QUALIFYING EXAM IN SYSTEMS ENGINEERING

Written Exam: MAY 24, 2022, 9:00AM to 1:00PM, EMB 105

Oral Exams: May 26, 2022
Time/Location TBA (~1 hour per student)

CLOSED BOOK, NO CHEAT SHEETS
BASIC SCIENTIFIC CALCULATOR PERMITTED
ALL EXAM MATERIALS STAY IN THE EXAM ROOM

GENERAL INSTRUCTIONS:

1) **Please write on every sheet:**
   a. Your Exam Number
   b. The page numbers (example: Page 1 of 4)

2) **Only write on 1 side.**
   Exams may be scanned and emailed to the faculty for grading.
   If using pencil, make sure it is dark.

COMPLETE THE REQUIRED SECTIONS AS BELOW:

The exam consists of **three topical sections**. Select **three** of the following **five sections**:

A. Dynamic Systems Theory (SE/EC/ME 501)

B. Continuous Stochastic Processes (EC505) or Discrete Stochastic Processes (EK500 and SE/ME 714)

C. Optimization (SE/EC 524)

D. Dynamic Programming and Stochastic Control (SE/EC/ME 710)

E. Nonlinear Systems and Control (SE/ME 762)

A. Dynamic Systems Theory (SE/EC/ME 501)

SE Linear Systems Qualifying Exam - 2022

1. Suppose that the matrix A is $n \times n$, B is $n \times m$, and C is $q \times n$ and that the system

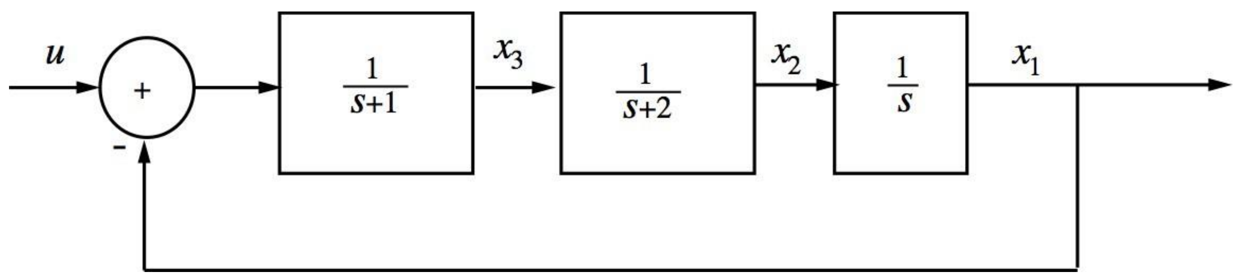
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

is *controllable* and *observable*. Prove that there are an $m \times n$ matrix K and an $n \times q$ matrix E suchthat the observer system

$$\dot{z} = (A - EC)z + ECy + Bu \tag{2}$$

together with (1) are asymptotically stable under the control $u = Kz$.

2. Design an observer for the system shown in the figure. The observer should be of second order with both eigenvalues equal to -3.



3. (a) Find the Jordan Normal Form $\mathbf{J_A}$ of

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & -4 \\ 8 & 5 & -4 \\ 12 & 6 & -5 \end{bmatrix}.$$

(b) Find a matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{J}_{\mathbf{A}}$.

(c) For the $\mathbf{J}_{\mathbf{A}}$ found in part (a), compute $e^{\mathbf{J}_{\mathbf{A}}t}$ in closed form.

B. Continuous Stochastic Processes (EC505) or Discrete Stochastic Processes (EK500 and SE/ME 714)

505 Question

Qualifying Examination
Discrete Stochastic Processes
Spring 2022

Problem 1. Suppose X and Y are independent standard normal random variables. Let (R, Θ) be the polar coordinates of (X, Y), i.e.,

$$\Theta(X, Y) = \arctan(Y/X) \qquad R = \sqrt{X^2 + Y^2}$$

Then, the joint probability density function of R and Θ is given by

$$f(r, \theta) = \frac{r}{2\pi} e^{-r^2/2}$$

a. Determine the marginal probability density functions of R and Θ. Are R and Θ independent?

b. Determine an algorithm for generating R using $U_1 \sim U[0, 1]$.

c. Determine an algorithm for generating Θ using $U_2 \sim U[0, 1]$.

d. Assuming U_1 and U_2 above are independent, determine an algorithm to generate standard normal random variables X and Y, discussed above.

Problem 2. Suppose $\mathbf{N} = \{N(t); t \geq 0\}$ is a homogeneous Poisson process with rate function λ. Let $A(t)$ and $Y(t)$ denote respectively the age and residual renewal time at t for the renewal process. Determine $P(Y(t) > x | A(t + x) > s)$, where $t, s, x > 0$.

Problem 3. Alice and Betty are looking for each other. Starting in Room 1, Alice moves between Room 1 and Room 2 according to a Markov chain with one-step transition matrix

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$$

Starting in Room 2, Betty moves between Room 1 and Room 2 (using a different door) according to a Markov chain with one-step transition matrix

$$\begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Once Alice and Betty are in the same room, they stop transitioning.

a. Define state (i, j) to be when Alice is in Room i and Betty is in Room j. How many classes are there? Which states are transient, null recurrent, positive recurrent?

   NOTE: For the remainder of the problem, aggregate the two states that they meet each other into a single state; thus, the system is now described by a 3-state Markov chain.

b. Determine P, the one-step transition probability matrix.

c. For $n = 1, 2, \ldots$, determine P(at step n, Alice is in Room 1 and Betty is in Room 2) using the fact that the eigenvalues of P are $\{1, 0.46, 0.1\}$. [Note at $n = 0$, this probability is 1.]

d. Determine the expected number of steps until Alice and Betty are in the same room.

C. Optimization (SE/EC 524)

Division of Systems Engineering | Boston University
Ph.D. Qualifying Exam | May 2022

Area Qualifying Exam in Optimization

Due: Tuesday, May 24, 1:00pm

Last Name	First Name	Student ID #

Honor Code: I have not received nor given any unauthorized aid during this exam in accordance with the indicated rules communicated by the SE Division. I further certify that all work is entirely my own and does not violate the Boston University Academic Conduct Code.

Signature: ______________________________

- You may consult the SE 524/674 textbook (Linear Optimization by Bertsimas and Tsitsiklis) and/or the SE 724 textbook (Nonlinear Programming by Bertsekas). Nothing else is allowed.
- Calculators and computing devices would not be needed. Communication devices and access to the Internet are *not* permitted.
- All work you want graded must go in this exam booklet or in a separate set of pages which should include this cover page as the first page, including the signed honor code.

*** Good Luck!!! ***

Problem	Points earned	out of
Problem 1		50
Problem 2		50
Total		100

Problem 1 $5 \times 10 = 50$ *points*

For each one of the following statements please state whether they are true or false, **with** a detailed justification (no rigorous proof is required but you are welcome to provide one). Grading will be done as follows:
Correct answer with correct justification: 5 points,
Correct answer with missing or wrong justification: 2 points,
No answer: 1 point,
Wrong answer: 0 points.

1. Consider the LP $\min_x\{\mathbf{c}^0\mathbf{x} \mid \mathbf{Ax} \geq \mathbf{b}\}$ and assume it is feasible. It has finite cost if and only if $\mathbf{c}$ can be written as a nonnegative combination of the rows of $\mathbf{A}$.

2. There exist nonempty polytopes (bounded polyhedra) of the form $\{\mathbf{x} \mid \mathbf{Ax} \leq \mathbf{b}\}$ in which every basic solution is also a vertex.

3. Consider an LP and let K be the largest number among the entries of $\mathbf{A}$, $\mathbf{b}$ (the right hand side vector), and $\mathbf{c}$ (the cost vector). Suppose we have an algorithm to find a feasible solution with cost that is within of the optimal. Assume that the algorithm runs in $O(\sqrt{K}\log(1/\epsilon))$ time. Then it is a polynomial time algorithm.

4. Consider a convex set and define *extreme points* in exactly the same way we defined them in polyhedra. The number of extreme points of any convex set is always finite.

5. Suppose that we have access to a procedure that solves systems of linear inequalities. Then we can solve an LP by a single call to that procedure, assuming than an optimal solution exists.
6. If a point $\mathbf{y} \in \mathbb{R}^n$ is not constrained in a convex polyhedral cone $\mathbf{K} \subset \mathbb{R}^n$ then $\mathbf{K}$ and $\mathbf{y}$ can be separated by a hyperplane.

7. There exists a polynomial time algorithm for the *assignment problem*.

8. The simplex method has been shown to be a polynomial-time algorithm.

9. In an uncapacitated network flow problem, every optimal solution has a tree structure.

10. Consider the optimization problem of minimizing $\mathbf{c}^0\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $|x_i| \geq \gamma$ for all i, where $\gamma \geq 0$. This problem can be formulated as a linear programming problem.

Problem 2 15 + 15 + 20 = 50 *points*

Consider the LP problem

$$\max \mathbf{c}^0\mathbf{x}$$
$$\text{s.t. } \mathbf{Ax} \le \mathbf{b} \quad (1)\ \mathbf{l} \le \mathbf{x} \le \mathbf{u},$$

where $\mathbf{c},\mathbf{l},\mathbf{u} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A} = (a_{ij})$ is an $m \times n$ matrix, and $\mathbf{x} \in \mathbb{R}^n$ is the vector of decision variables. Suppose that the elements of $\mathbf{A}$ are **uncertain**; specifically, we assume that $a_{ij} \in [\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}]$, where $\bar{a}_{ij}$ can be seen as a nominal value, and where $\hat{a}_{ij} \ge 0$ for all i,j. To ensure feasibility for all possible values of $\mathbf{A}$, we formulate the robust problem:

$$z_F = \max \mathbf{c}^0\mathbf{x}$$
$$\text{s.t. } \max_{\mathbf{a}_i \in \mathcal{U}_i} \{\mathbf{a}_i'\mathbf{x}\} \le b_i, \quad i = 1,\ldots,m, \qquad (2)\ \mathbf{l} \le \mathbf{x} \le \mathbf{u},$$

where the *uncertainty set* for the ith row, $\mathcal{U}_i$, is given by

$$\mathcal{U}_i \triangleq \{\mathbf{a}_i \mid a_{ij} \in [\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}],\ \forall j\}.$$

(a) Show that (2) can be written as

$$z_F = \max \mathbf{c}^0\mathbf{x}$$
$$\text{s.t. } \sum_{j=1}^n \bar{a}_{ij}x_j + \sum_{j=1}^n \hat{a}_{ij}|x_j| \le b_i, \quad i = 1,\ldots,m, \quad (3)\ \mathbf{l} \le \mathbf{x} \le \mathbf{u},$$

and formulate (3) as an LP.

(b) Let $\hat{\mathbf{x}}$ be an optimal solution of (3). Show that it is a feasible solution of (1) for every possible realization of $\mathbf{A}$.

(c) Suppose we restrict the robust formulation by limiting the total relative deviation of the a_{ij}'s from their nominal values, namely,

$$\sum_{j=1}^{n} \frac{|a_{ij} - \overline{a}_{ij}|}{\hat{a}_{ij}} \leq \Gamma_i, \qquad \forall i,$$

where $\mathbf{\Gamma} = (\Gamma_1,\ldots,\Gamma_m)$ is given. Let

$$\mathcal{R}_i(\Gamma_i) \triangleq \left\{ \mathbf{a}_i \mid a_{ij} \in [\overline{a}_{ij} - \hat{a}_{ij}, \overline{a}_{ij} + \hat{a}_{ij}], \forall j; \sum_{j} \frac{|a_{ij} - \overline{a}_{ij}|}{\hat{a}_{ij}} \leq \Gamma_i \right\}.$$

and define the *restricted robust problem*

$$\begin{aligned} z_R(\mathbf{\Gamma}) = \max\ & \mathbf{c}^0\mathbf{x} \\ \text{s.t. } & \max_{\mathbf{a}_i \in \mathcal{R}_i(\Gamma_i)} \{\mathbf{a}_i'\mathbf{x}\} \leq b_i, \quad i = 1, \ldots, m, \qquad (4) \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}. \end{aligned}$$

Reformulate problem (4) as an LP.

Hint: Consider the maximization problem in the left-hand-side of the constraint of (4) and write its dual. Use this to reformulate (4) as an LP.

D. Dynamic Programming and Stochastic Control (SE/EC/ME 710)

Systems PhD Written Qualifying Examination in Dynamic Programming. May24, 2022
Do *at least one* of the following problems. If you have spare time attempt to complete a second problem (45-55 minutes, closed book/notes)

Question 1

1.1 Consider first the stochastic DP problem with finite horizon N and positive scalar state and control variables, x_k, u_k, k=0,1,2,…,N. The DP Problem has:
dynamics, $x_{k+1} = x_k + 2u_k w_k$, with w_k a uniformly distributed random variable over $[0, x_k]$,
allowable control space U=$\{u_k : u_k \geq 0\}$, period cost $g_k(x_k, u_k) = x_k + \dfrac{1}{u_k}$, and terminal cost $g_N(x_N) = x_N$.
-Find the optimal control $u^*_{N-1} = \mu_{N-1}(x_{N-1})$ and cost to go function $J_{N-1}(x_{N-1})$

1.2 Consider next the imperfect state information problem (i.e., the state variable x_k is not observed perfectly) with the same dynamics and costs. The observation equation is $z_{k+1} = x_{k+1} + u_k v_{k+1}$ with v_{k+1} a zero mean random variable distributed normally as, $\mathbb{N}(0, \sigma_v)$.
Consider the probability distribution, $P_0(x_0 \mid z_o)$, after z_0 is observed, as given, i.e., $P_k(x_k \mid I_k)$ is known for k=0.
-Comment on the sufficient statistic $P_{x_k|I_k}$ of this imperfect state information problem, i.e. the information needed to determine the optimal control u_k. In particular, argue whether the sufficient statistic $P_{x_k|I_k}$ is the full probability distribution of x_k conditional upon the information vector $I_k = \{u_0, \dots, u_{k-1}; z_0, \dots z_k\}$, or whether it is simply the first moment of the distribution? Justify your answer.
- Develop an expression for the estimator Φ that describes the evolution of the sufficient statistic, $P_{x_{k+1}|I_{k+1}} = \Phi(P_{x_k|I_k}, u_k, z_{k+1})$. Make sure to derive as explicitly as you can, expressions for

- the prior probability distribution of x_{k+1}, namely, $P(x_{k+1} \mid u_k, P_{x_k|I_k})$, and
- the distribution of z_{k+1} conditional upon x_{k+1} and u_k.

Question 2

A cloud computing facility with n processors accepts jobs at a dynamically modified connection price $u(t)$ which is broadcasted in real time. When the broadcasted price $u(t) = u$, jobs arrive at exponentially distributed inter arrival times with mean τ_α=1/(1-u). The allowed prices that may be broadcasted must lie in the interval $0 \leq u \leq 1$. When a job arrives, it pays the cloud computing facility u dollars, is assigned to a processor that it employs for an exponential time with mean τ_s, and then departs. If a job arrives while there are n jobs being processed, it is turned away and the cloud computing facility is penalized by c dollars. The objective of the cloud computing facility is to maximize its average income over time derived at job arrival events.
-Formulate the Hamilton-Jacoby-Bellman equation and use it to characterize the optimal price u=$\mu(x)$ that the cloud computing facility ought to broadcast when the number of jobs being processed equals x, noting that $x \in \{x : 0 \leq x \leq n\}$. Make sure to describe the dynamics of x.

E. Nonlinear Systems and Control (SE/ME 762)

SE Qualifying Exam
Nonlinear Systems and Control
May 24, 2022

Problem 1 Give a precise mathematical definition of the uniformly asymptotic stability of an equilibrium point x_0 of $\dot{x}(t) = f(t, x(t))$.

Problem 2. Consider a simple pendulum system $\ddot{\theta} +$

$\dot{\theta} + \sin\theta = 0$

Using three different methods, show that the equilibrium point $\theta = 0$, $\dot{\theta} = 0$ is asymptotically stable.

Problem 3. Consider the following nonlinear control system $\dot{x}_1$

$= x_2 + x_1 \sin x_1,\ \dot{x}_2 = x_1 x_2 + u,\ y = x_1$

Design a state feedback control law such as the output y asymptotically tracks the reference signal $r(t) = \sin t$.