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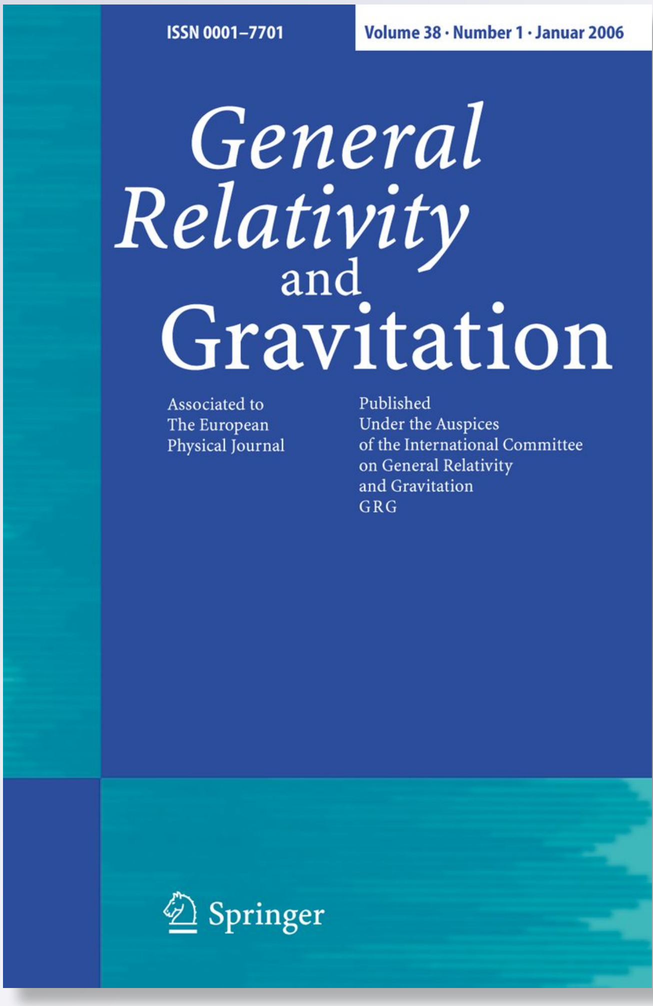

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RESEARCH ARTICLE

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Abstract It is proposed that compatible conformal and projective structures be taken as the basic space-time structures in general relativity, with the symmetry group restricted to unimodular diffeomorphisms. Models of classical massless fields, such as the Maxwell field, interact directly with the conformal structure; while classical bodies composed of massive particles, such as solids and fluids, interact directly with the projective structure. It is suggested that this difference is the classical limit of the respective quantum-gravitational interactions, which should reflect the differing approaches to the quantization of fields and particles when gravity is neglected. Models of general relativity based on compatible conformal and projective structures should be the basis for the exploration of ideal measurement procedures, both classical and quantum.

Keywords General relativity · Conformal structure · Projective structure · Quantum gravity

1 Introduction

Everyone knows that general relativity involves a pseudo-Riemannian metric,[1] and most expositions of the subject are based mathematically entirely on this structure. A few brave souls insist that it is the chrono-geometry that is associated with the pseudo-Riemannian structure, while the inertio-gravitational structure is associated with the (torsion-free) affine structure; and it is the compatibility conditions between the two

[1] By this I mean a metric tensor with Lorentz signature. Hereafter, any reference to the metric tensor should be so understood unless explicitly noted.

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structures that connect them.[2] But few indeed are those who insist on the importance for general relativity of a further decomposition of each of these two structures based on the conformal and projective structures, respectively.[3] So I shall offer some thoughts on this subject as a small homage to Josh Goldberg, from whom, in person and through his papers, I have learned so much.

2 Angular momentum: Goldberg's choice

A small link between Josh's work and my topic is provided by the following observation. In his important paper on "Conservation Laws in General Relativity" [4], he showed that there are a denumerably infinite number of superpotentials, tensor densities differing only in their (integer) weight, from which the Einstein field equations may be deduced. In a paper of mine [5], I used the one of weight $+1$ simply for reasons of expediency. But now I realize that it is the only one of these superpotentials that is conformally invariant (see below for details), so conformal invariance can be used as a criterion for selecting a unique superpotential. But this is *just* the superpotential (corresponding to his $n = -1$) that Josh finds he needs to define a gravitational angular momentum complex, in such a way that the total angular momentum transforms as a free antisymmetric tensor.[4] It will be interesting to understand just why this is the case; but surely it has something to do with the fact that a conformally-invariant boundary can be attached to the space-time.

3 Conformal and projective structures

From the point of view of G-structures and their first order prolongations,[5] neither the pseudo-Riemannian metric nor the affine connection is irreducible.[6] At each point of the four-dimensional space-time manifold, the general linear group in four dimensions $GL(4, \mathbf{R})$ is restricted to the pseudo-orthogonal group $O(3, 1)$ by the metric. $CO(3, 1)$, the conformal group, is equal to $O(3, 1) \times R^+$, the product of the pseudo-orthogonal group and the multiplicative group of real positive numbers. But $O(3, 1)$ is itself the intersection of $CO(3, 1)$ and $SL(4, \mathbf{R})$, the group determined by the volume

[2] Einstein in his later years was among the happy few. "[A]t first Riemannian metric was considered the fundamental concept on which the general theory of relativity and thus the avoidance of the inertial system were based. Later, however, Levi-Cività rightly pointed out that the element of the theory that makes it possible to avoid the inertial system is rather the infinitesimal displacement field Γ^l_{ik}. The metric or the geometric tensor field g_{ik} which defines it is only indirectly connected with the avoidance of the inertial system in so far as it determines a displacement field (see [1], 'Appendix II', p. 141).

[3] For some discussions of this point, see [2] and [3].

[4] See Section 4 of [4]. Goldberg notes that his complex differs from both the Landau-Lifshitz and the Bergmann-Thomson angular momentum complexes.

[5] For an excellent mathematically oriented introductory discussion of G-structures, see [6], Chapter I. Conformal and projective structures are discussed in Chapter IV. For a physically oriented introduction, see [7].

[6] For a short exposition, keyed to the problems of general relativity and including references to the relevant literature, see [8].

structure.[7] A symmetric linear connection ("affine structure" for short) is a representation of the full group $GL(4, \mathbf{R})$, or rather of its first prolongation. An *equiaffine structure* consists of a symmetric linear connection together with a *parallel volume*, i.e., a volume structure such that its covariant derivative w.r.t. the linear connection vanishes. A volume structure and a projective structure on a manifold together determine an affine structure that makes the volume parallel: "Hence a volume selects a class of affine parametrizations for the geodesics of a projective structure" [9].

It has been known for some time that the conformal and projective structures of a pseudo-Riemannian space-time determine the metric up to a constant factor;[8] and more recently it was shown by Ehlers, Pirani and Schild (EPS)[9] that there is a sort of inverse relation: A conformal and a projective structure are said to be compatible if the null geodesics of the conformal structure[10] form a subset of the autoparallel paths of the projective structure.[11] A unique affine connection within the class of projective structures then always can be defined (see section on *Compatibility of Conformal and Affine Structures*). A space with such a unique affine connection and a class of compatible conformal pseudo-metrics is called a *Weyl space*.[12]

But I find the emphasis placed by EPS on the parallel treatment (pun intended) of null and timelike geodesics to be misplaced. The conformal structure basically singles out the class of null hypersurfaces, the solutions of the conformally-invariant characteristic equation. At the conformal level, null geodesics are basically the bicharacteristic curves associated with a solution of the characteristic equation.[13] In more sophisticated mathematical language, this has been explained by Arnold [18], p. 220:

> The geometrical optics of rays and wave fronts defined by a system of hyperbolic partial differential equations is the geometry of a hypersurface in a contact space of the projectivised cotangent bundle of space time. This hypersurface, called the light hypersurface, is the set of zeroes of the (principal) symbol. In the theory of partial differential equations, the characteristics of this hypersurface in the contact manifold are, strangely, called the bicharacteristics.[14] These curves (and their projections to space-time and physical space) are called the physical rays. The Legendre submanifolds of the light hypersurface, projected to space-time, define in space-time the hypersurface of the big front.This hyper-

7 I am assuming the manifold to be orientable. Otherwise, I would have to replace $SL(4, \mathbf{R})$ with $SL^{\pm}(4, \mathbf{R})$.

8 See [10]. See also [11], pp. 304, and [12].

9 See [13].

10 Only the null geodesics are invariant under conformal transformations.

11 See [10]. See notes 8 and 9.

12 See [11], pp. 133–134.

13 I.e., the equation for the characteristic hypersurfaces. See, [14], especially pp. 1006. I follow the terminology of Hadamard. See [15], p. 75, [16], Chapter VII, "La Théorie Générale des Caractéristiques," Section 1, "Caractéristiques et bicaractéristiques," pp. 263–296. [17], Chapter 1, "Données de Cauchy en general," Section C, Caractéristiques, pp. 38–59.

14 As this remark indicates, one must be very careful in interpreting the meaning of the word "bicharacteristic." Some writers use it to describe the wave fronts (hypersurfaces), and some (following Hadamard), use it to describe the rays (curves).

surface defines the propagation of waves; its sections by the isochrones are the momentary fronts.

This observation is basically the translation into modern differential-geometric language of the age-old relation between the Huyghens principle and the Fermat construction of light rays. So the conformal structure is best associated physically with the propagation of massless fields, such as the electromagnetic. On a null hypersurface Σ (a conformal concept), the conformal two-structure[15] is closely related to the shear of the congruence of null geodesics generating the hypersurface (another conformal concept). Pirani and Schild have proved [20] that the projection of the conformal curvature tensor onto Σ and twice onto the tangent null direction k^{μ} is the second Lie derivative of the shear, as shown by the following conformally-invariant equation:

$$\mathcal{L}_k^2 \sigma_{AB} = 2C^{\kappa}_{\mu\nu\lambda} P^{\lambda}_{AB\kappa} k^{\mu} k^{\nu}, \tag{1}$$

where σ_{AB} is the conformal two-structure or shear of the congruence, $C^{\kappa}_{\mu\nu\lambda}$ is the Weyl or conformal curvature tensor, and $P^{\lambda}_{AB\kappa}$ is the projection operator onto a family of two-dimensional space-like surfaces on Σ orthogonal to the null congruence k^{μ} that rules Σ.

And indeed, if one wants to preserve the intimate association between definability of a concept in a theory and its measurability in principle by some ideal measuring device not incompatible with the foundations of the theory in question; then one must acknowledge that one can only measure such conformal-invariant properties of a congruence of null rays as its shear by means of optical devices that study the rate of change of the shape of the shadows cast by an obstacle placed in the path of a light beam.[16]

On the other hand, the projective structure is really associated with the behavior of particles traveling along a timelike path.[17] Indeed, one can study the projective structure along the path of a freely-falling particle, chosen as the fiduciary path; and then examining the relative displacements of a set of neighboring freely falling particles, such that displacements of this set form a linearly-independent basis at the tangent space of each point of the fiduciary path (i.e., n additional paths in an $(n + 1)$-dimensional space-time).

So the relation between conformal and projective structures reflects—and is reflected by—the relation between classical massless wave theories, which in practice means electromagnetism, and classical particle theories and their ensembles represented by the stress-energy tensors of ordinary matter.[18]

[15] See [19]. It provides a direct representation of the two degrees of freedom of the gravitational field.

[16] See, e.g., [21]. A hole in a screen may be used instead, of course, and diffraction effects neglected in either case (eikonal approximation).

[17] We make the usual distinction between paths, which are parameter-independent, and therefore projectively invariant; and curves, which are parametrized paths, and therefore not projectively invariant.

[18] Quantum mechanically, this goes over into the relation between massless boson fields, and massive fermion fields, which have the respective classical wave and classical particle limits. For further discussion of this question, with extensive quotations from Bohr's comments on the subject, see [22]. But in this paper,

Now I shall go into some more mathematical detail about the conformal and projective structures, and the remaining portions of the pseudo-Riemannian and affine structures, which have to do with the existence of a volume structure and a preferred affine parameter, respectively.

4 Breakup of the pseudo-Riemannian metric

Let $g_{\mu\nu}$ be a pseudo-Riemannian metric tensor on a four-dimensional differentiable manifold M. Consider the group of conformal transformations of the metric:[19]

$$g_{\mu\nu} \longrightarrow {}'g_{\mu\nu} = \varphi g_{\mu\nu}, \tag{2}$$

where φ is an arbitrary positive-definite scalar function on M. These transformations are said to leave the conformal structure of M invariant. Now $g_{\mu\nu}$ is not an irreducible geometric quantity under the conformal group, but can be constructed from two independent quantities: $\mathbf{g}_{\mu\nu}$ and γ. The first term $\mathbf{g}_{\mu\nu}$, is a pseudo-Riemannian tensor density of weight $-1/2$, with determinant -1; it is invariant under conformal transformations of the type in Eq. (2).

The second term γ is a scalar density of weight $+1/2$; it transforms under conformal transformations as:

$$\gamma \longrightarrow {}'\gamma = \varphi\gamma. \tag{3}$$

Conversely, if one starts from an arbitrary pair γ and $\mathbf{g}_{\mu\nu}$ and defines

$$g_{\mu\nu} = \gamma \mathbf{g}_{\mu\nu}, \tag{4}$$

then it is a pseudo-Riemannian metric tensor with determinant:

$$g = -\gamma^4, \tag{5}$$

and an inverse contravariant pseudo-Riemannian tensor:

$$g^{\mu\nu} = \gamma^{-1}\mathbf{g}^{\mu\nu}, \tag{6}$$

where $\mathbf{g}^{\mu\nu}$, the inverse of $\mathbf{g}_{\mu\nu}$, is a contravariant pseudo-Riemannian tensor density of determinant -1 and weight $+1/2$. It follows that γ^2 serves to define a volume element on the manifold, and that $\gamma\mathbf{g}^{\mu\nu}$ is a pseudo-tensor density of weight $+1$, which is needed for derivation of the Einstein equations from the usual variational principle.

I shall confine discussion primarily to the classical case, resisting the temptation to comment extensively on the issue of quantum gravity.

[19] We follow Schouten's kernel-index convention (see [11], pp. 2–3), according to which a symbol placed to the left of the kernel symbol denotes a different geometric object of the same type; while a symbol placed to the right of the kernel symbol denotes the components of the same object in a different coordinate system.

5 Breakup of the affine connection

Let $\Gamma^{\kappa}_{\mu\nu}$ be a symmetric affine connection on M, initially given quite independently of any metric. The group of projective transformations:

$$\Gamma^{\kappa}_{\mu\nu} \longrightarrow \Gamma^{\kappa}_{\mu\nu} + \frac{1}{5}(\delta^{\kappa}_{\mu} T_{\nu} + \delta^{\kappa}_{\nu} T_{\mu}), \tag{7}$$

where T_{ν} is an arbitrary convector field, leave the affine autoparallel paths unchanged, and are hence said to leave the projective structure invariant. Now a connection $\Gamma^{\kappa}_{\mu\nu}$ on M is not an irreducible geometric object under the group of projective transformations,[20] but can be constructed from two independent terms: $P^{\kappa}_{\mu\nu}$ and Γ_{μ}. The second term Γ_{μ} is the trace of the connection:

$$\Gamma_{\mu} = \Gamma^{\kappa}_{\mu\kappa}, \tag{8}$$

and transforms under projective transformations as:

$$\Gamma_{\mu} \longrightarrow \Gamma_{\mu} + T_{\mu}. \tag{9}$$

The first term:

$$P^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \frac{1}{5}(\delta^{\kappa}_{\mu} \Gamma_{\nu} + \delta^{\kappa}_{\nu} \Gamma_{\mu}), \tag{10}$$

is traceless and invariant under projective transformations. The $P^{\kappa}_{\mu\nu}$, sometimes called the projective parameters,[21] are not themselves the components of a connection, but they do form a geometric object, obeying a more complicated transformation law.[22]

Conversely, given any $P^{\kappa}_{\mu\nu}$ (symmetric in the lower indices) and Γ_{μ} obeying the appropriate transformation laws, then:

$$\Gamma^{\kappa}_{\mu\nu} = P^{\kappa}_{\mu\nu} + \frac{1}{5}(\delta^{\kappa}_{\mu} \Gamma_{\nu} + \delta^{\kappa}_{\nu} \Gamma_{\mu}) \tag{11}$$

will form the components of a symmetric affine connection.

6 Compatibility of conformal and affine structures

If a conformal and a projective structure are present on M, this enables the classification of all autoparallel path elements as either spacelike, timelike or null. The two

[20] For the definitions of geometric objects and geometric quantities, see [11], Section II.3, pp. 67–72.

[21] See [23] and the section *Unimodular Relativity*.

[22] See [11], "Projective connexions," Section VI.4, pp. 300–303. For the transformation law, see p. 301.

structures are said to be compatible if all the null geodesics of the conformal structure[23] are also autoparallel paths of the projective structure.[24] A unique projective connection within the class of projective structures can then always be defined (up to a linear transformation of the affine parameter) by the demand that the equation for an affine autoparallel curve take the form

$$D_\lambda V^\kappa = 0,\ V^\kappa = \frac{dx^\kappa}{d\tau}, \tag{12}$$

where D_λ stands for the covariant derivative w.r.t. the affine connection of the tangent vector to the curve $V^\kappa(\tau)$, and τ is the preferred affine parameter. A space with such a unique affine connection and a class of compatible conformal pseudo-metrics is called a *Weyl space*.[25]

If in a Weyl space, the ratio of the magnitudes of two (non-null) vectors at any point, as defined by the conformal structure, is preserved by parallel transport independently of the path taken, then a unique pseudo-Riemannian metric exists (up to a constant factor) that determines both the conformal and projective structures.[26] As we shall see in the section *Unimodular Relativity*, there is another, more geometrically intuitive way of formulating compatibility that eliminates the detour through Weyl spaces.

7 Breakups of the Riemannian and affine curvature tensors

The Riemannian curvature tensor $R_{[\mu\nu][\kappa\lambda]}$, formed from the pseudo-Riemannian metric and its first and second derivatives, can of course be written in terms of $\mathbf{g}_{\mu\nu}$ and γ, and their derivatives. There is a decomposition of this tensor into the sum of the Weyl or conformal curvature tensor $C_{[\mu\nu][\kappa\lambda]}$, the traceless part of the Ricci tensor $S_{\mu\nu} = R_{\mu\nu} - 1/4Rg_{\mu\nu}$, and the Ricci scalar R:[27]

$$R_{[\mu\nu][\kappa\lambda]} = C_{[\mu\nu][\kappa\lambda]} + E_{[\mu\nu][\kappa\lambda]} + 1/12RG_{[\mu\nu][\kappa\lambda]}, \tag{13}$$

where:

$$E_{[\mu\nu][\kappa\lambda]} = 1/2(S_{\mu\kappa}g_{\lambda\nu} - S_{\mu\lambda}g_{\kappa\nu} + S_{\nu\lambda}g_{\kappa\mu} - S_{\nu\kappa}g_{\lambda\mu}) \tag{14}$$

$$G_{[\mu\nu][\kappa\lambda]} = g_{\mu\kappa}g_{\lambda\nu} - g_{\mu\lambda}g_{\kappa\nu} \tag{15}$$

The Weyl or conformal curvature tensor proper, $C^{\nu}_{\mu[\kappa\lambda]} = \gamma^{-1}\mathbf{g}^{\nu\sigma}C_{[\sigma\mu][\kappa\lambda]}$, is invariant under conformal transformations; so it cannot depend on γ but only on $\mathbf{g}_{\mu\nu}$. This means that $C_{[\sigma\mu][\kappa\lambda]}$ must be γ times a function of $\mathbf{g}_{\mu\nu}$ and its derivatives.

23 Only the null geodesics are invariant under conformal transformations.

24 In order to avoid confusion, I shall confine use of the term "geodesic" to metric or conformal structures, and use of autoparallel to affine or projective structures.

25 See [10,11], pp. 133–134.

26 See [10,13].

27 For the conformal curvature tensor, see [11], "Conformal transformation of a connexion in V_n," Section VI.5, pp. 304–312. The definition of the conformal curvature tensor is on p. 306.

Similarly, the affine curvature tensor $A^{\nu}_{\mu[\kappa\lambda]}$, formed from the affine connection and its first derivatives, can be written in terms of $P^{\kappa}_{\mu\nu}$ and Γ_{μ}, and their derivatives. There is a decomposition of this curvature tensor into the sum of the projective curvature tensor $P^{\nu}_{\mu[\kappa\lambda]}$, which is also traceless, and the analogue $A_{\mu\nu}$ of the Ricci tensor (which need not be symmetric!) for the affine curvature tensor:[28]

$$A^{\nu}_{\mu[\kappa\lambda]} = P^{\nu}_{\mu[\kappa\lambda]} + 2(\delta^{\nu}_{\mu} P_{[\kappa\lambda]}) + 2(\delta^{\nu}{}_{[\kappa} P_{\lambda]\mu}) \tag{16}$$

where:

$$P_{\kappa\lambda} = (4/15)A_{\kappa\lambda} - (1/15)A_{\lambda\kappa}. \tag{17}$$

The projective curvature tensor $P^{\nu}_{\mu[\kappa\lambda]}$ is invariant under projective transformations, so it cannot depend on Γ_{μ}.

If the homogeneous Einstein field equations hold, the difference between these curvature tensors is purely conceptual, in the presence of matter or radiation fields, they are mathematically different quantities. This difference may have consequences for a quantum theory of interacting gravitational fields and non-gravitational fields and matter (see the concluding section).

8 Unimodular relativity

In 1971, Anderson and Finkelstein discussed a variant of general relativity, which they named "unimodular relativity" [24]. Here, the subgroup of the full diffeomorphism group, under which the theory is invariant, is confined to the unimodular group of point transformations, i.e., those with determinant $+1$.[29] This means that, in the tangent space at each point, the symmetry group is reduced from the general linear group $GL(4, \mathbf{R})$ to the special linear or unimodular group of order four, $SL(4, \mathbf{R})$, consisting of matrices with determinant $+1$.[30] Anderson and Finkelstein confined their considerations to conformal transformations, and introduced the conformally-invariant metric tensor density $\mathbf{g}_{\mu\nu}$. Their work inaugurated a tradition of work on unimodular relativity, some notable recent exemplars of which are [25–27]. None of these works discuss the projective structure, but it is rather easy to extend the unimodular theory to include it. Indeed, under unimodular transformations, both the conformally invariant parameters $C^{\kappa}_{\mu\nu}$ of J.M. Thomas (see [28] and [11], pp. 315–317) and the projectively invariant parameters $P^{\kappa}_{\mu\nu}$ of T.Y. Thomas[31] (see [23] and [11], pp. 300–303) transform like the components of a symmetric affine connection, and so may be used to define covariant differentiation operators, etc. Compatibility between the conformal

[28] For the projective curvature tensor, see [11], "Projective transformations of a symmetric connection," Section V.1, pp. 287–292. The definition of the projective curvature tensor is on p. 289.

[29] Anderson and Finkelstein use the language of coordinate transformations, but that is easily remedied.

[30] Schouten calls this group "the special affine group," and points out that: "Then the differences between tensors, tensor densities and tensor Δ-densities vanish" [11], p. 12.

[31] In this case, Thomas speaks of an "equi-projective geometry of paths."

and projective structures is established by an equation relating these two connections and the volume element. Each such compatible conformal and projective structure defines a unique pseudo-Riemannian structure without any need for detour through a Weyl space. The compatibility conditions and field equations can be derived from a corresponding modification of the Palatini-style variational principle for general relativity.[32]

9 Conclusion: A bold hypothesis

There is no unique approach to quantum gravity, nor need there be. Quantum theories are attempts to take into account the effects of the existence of the quantum of action h on physical systems. But we only deal with physical systems by means of theoretical models, and different models of the same system will give rise to different quantization procedures. Inter-theory relations between different models will then result in relations between the different quantization schemes. For some example of this approach, see [3]. Each such model must be associated with ideal measurement procedures for each of the observables postulated by the model. Ultimately, of course, the only test of the validity of any quantization procedure is the utility of these ideal measurements in helping us to understand and predict the results of various actual measurement procedures.

All of this suggests the following hypothesis: In general relativity, massless classical fields (in practice, this means electromagnetic theory) interact with the space-time structures through the conformal structure; while massive classical bodies interact through the projective structure. Given the difference between the quantum theories in these two cases-notably the difference between integral and half-integral spins and the bosonic versus fermionic statistics that follow - it is hard to believe that this hypothesis would not have significant consequences for a quantum version of general relativity. For example, this approach suggests that the interaction between the gravitation and electromagnetism can be treated at the field level, not only classically, but even quantum field-theoretically. Just as Bohr and Rosenfeld were able to analyze the quantum limitations on the measurability of two components of the electromagnetic Maxwell field in Minkowski space (absence of a gravitational field) [30], and Bergmann and Smith the quantum limitations on measurability of two components of the (linearized) Riemann tensor in empty space-time (absence of an electromagnetic field) [31]; so it should be possible to investigate the quantum limitations on the measurability of one component of the electromagnetic field and one component of the conformal curvature tensor when both are present. It should only be necessary to bring gravitational quanta ("gravitons") into the story when the gravitational field is interacting with massive objects that include fermions or other entities capable of interacting with the projective curvature tensor. In the case of superfluids, for example, one might be able to investigate such interactions between phonons and the gravitational projective curvature tensor.

[32] For details, see [29].

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