PhD QUALIFYING EXAM IN SYSTEMS ENGINEERING

Written Exam: MAY 26, 2020, 9:00AM to 1:00PM
Via Zoom (link below) and Blackboard. Session will be recorded.

Oral Exam: May 28, 2020
Via Zoom. Link and schedule below (see No. 5)

CLOSED BOOK, NO CHEAT SHEETS
BASIC SCIENTIFIC CALCULATOR PERMITTED

GENERAL INSTRUCTIONS:

1) **Log into Blackboard, Systems PhD Written Qualifying Exam.**

2) **Log into Zoom and turn on your Video. The session will be recorded.**

3) **Questions will appear automatically after 8:55AM. Sign the Honor Code on each section.Complete three of the four topical sections.**
   a. Dynamic Systems Theory (SE/EC/ME 501)
   b. Discrete Stochastic Processes (EK500 and SE/ME 714)
   c. Optimization (SE/EC 674)
   d. Nonlinear Systems and Control (SE 762)

4) **Please write on every sheet and only write on one side:**
   a. Your UID Number
   b. The Section and page numbers (example: Section A, Page 1 of 4)
   c. **Upload responses before 1:30PM.**

5) **Oral Exam Schedule**
   https://bostonu.zoom.us/j/97087823824?pwd=RTR5NndWUWo3cG42OWJqSjhGNHBTdz09
   Meeting ID: 970 8782 3824, **Password:** 898447

Time	Room	Last Name	First Name	Fac1	Fac2	Fac3	Advisor (invited)
10-11AM	1						
10-11AM	2						
11-12PM	3						
11-12PM	4						
12-1PM	5						
12-1PM	6						
1-2PM	7						

Qualifying Examination
Discrete Stochastic Processes
Spring 2020

Problem 1. Let $\{N_j(t); t \geq 0\}$ $j = 1, 2$ be two homogeneous Poisson processes with rates λ_1 and λ_2 respectively. Assume they are independent from each other. Determine an expression for the probability

$$P\left(2^{nd} \text{ arrival in } N_1(t) \text{ occurs before } 2^{nd} \text{ arrival in } N_2(t)\right)$$

Problem 2. Let $\{X_n; n \geq 1\}$ be a renewal process such that $X_n \sim U[0, 4]$, i.e., X_n has a continuous uniform distribution on $[0, 4]$. Determine

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P\left(Y(u) > 2\right) du.$$

where $Y(t) = S_{N(t)+1} - t$ is the residual lifetime at time t.

Problem 3. Suppose two people are standing on opposite sides (Alice on the North side and Betty on the South side) of a one-way street on which cars are passing them according to a Poisson process with rate λ. Immediately after a car passes, Alice instantaneously crosses the street if and only if the interarrival time for that car was greater than a constant $T_A > 0$; Betty instantaneously crosses the street if and only if the interarrival time for the same car was greater than a constant $T_B > 0$. Assume that $T_A \leq T_B$. Draw and label the resultant Markov chain, where, for example, the state (S, N) means Alice is on the South side of the street and Betty is on the North side. Thus, the initial state is $X(0) = (N, S)$, and if the first car passes after time T_B, then $X(1) = (S, N)$.

(a) Is the one-step transition matrix P symmetric?

(b) If $T_A = T_B$, specify the classes and if they are transient, null recurrent, or positive recurrent.

(c) If $T_A = T_B$, does at least one stationary distribution exist? If so, is it unique?

(d) If $T_A < T_B$, specify the classes and if they are transient, null recurrent, or positive recurrent.

(e) If $T_A < T_B$, does at least one stationary distribution exist? If so, is it unique? If a stationary distribution exists, determine it.

(f) If $T_A < T_B$, determine the expected number of cars that will have passed when Alice and Betty are on the same side of the street for the first time.

Honor Code Statement: I have not received nor given any unauthorized aid during this exam in accordance with the indicated rules outlined above. I further certify that all work is entirely my own and does not violate the Boston University Academic Conduct Code. Signature and UID:

Systems Engineering Control Qualifying Exam — May 2020

I have not received any unauthorized aid during this exam in accordance with the rules of the Boston University Academic Conduct Code. I further certify that all work is entirely my own and does not violate the Boston University Academic Conduct Code.

_____

Signature — UID — date

Systems Engineering Control Qualifying Exam — May 2020

1. Compute e^{At} where

$$(a)\ \ A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{pmatrix}, \ \text{and}\ \ (b)\ \ A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. Consider the system $\dddot{x} = u, \quad y = x$.

   (a) Write the system in (first order) state-space form.

   (b) Is the system you have written *controllable?* *Observable?*

   (c) Design a state feedback control law such that the closed loop system has all poles at -1.

   (d) Design a full-state observer with poles at $-1, -2, -3$.

3. Find a polynomial having the least possible degree such that all coefficients are positive, but the polynomial has at least one root in the right half plane.

4. (a) For the finite dimensional linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices, consider the problem of steering the system from $x_0 \in \mathbb{R}^n$ to $x_1 \in \mathbb{R}^n$ in $T > 0$ units of time so as to minimize

$$\eta = \int_0^T \|u(t)\|^2 \, dt.$$

(a) Prove that the optimal value of η is

$$\eta_0 = [x_0 - \Phi(0,T)x_1]^{\mathrm{T}} W(0,T)^{-1} [x_0 - \Phi(0,T)x_1].$$

where $W(0,T)$ is the *controllability Grammian:*

$$W(0,T) = \int_0^T \Phi(0,t)B(t)B(t)^T\Phi(0,t)^T \, dt.$$

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(b) Consider two finite dimensional linear systems in $\mathbb{R}^n$:

$$\dot{x} = Ax + b_1 u, \text{ and}$$
$$\dot{x} = Ax + b_2 u,$$

where A is an $n \times n$ constant matrix, and b_1, b_2 are non-zero n-vectors such that the (A, b_1) system is controllable, but the (A, b_2) system is not controllable. Let $B = (b_1 \vdots b_2)$ be the $n \times 2$ matrix whose columns are the vectors b_1 and b_2. Prove that the two-input system

$$\dot{x} = Ax + Bu \ (= Ax + b_1 u_1 + b_2 u_2)$$

is controllable.

(c) Let x_0, x_1 be arbitrary points in $\mathbb{R}^n$. Compare the costs of steering each of the three systems in part (b) from x_0 to x_1 in T-units of time. Are the costs all defined, and among them, which has the smallest value?

5. Recall that a SISO linear system $\dot{x} = Ax + bu$, $y = cx$ is said to be in *standard controllable form* if

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad c = (c_0, \cdots, c_{n-1}).$$

With this same form of the matrix A and

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad c = (1, 0, \cdots, 0),$$

the system is said to be in *standard observable form*.

Let $g(s) = \frac{6s+8}{s^3+9s^2+23s+15}$.

(a) Write down the standard controllable realization.

(b) Write down the standard observable realization.

Division of Systems Engineering | Boston University
Ph.D. Qualifying Exam | May 2020

Area Qualifying Exam in Optimization

Due: Tuesday, May 26, 1:30pm

Last Name	First Name	Student ID #

Honor Code: I have not received nor given any unauthorized aid during this exam in accordance with the indicated rules communicated by the SE Division. I further certify that all work is entirely my own and does not violate the Boston University Academic Conduct Code.

Signature: _____

- You may consult the SE 524/674 textbook (Linear Optimization by Bertsimas and Tsitsiklis) and/or the SE 724 textbook (Nonlinear Programming by Bertsekas). Nothing else is allowed.
- Calculators and computing devices would not be needed. Communication devices and access to the Internet, other for the purposes of downloading and uploading this exam, are *not* permitted.
- All work you want graded must go in this exam booklet or in a separate set of pages which should include this cover page as the first page, including the signed honor code.
- If you can not scan your solutions and you can only take photos using a cell phone, please assemble everything into a single pdf file to upload. Make sure each page of your solution is scaled to cover an entire page of your pdf file and is legible.

*** Good Luck!!! ***

Problem	Points earned	out of
Problem 1		50
Problem 2		50
Total		100

Problem 1 $5 \times 10 = 50$ *points*

For each one of the following statements please state whether they are true or false, **with** a detailed justification (no rigorous proof is required but you are welcome to provide one). Grading will be done as follows:
Correct answer with correct justification: 5 points,
Correct answer with missing or wrong justification: 2 points,
No answer: 1 point,
Wrong answer: 0 points.

1. Consider the *compressed sensing* problem where one wants to find a solution $\mathbf{x}$ to an underdetermined system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, for some given matrix $\mathbf{A}$ and vector $\mathbf{b}$. The *basis pursuit* problem $\min_{\mathbf{x}}\{\|\mathbf{x}\|_1 \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ can be formulated as a linear programming problem. ($\|\mathbf{x}\|_1 = \sum_i |x_i|$ is the ℓ_1 norm of $\mathbf{x} = (x_1, \ldots, x_n)$.)

2. Consider a bipartite graph with n nodes on the left and n on the right. We want to match each node on the left with exactly one node on the right to minimize an overall cost. We can formulate this problem as follows:

$$\begin{array}{rll} \min \sum_{i=1}^n & \sum_{j=1}^n c_{ij} f_{ij} & \\ \text{s.t.} & \sum_{i=1}^n f_{ij} = 1, & \forall j = 1, \ldots, n, \\ & \sum_{j=1}^n f_{ij} = 1, & \forall i = 1, \ldots, n, \\ & f_{ij} \in \{0, 1\}, & \forall i, j = 1, \ldots, n, \end{array} \tag{1}$$

   where f_{ij} is the indicator variable of matching node i on the left with node j on the right and c_{ij} is the cost of this matching. Solving the linear programming relaxation of the above with the simplex method produces an optimal solution to problem (1).

3. The primal-dual path following interior point method will converge faster than the simplex method in any given linear programming instance.

4. Your friend who works on photonic devices and does not know much about optimization asks you what would you recommend for solving the problem $\min_{\mathbf{x}}\{\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{c}'\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$

for $\mathbf{Q} \succ 0$ (positive definite) and given $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$. Your best advice is to use a gradient method.

5. Consider the linear program $\min_x\{\mathbf{c}'\mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ and assume it is feasible. It has finite cost if and only if $\mathbf{c}$ can be written as a nonnegative combination of the rows of $\mathbf{A}$.

6. When "re-optimizing" a linear program (i.e., resolve the LP by adding a constraint, or perturbing the right hand side vector, or perturbing the cost vector) we prefer to use the simplex method rather than an interior point method.

7. For a linear program with a modest number of decision variables and many constraints, the decomposition method one should use is the delayed column generation method.

8. If a standard form linear programming problem is unbounded, then the right hand side vector $\mathbf{b}$ can be appropriately adjusted to make the optimal cost finite.

9. Let $\mathbf{x}$ be a basic feasible solution (bfs) of a polyhedron $\mathcal{P}$. Then, there exists some choice of a cost vector $\mathbf{c}$ such that $\mathbf{x}$ is the unique optimal solution of the linear program $\min_{\mathbf{x}}\{\mathbf{c}'\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}$.

10. A problem in which a piecewise linear and concave function has to be minimized subject to linear constraints can be transformed into a linear programming problem.

Problem 2 $10 + 15 + 10 + 15 = 50$ *points*

Consider the linear program

$$\begin{aligned} Z_0 = \text{minimize} \quad & \mathbf{c}'\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{2}$$

where $\mathcal{X}$ is a polyhedron. We assume that $\mathcal{X}$ is an "easy" set to optimize over and that without the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ the problem is easily solved by exploiting the special structure of $\mathcal{X}$. We will develop a decomposition approach for solving (2).

1. Let $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ and $\{\mathbf{w}^1, \ldots, \mathbf{w}^l\}$ be the sets of extreme points and extreme rays of $\mathcal{X}$, respectively. Using the resolution theorem, write an $\mathbf{x} \in \mathcal{X}$ in terms of the extreme points and extreme rays of $\mathcal{X}$ and substitute $\mathbf{x}$ in (2) using this expression. Write down the equivalent to (2) optimization problem you obtain by this approach. Indicate which are the new decision variables. We will be referring to this problem as the (full) *master problem (MP)*.

2. Follow exactly the same procedure as in Part 1 but now write $\mathbf{x}$ in terms of subsets $\{\mathbf{x}^i : i \in I\}$ and $\{\mathbf{w}^j : j \in J\}$ of extreme points and extreme rays of $\mathcal{X}$, where the cardinalities of I and J satisfy $|I| \leq k$ and $|J| \leq l$. Write down the resulting optimization problem to which we will be referring as the *restricted master problem (RMP)*. Suppose

next we solve the RMP using the primal-dual path following interior point method and stop when the duality gap drops below ϵ_0. At termination we have a feasible primal-dual pair $(\mathbf{x}_u, \mathbf{p}_u)$ where $\mathbf{x}_u$ is the primal solution and $\mathbf{p}_u$ is the dual vector corresponding to the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$. Let $Z_u = \mathbf{c}'\mathbf{x}_u$. Find a tight upper bound on Z_0 in terms of Z_u.

3. Next dualize the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ in (2) and consider the dual function

$$L(\mathbf{p}) = \inf_{x \in \mathcal{X}} [\mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x})]. \tag{3}$$

Show that $L(\mathbf{p})$ is a lower bound on Z_0 for all $\mathbf{p}$.

4. Consider a decomposition algorithm for solving (2) which operates as follows. At each iteration we solve the RMP as described in Part 2; this yields an upper bound Z_u on Z_0. Next solve (3) with $\mathbf{p} = \mathbf{p}_u$; as you proved in $(\mathbf{c})$ this yields a lower bound, say Z_l, on Z_0. If $Z_u - Z_l \leq \epsilon$ then stop; we have a solution of (2) which is ϵ-close to optimal. Otherwise, i.e., if $Z_u - Z_l > \epsilon$, describe how we can generate columns to add to the RMP in order to re-solve it and continue to iterate.

Nonlinear Systems and Control

Problem 1. Show that if $f(t,x) = f(t+T,x), \forall t$, then $x(t,x_0,t_0) = x(t-T,x_0,t_0-T), \forall t$.

Problem 2. Consider a mass-spring system. With a linear spring and nonlinear viscous damping described by $c_1 y + c_2 \dot{y}|\dot{y}|$, the system is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

Problem 3. Consider the system

$$\dot{x}_1 = -x_1 + x_1^2 x_2, \quad \dot{x}_2 = u$$

Design a state feedback control law to globally stabilize the origin.

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Signature and UID:

General Guidance

- **8:30AM, Tuesday, May 26**. This is the time recommended to enter the Zoom meeting and Blackboard so that you can ask the proctor questions and be ready for the questions which will be released automatically at 8:50AM and close at 1:00PM (will allow upload until 1:30PM).
- **Blackboard**
  - Systems PhD Written Qualifying Exam
- **Zoom**
  - https://bostonu.zoom.us/j/97083918632?pwd=elkyeWRsK3VpR01MVzhvWnE2L3M1QT09
  - Meeting ID: 970 8391 8632
  - Password: 607166
  - The Zoom session will be recorded.
  - Your camera should show your face and your work surface.
  - Virtual backgrounds are not permitted.
- **Exam timeframe**: 9:00AM-1:00PM
- **Exam Responses** must be uploaded to Blackboard by 1:30PM.
- **Other Info:**
  - A basic scientific calculator is permitted.
  - Bathroom breaks are permitted.
  - No other smart devices are permitted (i.e., no cell phones)
- **You must sign the honor code on each exam question**. The honor code is:
  - Honor Code Statement: I have not received nor given any unauthorized aid during this exam in accordance with the indicated rules outlined above. I further certify that all work is entirely my own and does not violate the Boston University Academic Conduct Code.
    Signature and UID:
- **Exam Responses**
  - Make sure your handwriting is clear and legible.
  - Only write on one side of each exam response sheet. You will need to upload to Blackboard no later than 1:30PM.
  - Number the pages by Section, NOT by the entire exam. For example: Section 2, Problem 2, Page 1 of 4.
- **Clarification Needed During the Exam?** Send a private chat to the Exam Proctor if clarification is needed on anything.
  - Please include your cell phone number in case the faculty member wants to call and talk to you rather than chat via Zoom.
  - The Exam Proctor will contact the faculty and the faculty will send clarification via the proctor, enter the Zoom and chat with you directly, or call you.