The Einstein challenge for differential geometry

The case of Weyl and Cartan

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Einstein in 1916

about Christoffel symbols:

also eine geodätische Linie. Da nun die geodätische Linie unabhängig vom Bezugssystem definiert ist, wird ihre Gleichung auch die Bewegungsgleichung des materiellen Punktes in bezug auf K_1 sein. Setzen wir

$$\Gamma^{\tau}_{\mu\nu} = - \begin{Bmatrix} \mu\,\nu \\ \tau \end{Bmatrix}, \tag{45}$$

so lautet also die Gleichung der Punktbewegung inbezug auf K_1

$$\frac{d^2 x_\tau}{d s^2} = \Gamma^{\tau}_{\mu\nu} \frac{d x_\mu}{d s} \frac{d x_\nu}{d s}. \tag{46}$$

Wir machen nun die sehr naheliegende Annahme, daß dieses allgemein kovariante Gleichungssystem die Bewegung des Punktes im Gravitationsfeld auch in dem Falle bestimmt,

. . .

Verschwinden die $\Gamma^{\tau}_{\mu\nu}$, so bewegt sich der Punkt geradlinig und gleichförmig; diese Größen bedingen also die Abweichung der Bewegung von der Gleichförmigkeit. Sie sind die Komponenten des Gravitationsfeldes.

§ 14. Die Feldgleichungen der Gravitation bei Abwesenheit von Materie.

components of the gravitational field.

1916. № 7.

ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 49.

1. ***Die Grundlage der allgemeinen Relativitätstheorie;***

von A. Einstein.

Die im nachfolgenden dargelegte Theorie bildet die denkbar weitgehendste Verallgemeinerung der heute allgemein als „Relativitätstheorie" bezeichneten Theorie; die letztere nenne ich im folgenden zur Unterscheidung von der ersteren „spezielle Relativitätstheorie" und setze sie als bekannt voraus. Die Verallgemeinerung der Relativitätstheorie wurde sehr erleichtert durch die Gestalt, welche der speziellen Relativitätstheorie durch Minkowski gegeben wurde, welcher Mathematiker zuerst die formale Gleichwertigkeit der räumlichen Koordinaten und der Zeitkoordinate klar erkannte und für den Aufbau der Theorie nutzbar machte. Die für die allgemeine Relativitätstheorie nötigen mathematischen Hilfs-

Response of mathematicians

- Levi-Civita 1917: Geometrical analysis/interpretation of Christoffel symbols as infinitesimal parallel displacement for manifolds embedded in euclidean $\mathbb{R}^{n+}$
- Levi-Civita 1917, generalization to any Riemannian manifold: affine connection Γ (derived from metric)
- Weyl 1918: affine connection as abstract structure in diff'ble manifold (not necessarily metric)
- Weyl 1918, physical interpretation: gravito-inertial guiding field (compare Einstein's 1916 remark), important for his view of GR.
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"Postmature" development (if any) – but not only

- John Stachel 2007 ("Newstein ..."): postmature development

  " 'affine connection' is a postmature concept, the absence of which during the course of development of the general theory of relativity had a crucial negative influence on its development and subsequent interpretation.", p. 1043)

- ...interesting point of view from philosophy of science,

- From a historian's point of view it was, perhaps even more, a starting point for a blossoming study of diverse differential geometric structures.

- This shall be the main subject of my talk ...

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Outline

1. New structures: (i) Gauge geometry and gauge field(s)

- Weyl 1918: purely infinitesimal geometry without direct comparison of lengths (or physical quantities) at different points of *"the world"*.
- Basic concepts: conformal structure plus *scale* gauge; both together imply a *unique affine connection* conformal curvature (Weyl tensor).
- Electromagnetism as gauge field (scale !): geometrically unified field theory and attempt to improve Hilbert-Mie theory of matter.
- Twisted history of reception and criticism by physicists and philosophico-mathematical "analysis of the problem of space" (PoS) by Weyl (1920–1923).
- After rise of QM shift of gauge concept from scale to phase; for Weyl conceptually bound to general relativistic framework of Driac field (1929 Weyl and Fock).
- Later (1940s) reflection on relationship and difference between mathematical and physical automorphisms.

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(ii) Conformal and projective structures

- Schouten (1920/21): study of conformal differential geometry.
- Weyl 1921: conformal and projective structure ("Beschaffenheit"), conceptual role in GR (causal and inertial structure), projective curvature tensor.
- Cartan 1922ff.: conformal and projective Cartan spaces, part of wider program (below).
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  Darboux's diff. geometry ("trièdres"/3-frames)
  + Lie "groups" (Lie algebras)+ diff. forms
  + E. & F. Cosserat's theory of generalized elastic media
  + Einstein' theory.
- Cartan 1921ff.: construction of generalized spaces from infinitesimal Klein spaces, using differential forms and infinitesimal Lie groups
- Connections with values in Lie algebra (modernized language) two kinds of curvature: isotropy component (known before in different disguise) and a kind of translational curvature, called torsion by Cartan (role in geometrized Cosserat theory).
- Huge reseach program with repercussions
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2. Weyl's gauge geometry

Weyl's approach: 'localizing' the automorphisms of SR

- Weyl considered the similarities as the automorphisms of Euclidean geometry: No natural unit of length given. In 1918 he transferred this idea to SR and Minkowski space thus $\mathfrak{W} := \mathbb{R}^4 \rtimes (\mathbb{R}^+ \times SO(1,3))$ the automorphisms of SR.
- In 1921 (4th edition RZM and sep. publication) he gave a physically founded structural description of the automorphisms of SR:
  - Inertial trajectories of SR specify a line structure, and a mathematical image ("Bildraum") such that these lines are straight, up to projective transformations.
  - Light propagation specifies an infinite plane E_∞ by a family of "parallel" cones which are projections of a conic section in E_∞.
  - Both together lead to the restricted conformal group of Minkowski space $\mathfrak{W}$ as the automorphisms of SR.

– Consequence:

"To distinguish 'normal' co-ordinate systems among all others in the special theory of relativity, . . . we may dispense with not only rigid bodies but also with clocks."

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→ *"To distinguish 'normal' co-ordinate systems among all others in the special theory of relativity, . . . we may dispense with not only rigid bodies but also with clocks."*

Anhang I.

(Zu Seite 161 und 207.)

Um in der speziellen Relativitätstheorie die »normalen« Koordinatensysteme vor allen andern auszuzeichnen, in der allgemeinen die metrische Fundamentalform zu bestimmen, kann man nicht bloß der starren Körper, sondern auch der Uhren entraten.

In der *speziellen* Relativitätstheorie wird durch die Forderung, daß bei der den Koordinaten x_i entsprechenden Abbildung eines Weltstücks auf den vierdimensionalen Euklidischen Bildraum mit den Cartesischen Koordinaten x_i die Weltlinien der kräftefrei sich bewegenden Massenpunkte in *Gerade* übergehen sollen (Galileisches Trägheitsprinzip), dieser Bildraum *bis auf eine projektive Abbildung* festgelegt. Denn es gilt der Satz, daß die projektiven die einzigen stetigen Abbildungen eines Raumstücks sind, durch welche Gerade in Gerade übergeführt werden. Er ergibt sich sogleich, wenn wir in der Möbiusschen Netzkonstruktion (Fig. 12) die unendlichferne durch eine unser Raumstück durchschneidende Gerade ersetzen (Fig. 15). Der Vorgang der Lichtausbreitung legt dann in unserm vierdimensionalen projektiven Bildraum das *Unendlich-Ferne* und die *Metrik* fest; denn seine (dreidimensionale) »unendlichferne Ebene« E ist dadurch gekennzeichnet, daß die Lichtkegel die von den verschiedenen Weltpunkten aus gewonnenen Projektionen eines und desselben in E gelegenen zweidimensionalen Kegelschnitts sind.

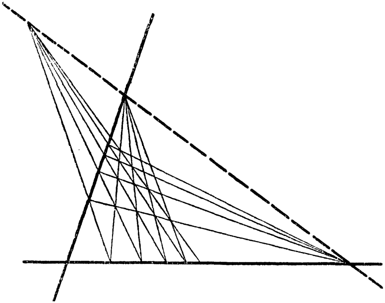

Fig. 15.

In der *allgemeinen* Relativitätstheorie gestalten sich diese Schlüsse am einfachsten so. Der vierdimensionale Riemannsche Raum, als welchen sich Einstein die Welt denkt, ist ein Sonderfall des allgemeinen metrischen Raums (§ 16). Bei dieser Auf-

Consequences for GR

- For general relativity Weyl proposed to weaken the Riemannian metric g to its conformal class $c := [g]$ and added a real valued "length" (scale) connection φ dependent on the choice of the representative $g \in c$: Weylian metric (g, φ), up to rescaling:
- Choosing a representative $g \in c$ meant to "gauge" the measurement units. Changing the representative $\tilde{g} = \Omega^2 g$ was accompanied by a gauge transformation of the scale connection $\tilde{\varphi} = \varphi - d \log \Omega$.
- The scale connection φ entered the specification of a compatible affine connection $\Gamma = \Gamma(g, \varphi)$. The latter is uniquely determined and gauge independent.
- Generalization of Einstein gravity to a Weyl geometric theory of gravity; built upoin it a unified field theory including electromagnetism (φ as e.m. potential).

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Consequences for math: new problem of space (PoS)

Rigid motions of classical problem of space no longer useful in GR.
Weyl started to analyze the PoS in the light of GR.
Skipping details, he postulated (1920–1923) ...

- a pair of groups $G \subset H \subset GL(n, \mathbb{R})$ with $H =$ normalizer of G, spacetime point dependent by conjugation in $GL(n, \mathbb{R})$ (generalized "congruences" and "similarities");
- ... and a linear connection Λ ("metric connection") as "congruent transference" between distinct infintesimally close points.
- Λ not unique (could be combined with "localized" congruences). Weyl demanded that among all equivalent forms there is exactly one symmetric (torsion free, in Cartan's terms) affine connection.

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Two main principles of Weyl's PoS and its result

P1 ("Principle of freedom") G allows the "widest concievable range of possible congruence transfers" in one point.

P2 ("Principle of coherence"): To each congruent transfer Λ there exists exactly one equivalent affine connection.

- Surprisingly P1 turned out to be superfluous mathematically (Scheibe 1954).
- Result of the analysis after evaluating algebraic consequences of geometrical postulates: Only special orthogonal groups of any signature satisfy the conditions for G and P1, P2..
- Consequence: justification of Weyl geometry with its peculiar gauge structure rather than (semi-)Riemannian geometry as proper framework for "Space".
- A late argument in the aftermath of the Einstein-Weyl debate of 1918 about "absolute" versus "relative" standards of length/time.

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– The insights of quantum mechanics convinced Weyl in the middle of the 1920s that Einstein's 1918 claim of a "universal bureau of standards (universelles Eichbüro)" of nature was basically correct.

> *"The atomic constants of charge and mass of the electron and Planck's quantum of action h, which enter the universal field laws of nature, fix an absolute standard of length, that through the wave lengths of spectral lines is made available for practical measurements." (Weyl 1949)*

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3. Cartan's spaces

Cartan's first publication on "non-holonomous spaces"

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- 1922: Series of notes in the C.R.
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Motivation for Cartan's program

- Toronto ICM 1924: Cartan discussed the gap between Klein's conception of a geometry characterized by the principle of homogeneity (Klein spaces $S = G/H$)
  and Riemannian geometry which is only *isotropic* (not homogeneous) in infinitesimal neighbourhoods.

- This gap had been filled only partially, by Levi-Civita's introduction of parallel discplacement, due to the challenge of GR:

  Or, c'est le développement même de la théorie de la relativité, liée par l'obligation paradoxale d'interpréter dans et par un Univers non homogène les résultats de nombreuses expériences faites par des observateurs qui croyaient à l'homogénéié de cet Univers, qui permit de combler en partie le fossé qui séparait les espaces de Riemann de l'espace euclidien. Le premier pas dans cette voie fut l'oeuvre de M. Levi-Civita, par l'introduction de sa notion de parallélisme. (Cartan 1924 (ICM Toronto), 86)

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The theory of relativity, facing the paradoxical task of interpreting in a non-homogeneous universe, and by means of it, the multiple experiences made by observers who used to believe ("qui croyaient") in the homogeneity of this universe, allowed to partially fill the gap between Riemannian geometry and euclidean geometry [in the Kleinian sense, ES].
M. Levi-Civita made the first steps in this direction by introducing the concept of parallelism. "
(Cartan 1924, O.C. 594)

Cartan's program: infinitesimal homogeneity

- Cartan proposed *infinitesimally homogeneous*, but *"non-holonomous"*, spaces as a second step for bridging the gap.
- Infinitesimally homogeneous: Cartan parametrized infinitesimal neighbourhoods of points p in a manifold (space) S by a homogeneous space in Klein's sense (of correct dimension). This was expressed by point dependent *"répères"/frames* (later Cartan gauge (Sharpe)). Result in modernized notation:

$$T_pS \cong G/H$$

- For any infinitesimal displacement δx (modernized, $\xi \in T_pM$) Cartan specified an infinitesimal element of the group by a set of differential (Pfaffian) forms. He called the whole set a "connection" ("affine, projective ...").
- Modernized: a differential form with values in $\mathfrak{g}$ with specific transformation rules, now Cartan connection. Pointwise:

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Curvature and torsion

- By "non-holonomous" Cartan meant that, after traversing an infinitesimal closed loop (in a surface element), the total "homogeneity operation" was different from the one in the corresponding Klein geometry.
- This difference could be expressed by a collection of Pfaffian 2-forms; some of them characterized the deviation
  - in $\mathfrak{h} = Lie(H)$ (isotropic or "rotational" curvature),
  - and a deviation in a transversal Liealgebra $\mathfrak{l}$ (torsion), a generalized translational curvature. ($\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{h}$ with specific properties – "reductivity").
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Physics and mathematics

From the very beginning of his presentations (C.R. Feb. 1922 etc.), Cartan discussed the relation of his new spaces to physical theories:

- Einstein's theory of general relativity,
- the "beaux travaux de MM. E. et F. Cosserat sur, *l'action euclidienne*": a generalized elasticity theory of media with non-symmetric tensor of tension (E. & M. Cosserat 1909).
- and Weyl's generalized spaces (which fitted in Cartan's framework and were again generalized – by including torsion).

Background info: Cosserat media (1909)

- In 1909 Eugène and François Cosserat demanded to consider action principles of mechanics, invariant under the group of full Euclidean motions ("action euclidienne").
- In particular they studied a *variational principle* for elastic media with an action density dependent on an infinitesimal "trièdre trirectangulaire" (orthonormal frame) and *invariant under infinitesimal Euclidean motions*.
- The resulting stress tensor was no longer necessarily symmetric.
- An "infinitesimal" surface element (on the boundary of an elastic body under external influences) could be subject to an elastic torque in addition to an elastic force. In the language of the Ecole Polytechnique tradtion described by a "couple" of forces (parallel, inversely oriented, different lines of action).

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Elastic media in the light of the Einstein equation (1922)

In his paper on "les équations d'Einstein" (1922) Cartan analyzed which quadratic differential forms $G_{ij}dx^i dx^j$, satisfying a "conservation law" like the Einstein tensor, can be formed from up to *second partial derivatives of the metric* $g_{ij}dx^i dx^j$. He showed:

- G_{ij} is determined up to three arbitrary constants.
- G_{ij} can be expressed in terms of infinitesimal rotations of the Levi-Civita connection (Ricci coefficients).
- The classical stress tensor of elastic media in Euclidean space can be re-written, by differential 2-forms Ω_ξ, depending on a direction vector ξ (Ω_ξ gives the norm of the force in direction ξ applied on a surface element $dx \wedge dy$."Conservation law": $d\Omega_\xi = 0$
- In Einstein gravity, Cartan expressed the coefficients of Ω_ξ by the Einstein tensor G_{ij} – Einstein equation! – and thus, indirectly, by the Ricci coefficients.

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Origin of the terminology "torsion" (1921/22)

- By formal analogy, Cartan expressed the stress tensor of *elastic media in Euclidean space* in terms of Ricci coefficients of an auxiliary metric (reading Einstein's equ. in 3 dim. from right to left).
- Generalizing the approach to "non-holonomous" spaces, a translational component appears in the connection, in addition to the rotational one (announced in CR notes,
explained in 1923/24 paper. part II)
- The corresponding stress tensor becomes asymmetric, if the second curvature, mentioned above, does not vanish.
The asymmetry corresponds to a torque in addition to a force (on any surface element) like in the Cosserats' analysis.
- For stress, the second curvature leads to a torque like the rotational curvature (Ricci coefficients) corresponds to surface force.
Thus Cartan's surprising terminological choice "torsion" for a translational (!) curvature component of the infinitesimal group.

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Short outlook

Weyl and Cartan: geometry and physics

- For both mathematicians discussed here, Weyl and Cartan, the study of general differential geometry was closely intertwined with physical considerations.
- The challenge of general relativity was crucial for their mathematical and conceptual innovations.
  Also the backreaction of their respective works on physical theories was considerable (in particular in the long run).
- But there were considerable differences in
  - the way they considered the new PoS,
  - and the relationship between math. considerations and physics.

PoS – Weyl and Cartan

- Weyl: PoS the conceptual challenge to consequently "infinitesimalize" differential geometric structures.
  Guiding question: Which automorphism groups for SR and GR?
  Inbuilt principle: Unique (torsion free) affine connection.
  Clue of his contribution gauge idea.
- Cartan: PoS the challenge to close the gap between Kleinian type homogeneous spaces and Riemannian geometry.
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- Key ideas gauge geometry/fields and torsion not at all "postmature", but – if anything of that kind – rather "premature": .

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Mathematics and physics

- For Weyl the mathematical generalizations in GR and differential geometry were *motivated*, sometime heuristically *guided, by philosophical and/or physical considerations*.
- Cartan's work in differential geometry developed "organically" from the symbolical tools of the French tradition in geometry (Darboux etc.) and his own contributions to differential forms and Lie groups. *Physics* was a *challenge/trigger* for going ahead and a rich *field of application*, rather than a motivational force driving his mathematical innovations.
- Not to forget: There were many mathematicians worldwide contributing to the interplay of differential geometry and GR.

– The field is open for further studies! –

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