

Evaluate by the residue theorem:

$$1. \int_0^{2\pi} \frac{d\theta}{1+a \sin \theta}, \quad |a| < 1 \quad \left[ \frac{2\pi}{\sqrt{1-a^2}} \right].$$

$$3. \int_0^{2\pi} \frac{\cos 2\theta d\theta}{1-2p \cos \theta + p^2}, \quad |p| < 1 \quad \left[ \frac{2\pi p^2}{1-p^2} \right].$$

$$5. \int_0^\pi \frac{\sin^4 \theta d\theta}{a + \cos \theta}, \quad a > 1, \quad \left[ \pi \left( \frac{3}{2}a - a^3 + (a^2 - 1)^{\frac{3}{2}} \right) \right].$$

$$7. \int_0^\infty \frac{dx}{(1+x^2)(4+x^2)} \quad \left[ \frac{\pi}{12} \right].$$

$$11. \int_0^\infty \frac{dx}{(1+x^2)^2} \quad \left[ \frac{\pi}{4} \right].$$

$$15. \int_0^\infty \frac{x \sin 2x dx}{1+x^2} \quad \left[ \frac{\pi}{2e^2} \right].$$

$$17. \int_{-\infty}^\infty \frac{\cos kx dx}{x-a}, \quad (a, k \text{ real}, k > 0) \quad [-\pi \sin ka].$$

$$19. \int_{-\infty}^\infty \frac{dx}{x(x-ai)}, \quad (a > 0) \quad \left[ \frac{\pi}{a} \right].$$

### ASSIGNMENT 3

#### Problems 1H

Find the eigenvalues and eigenfunctions of:

1.  $y'' + \lambda y = 0$ ,  $y(0) = 0$ ,  $y'(1) = 0$  [ $\lambda_n = \left(\frac{(2n+1)\pi}{2}\right)^2$ ,  $n = 0, 1, 2, \dots$ ,  $y_n = \sin(\sqrt{\lambda_n}x)$ ].
2.  $y'' + \lambda y = 0$ ,  $y'(0) = 0$ ,  $y'(\pi) = 0$  [ $\lambda_n = n^2$ ,  $n = 0, 1, 2, \dots$ ,  $y_n = \cos(nx)$ ].
3. Transform the equation  $y'' + 2y' + (1-\lambda)y = 0$ , ( $y(0) = 0$ ,  $y(1) = 0$ ) into Sturm-Liouville form by multiplying by  $e^{2x}$ . Calculate the eigenvalues and eigenfunctions and show that  $\int_0^1 e^{2x} y_n(x) y_m(x) dx = 0$ ,  $m \neq n$ . [ $\lambda_n = -n^2\pi^2$ ,  $n = 1, 2, 3, \dots$ ,  $y_n = e^{-x} \sin(n\pi x)$ ].

#### Problems 4A

1. If

$$f(x) = \begin{cases} x, & 0 < x < \ell/2, \\ \ell - x, & \ell/2 < x < \ell, \end{cases}$$

in the boundary value problem for the diffusion equation (4.3.8), show that

$$u = \frac{4\ell}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left(\frac{(2n+1)\pi x}{\ell}\right) \exp\left(\frac{-(2n+1)^2 \pi^2 \kappa t}{\ell^2}\right), \quad t > 0.$$

3. When  $f(x) = x$  in problem 2, deduce that

$$u = \frac{\ell}{2} - \frac{4\ell}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{(2n+1)\pi x}{\ell}\right) \exp\left(\frac{-(2n+1)^2 \pi^2 \kappa t}{\ell^2}\right)}{(2n+1)^2}, \quad t > 0.$$

5. Show that the solution of the *Dirichlet* problem for  $u$ :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the rectangular domain  $0 < x < a$ ,  $0 < y < b$ , where  $u = 0$  on each side of the rectangle *except* that along the  $x$ -axis, where  $u = f(x)$  ( $0 < x < a$ ,  $y = 0$ ), can be expressed in the form

$$u = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right), \quad \text{where } A_n = \frac{2}{a} \int_0^a \frac{f(x) \sin(n\pi x/a) dx}{\sinh(n\pi b/a)}.$$

7. Obtain all solutions of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = u$$

of the form  $u(x, y) = (A \cos \lambda x + B \sin \lambda x) f(y)$ , where  $A$ ,  $B$ ,  $\lambda$  are constants. Show that the particular solution that satisfies  $u(0, y) = 0$ ,  $u(\pi, y) = 0$ ,  $u(x, 1) = x$  ( $0 < x < \pi$ ) is given by

$$u = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{(1+n^2)(1-y)} \sin nx.$$

9. Show that

$$u(x, t) = \frac{1}{2}a - \frac{4a}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos\{(2n+1)\pi x/a\}}{(2n+1)^2} e^{-t[(2n+1)\pi/a]^2}$$

satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial x}(\pm a, t) = 0, \quad u(x, 0) = |x|, \quad |x| < a.$$

11. Show that when the conditions of Problem 10 are replaced by (i)  $u(0, t) = 1$  and  $u(1, t) = 0$  for  $t > 0$ , (ii)  $u(x, 0) = \cos(\frac{\pi}{2}x)$  for  $0 \leq x \leq 1$ ,

$$u(x, t) = 1 - x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n(4n^2 - 1)} e^{-n^2 \pi^2 t}.$$

13. Show that the solution  $u(x, y)$  of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi,$$

where  $\partial u / \partial x = 0$  at  $x = 0$  and  $x = \pi$ ,  $u(x, \pi) = 0$ , and  $u(x, 0) = f(x)$ , is given by

$$u = A_0 \frac{(\pi - y)}{2\pi} + \sum_{n=1}^{\infty} A_n \frac{\cos nx \sinh n(\pi - y)}{\sinh n\pi}, \quad A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx.$$

15. If  $u(x, t)$  satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0, \quad \text{subject to } u(0, t) = 0, \quad \frac{\partial u}{\partial x}(a, t) = 0, \quad u(x, 0) = \sin^3\left(\frac{\pi x}{2a}\right),$$

show that

$$u(x, t) = \frac{3}{4} \sin\left(\frac{\pi x}{2a}\right) e^{-\pi^2 t/4a^2} - \frac{1}{4} \sin\left(\frac{3\pi x}{2a}\right) e^{-9\pi^2 t/4a^2}.$$

### Problems 1J

Show that:

1.  $\delta(x - y) = \delta(y - x).$

3. If  $xf(x) = 0$  for all values of  $x$ , then  $f(x) = A\delta(x)$ , where  $A$  is an arbitrary constant.  $\left[ \int_{-\infty}^{\infty} g(x)f(x)dx = \int_{-\infty}^{\infty} f(x) \left\{ x \left( \frac{g(x)-g(0)}{x} \right) + g(0) \right\} dx = 0 + g(0) \int_{-\infty}^{\infty} f(x)dx = \text{constant} \times g(0), \therefore f(x) = \text{constant} \times \delta(x). \right]$

5.  $F(x)\delta(x - a) = F(a)\delta(x - a).$

7.  $\frac{d}{dx}H(f(x)) = \frac{df(x)}{dx}\delta(f(x)).$

9.  $\frac{d^2|x|}{dx^2} = 2\delta(x).$

11.  $\int_{-\infty}^{\infty} \delta^{(n)}(x - a)f(x)dx = (-1)^n f^{(n)}(a).$

13.  $\lim_{\epsilon \rightarrow 0} \frac{-2\epsilon x}{\pi(x^2 + \epsilon^2)^2} = \delta'(x).$

# ASSIGNMENT 4

## Problems 4F

Verify the following Fourier transform pairs:

1.  $f(x) = e^{-|x|}, \quad \hat{f}(k) = \frac{\sqrt{2}}{\sqrt{\pi}(1+k^2)}.$
3.  $f(x) = e^{-ax^2}, \quad \hat{f}(k) = \frac{1}{\sqrt{2a}}e^{-k^2/4a}, \quad a > 0.$
5.  $f(x) = \delta(x), \quad \hat{f}(k) = \frac{1}{\sqrt{2\pi}}.$
7.  $f(x) = \begin{cases} x, & 0 < x < a, \\ 0, & x > a, \end{cases} \quad \hat{f}_c(k) = \sqrt{\frac{2}{\pi}} \frac{[ka \sin ka + \cos ka - 1]}{k^2}.$
9.  $f(x) = e^{-x}, \quad \hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \frac{k}{1+k^2}.$
11.  $f(x) = x^n e^{-ax}, \quad n = \text{positive integer}, \quad a > 0, \quad \hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \frac{n!}{(k^2 + a^2)^{n+1}} \text{Im}(a+ik)^{n+1}.$
13.  $f(x) = e^{-ax} \quad (a > 0), \quad \hat{f}_c(k) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + k^2}.$
15.  $f(x) = \frac{e^{-ax}}{x} \quad (a > 0), \quad \hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \tan^{-1} \left( \frac{k}{a} \right).$
17.  $f(x) = \cos(x^2/2), \quad \hat{f}_c(k) = \frac{1}{\sqrt{2}} (\cos(k^2/2) + \sin(k^2/2)).$
19.  $f(x) = \frac{1}{x}, \quad \hat{f}_s(k) = \sqrt{\frac{\pi}{2}}.$

## Problems 4G

1. The steady temperature distribution  $u(x, y)$  in  $x > 0, y > 0$  is governed by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{and} \quad \begin{cases} u = 0, & x = 0, \quad 0 < y < \infty, \\ \partial u / \partial y = -\sigma \delta(x - a), & (a > 0), \quad 0 < x < \infty, \quad y = 0. \end{cases}$$

Use the sine transform to show that

$$u = \frac{\sigma}{2\pi} \ln \left( \frac{y^2 + (a+x)^2}{y^2 + (a-x)^2} \right), \quad x > 0, \quad y > 0.$$

3. Show that the bounded solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{and} \quad \begin{cases} u = 0, & x = 0, \quad 0 < y < \infty, \\ u = u_0 x / (1 + x^2), & 0 < x < \infty, \quad y = 0. \end{cases}$$

is

$$u(x, y) = \frac{u_0 x}{(1+y)^2 + x^2} \equiv \text{Re} \left( \frac{u_0}{z+i} \right), \quad z = x + iy.$$

6. Find the function  $u(x, y)$  that is bounded in  $y > 0$  and satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad y > 0, \quad -\infty < x < \infty, \quad \text{and} \quad u = \delta(x - \xi) \quad \text{on} \quad y = 0. \quad \left[ u = \frac{y}{\pi[(x - \xi)^2 + y^2]} \right]$$

7. If  $u(x, y)$  is bounded in  $y > 0$  and satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad y > 0, \quad -\infty < x < \infty, \quad \text{and} \quad u = f(x) \quad \text{on} \quad y = 0,$$

show that

$$u = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{[(x - \xi)^2 + y^2]}.$$

8. If  $\nabla u(x, y)$  is bounded in  $y > 0$  and satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad y > 0, \quad -\infty < x < \infty, \quad \text{and} \quad \frac{\partial u}{\partial y} = \delta(x - \xi) \quad \text{on} \quad y = 0,$$

show that

$$\nabla u = \frac{(x - \xi, y)}{\pi[(x - \xi)^2 + y^2]}, \quad u = \frac{1}{2\pi} \ln[(x - \xi)^2 + y^2] + \text{constant}, \quad y > 0.$$

9. If  $\nabla u(x, y)$  is bounded in  $y > 0$  and satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad y > 0, \quad -\infty < x < \infty, \quad \text{and} \quad \frac{\partial u}{\partial y} = f(x) \quad \text{on} \quad y = 0,$$

show that

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) \ln[(x - \xi)^2 + y^2] d\xi + \text{constant}, \quad y > 0.$$

10. Use the cosine transform to show that, if  $u(x, y)$  is bounded in  $y > 0$  and satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad y > 0, \quad -\infty < x < \infty, \quad \text{and} \quad u = H(a - |x|), \quad a > 0, \quad \text{on} \quad y = 0,$$

then

$$u = \frac{1}{\pi} \left\{ \tan^{-1} \left( \frac{a + x}{y} \right) + \tan^{-1} \left( \frac{a - x}{y} \right) \right\}, \quad y > 0.$$

11. Show that the solution  $u(x, y)$  of

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad x > 0, \quad 0 < y < a, \\ u(x, 0) &= f(x), \quad 0 < x < \infty, \\ u(x, a) &= 0, \quad 0 < x < \infty, \\ u(0, y) &= 0, \quad 0 < y < a, \end{aligned}$$

is given by

$$u = \frac{2}{\pi} \int_0^\infty f(\xi) d\xi \int_0^\infty \frac{\sinh[k(a - y)]}{\sinh ka} \sin kx \sin k\xi dk.$$

12. If  $u(x, t)$  is bounded and satisfies

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, \quad x > 0, \quad t > 0, \\ u(x, 0) &= xe^{-x^2/4}, \\ u(0, t) &= 0, \\ \text{show that } u &= \frac{x}{(1+t)^{\frac{3}{2}}} e^{-x^2/4(1+t)}.\end{aligned}$$

## REVIEW TOPICS

### VECTOR/MULTIVARIATE CALCULUS

Vector algebra, applications to geometry, simple linear equations (MMMS: §2.1)

Scalar and vector fields, div, grad and curl (§§2.2 - 2.4)

Divergence theorem, Stokes theorem (§2.4)

Green's identities (§2.5)

How to evaluate line and surface integrals (§2.7)

### COMPLEX VARIABLES AND INTEGRATION

Algebra of complex numbers (§3.1)

Functions of a complex variable, definition of derivative (§3.2)

Cauchy-Riemann equations (§3.2)

Integration in the complex plane (§3.3)

Cauchy's theorem (§3.4)

Taylor and Laurent expansions (§§3.5 - 3.7)

Poles, residue theorem (§§3.8, 3.9)

Applications to evaluate simple trig and infinite integrals (§3.10)

Principal value integrals, Fourier integrals (§3.10)

The ML-theorem (§3.3)

### PARTIAL DIFFERENTIAL EQUATIONS AND TRANSFORM THEORY

Well posed problems (§4.2)

D'Alembert's solution of the wave equation (§4.1)

Method of separation of variables (§4.3)

Sturm-Liouville equation, eigenvalues and eigenfunctions (§1.9)

Eigenfunction expansions (generalized Fourier series) (§§1.9, 4.3)

Dirac delta function, Heaviside and sgn functions, and their derivatives,  $\epsilon$ -sequences (§1.11)

Use of  $e^{-\epsilon|x|}$  and  $e^{-\epsilon x}$  (§§1.11, 4.9, 4.10)

Fourier, Sine and Cosine transforms, inversion formulae (§4.9)

Applications to partial differential equations (§4.10)

Time: 20 minutes

1. Evaluate  $\int_0^{\frac{\pi}{2}} \cos^4 x \sin 2x \, dx$ . [Ans.  $\frac{1}{3}$ ]
2. Evaluate  $\int_V \{(x^2 + y^2 + z^2)^{\frac{1}{2}} + 3x^2\} \, dx \, dy \, dz$  where  $V$  is the sphere  $x^2 + y^2 + z^2 = R^2$ .  
[ $\pi R^4 + \frac{4\pi}{5} R^5$ ]
3. Evaluate  $\int_0^1 dy \int_{y-1}^{1-y} y \, dx$ . [ $\frac{1}{3}$ ]
4. Find the general solution of  $d^2y/dx^2 + k^2y = 2x$ , where  $k > 0$  is constant.  
[ $y = A \cos kx + B \sin kx + 2x/k^2$ ]
5. Solve  $dy/dx + x^3y = 0$ , given that  $y = 1$  when  $x = 1$ . [ $y = e^{\frac{1}{4}(1-x^4)}$ ]
6. Evaluate  $\frac{d}{dx} \left( \frac{\tan x}{1+x} \right)$ .  $\left[ \frac{\sec^2 x}{1+x} - \frac{\tan x}{(1+x)^2} \right]$
7. Evaluate  $\lim_{x \rightarrow 0} \frac{\sin^4 x}{(e^x - 1 - x)^2}$ . [4]