On Formalism Freeness: A Meditation on Gödel's 1946 Princeton Bicentennial Lecture

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What is formalism freeness?

- By a formalism, or a **logic**, we mean a combination of a list of symbols, commonly called a signature, or a **vocabulary**; **rules for building terms and formulas**, a list of **axioms**, a list of **rules of proof**, and finally a **definition of the associated semantics**.
- Logic (in this sense) minus semantics: a "deductive theory" according to the *Polish school.*

- With this concept of formalism we associate **"formalism freeness"** with the **suppression of any or all of the above aspects of a logic, except semantics**. The position we take here is that the associated semantics cannot be suppressed.

Gödel's Princeton Bicentennial Lecture, 1946[1]

- "Tarski has sketched in his lecture the great importance (and I think justly) of the concept of general recursiveness (or Turing computability). It seems to me that this importance is largely due to the fact that with this concept one has succeeded in giving a absolute definition of an interesting epistemological notion, i.e. **one not depending on the formalism chosen**." (Gödel, CW II, p.150)

- “In all other cases treated previously, such as demonstrability or definability, one has been able to define them only **relative to a given language**, and for each individual language it is clear that the one thus obtained is not the one looked for.”

- “For the concept of computability, however, although it is merely a special kind of demonstrability or decidability, the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion.”

- “This, I think, should encourage one to expect the same thing to be possible also in other cases (**such as demonstrability or definability**). It is true that for these other cases there exist certain negative results, such as the incompleteness of every formalism or the paradox of Richard. But close examination shows that these results do not make a definition of the absolute notions concerned impossible under all circumstances, but only exclude certain ways of defining them, or at least, that certain very closely related concepts may be definable in an absolute sense.”

Gödel remarked on this already in 1936

- "It can, moreover, be shown that a function computable in one of the systems S^i, or even in a system of transfinite order, is computable already in S^1. Thus the notion 'computable' is in a certain sense 'absolute', while almost all metamathematical notions otherwise known **(for example, provable, definable, and so on)** quite essentially **depend upon the system adopted**". ("On the Length of Proofs", CW I, p. 399.)

Broadened Absoluteness Claim[1]

- In 1965, Gödel replaces "computable in a higher order system" with a broader notion:
- "To be more precise: a function is computable in any formal system containing arithmetic if and only if it is computable in arithmetic, where a function *f* is called computable in *S* if there is in *S* a computable term representing *f.*" (CW II*,* p. 150, footnote added to reprinting of the lecture in *The Undecidable*.)

1 Sieg, "Gödel and Computability", 2006 and 2012

The Problem with Provability: Incompleteness

- "Let us consider the concept of demonstrability. It is well know that whichever way you make it precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident as those with which you started, and that this process of extension can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps…"

The Problem with Definability: Diagonalizability

- Definability (in set theory) is not itself definable.
- Leads to paradox: “Take the least undefinable ordinal…”

Gödel’s “expectation”

- One can extend Turing’s analysis of the notion of computability to definability and provability.
- Avoid such entanglement with formal languages and formal theories that imports undecidability and non-absoluteness.
- Introduce a *formalism free*---in Gödel’s words *formalism independent*---notion of definability and provability…

- Turing’s analysis was paradigmatic and decisive for Gödel. Settled the question of the generality of the incompleteness theorems, by clarifying the notion of “formal system.”

- “….due to A. M. Turing’s work, **a precise and unquestionably adequate definition of the general concept of formal system can now be given**…A formal system can simply be defined to be any mechanical procedure for producing formulas, called provable formulas." (Gödel in 1963, CW III, p. 369.)

The Turing analysis

"…makes the whole idea of a formal system plain. For it is everyday, perspicuous, simple, direct, or "commonsensical", and the focus is on the user, the human end."

(J.Floyd, "The Varieties of Rigorous Experience")

- The informal notion of computability is so clearly captured by the notion of a Turing Machine, that the Church-Turing thesis can actually be viewed as a theorem.

(Gandy, “The Confluence of Ideas in 1936”, 1988)

Polish School

- Lesniewski: A deductive theory is something that one “practices.”

- Tarski: A deductive theory is “performed.”

“There is a suggestive analogy with Turing’s work”. (W.Hodges, “Tarski’s Theory of Definition”)

- Whereas the informal notion of computability is captured by the notion of “Turing Machine”,
- …for Gödel the intuitive concept (of definability) to be made precise---by a similarly “unquestionably adequate” analysis---is: “Comprehensibility by our mind.”

The Bicentennial Lecture in Detail

- Three *epistemological*[1] notions: computability, provability, definability.
- Each come with their own paradoxes.
- For each notion, we want transcendence (of a kind)---but we also wish to avoid undefinability in set theory.

1 In 1936 “metamathematical”

Gödel's two notions of definability

- Two canonical inner models:
  - Constructible sets
    - Model of ZFC
    - Model of GCH
    - *Definable*
  - Hereditarily ordinal definable sets (in the lecture OD)
    - Model of ZFC
    - CH? – independent
    - *Definable* (Levy Reflection)

First: Constructibility

- Constructible sets (L):

$$
\begin{aligned}
L_0 &= \emptyset \\
L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\
L_\nu &= \textstyle\bigcup_{\alpha<\nu} L_\alpha \text{ for limit } \nu
\end{aligned}
$$

Gödel's criticism of L

- For Gödel, L is not the right notion, for the interesting reason that *not all reals live in L*:

  "…you can actually define sets, and even sets of integers, for which you cannot prove that they are constructible (although this can of course be consistently assumed). For this reason, I think constructibility cannot be considered as a satisfactory formulation of definability."

- For many set theorists L is not the right notion of definability because it is too “small”---large cardinals don’t live in L.
- There is also a second, “mathematical” definition of L, via rudimentary functions

We will argue that L is very close to being the right notion, but that certain “L-like” inner models come even closer.

Second: Ordinal definability

- Hereditarily ordinal definable sets (HOD):
- Take the ordinals as primitive.
  - A set is ordinal definable if it is of the form

$$\{a : \varphi(a,\alpha_1,\ldots, \alpha_n)\}$$

    where $\varphi(x,y_1,\ldots, y_n)$ is a first order formula of set theory.
  - A set is hereditarily ordinal definable if it and all elements of its transitive closure are ordinal definable.

- Ordinal definability is itself ordinal definable, by the Levy Reflection Principle:
- A set is ordinal definable over V if and only if it is ordinal definable over a V_α.

$$x = \{a : V_\alpha \models \phi(a, \vec{\beta})\}$$

- So we have a "nondiagonalizable notion"
- Adding OD as a predicate does not yield new ordinal definable sets.

HOD is not generically absolute

- A Gödelian would argue that HOD is perhaps not the right notion of definability because it violates generic absoluteness--- its theory can be changed by forcing.
- It also lacks the conceptual clarity of constructibility.

Meaning of “adequate”

- Computability unites into one concept: absoluteness, nondiagonalizability, formalism independence, and conceptual clarity (in that all known concepts of computability are equivalent to Turing computability---Gandy’s “theorem”)
- For definability (in set theory), these bifurcate: L is non-diagonalizable, absolute, generically absolute and conceptually clear, but too small; HOD is nondiagonalizable and “large”, but not generically absolute and conceptually complex.

- Gödel did not consider the “formalism independence” of L and HOD, although they exhibit such.

But what is formalism independence in the context of set theoretic definability? What is it that varies, with respect to which L and HOD stabilize? (in analogy with computability)

- Myhill-Scott: Hereditarily ordinal definable sets (HOD) can be seen as the constructible hierarchy based on second order logic (in place of first order logic), i.e. use the **real** power set operation in the construction. (1954)

An interesting construction

$$
\begin{aligned}
L'_0 &= \emptyset \\
L'_{\alpha+1} &= \mathrm{Def}_{SOL}(L'_\alpha) \\
L'_\nu &= \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu
\end{aligned}
$$

- Chang considered a similar construction with the infinitary logic $\mathcal{L}_{\omega_1\omega_1}$ in place of first order logic. It is *not* a fragment of SOL (for cardinality reasons: SOL has only countable many formulas, whereas in the Chang model you can define any countable structure).

- If V=L, then V=HOD=Chang's model=L.
- If there are uncountably many measurable cardinals then AC fails in the Chang model. (Kunen.)

Part 2: Implementation

(Joint work with Magidor and Väänänen)

C($\mathcal{L}$*)

- $\mathcal{L}$* any logic. We define C($\mathcal{L}$*):

$$
\begin{array}{lcl}
L'_0 & = & \emptyset \\
L'_{\alpha+1} & = & \mathrm{Def}_{\mathcal{L}^*}(L'_\alpha) \\
L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu
\end{array}
$$

- C($\mathcal{L}$*) = the union of all L'_α

Looking ahead

- For a variety of logics C($\mathcal{L}$*)=L
  - Gödel's L is very robust, not limited to first order logic.
- For a variety of logics C($\mathcal{L}$*)=HOD
  - Gödel's HOD is robust, not limited to second order logic
- For some logics C($\mathcal{L}$*) is a potentially interesting new inner model.

Formalism freeness of L

- $Q_1 x\varphi(x) \Leftrightarrow \{a : \varphi(a)\}$ is uncountable
- $C(\mathcal{L}(Q_1)) = L$.
- In fact: $C(\mathcal{L}(Q_\alpha)) = L$, where

  $$Q_\alpha x\varphi(x) \Leftrightarrow |\{a : \varphi(a)\}| \geq \aleph_\alpha$$

- Other logics, e.g. weak second order logic, “absolute” logics, etc.

Formalism freeness of L (contd.)

- A logic $\mathcal{L}^*$ is absolute if "$\varphi \in \mathcal{L}^*$" is Σ_1 in φ and "$M \models \varphi$" is Δ_1 in M and φ in ZFC.
  - First order logic
  - Weak second order logic
  - $\mathcal{L}(Q_0)$: ``there exists infinitely many
  - Finite fragments of $\mathcal{L}_{\omega_1\omega}$, $\mathcal{L}_{\infty\omega}$: infinitary logic
  - Finite fragments of $\mathcal{L}_{\omega_1 G}$, $\mathcal{L}_{\infty G}$: game quantifier logic

HOD: What Myhill-Scott really prove

- In second order logic $\mathcal{L}^2$ one can quantify over arbitrary subsets of the domain.
- A more general logic $\mathcal{L}^{2,F}$: in domain M can quantifier only over subsets of cardinality κ with F(κ) ≤ |M|.
- F any function, e.g. F(κ)=κ, κ^+, 2^κ, $\beth_\kappa$, etc

Theorem

- For all F: $C(\mathcal{L}^{2,F})$=HOD
- Third, fourth order, etc logics give HOD. (Definability reasons)

Other generalized quantifiers

- $Q_1^{MM}xy\varphi(x,y)$ ⇔ there is an uncountable X such that $\varphi(a,b)$ for all a,b in X
  - Can express Suslinity of a tree. (No uncountable branches, no uncountable antichains)
  - Can be badly incompact; is countably compact (i.e. w.r.t. countable theories) if V=L. L-Skolem down to $\aleph_1$.
- $Q_0^{cf}xy\varphi(x,y)$ ⇔ $\{(a,b) : \varphi(a,b)\}$ is a linear order of cofinality ω
  - Fully compact extension of first order logic. (Whatever the size of the vocabulary, if a theory of this logic is finitely consistent, then it is consistent.) L-Skolem down to $\aleph_1$. **Satisfies ZFC.**

- *aa* logic, Härtig quantifier…
- Cofinality ω was essential in Shelah's *provable* results on size of $\aleph_\omega{}^{\aleph_0}$.
- In contemporary model theory, both generalized quantifiers and infinitary languages have reemerged, due to work of Zilber and others.

Theorems

- $C(\mathcal{L}(Q_1^{MM})) = L$, assuming $0^{\#}$.
- Why? If there is an uncountable homogeneous set in V (w.r.t. a definable relation) then there is one in L. Roughly follows from the fact that ω_1 is weakly compact in L.
- So assuming large cardinals, L “reads” $\mathcal{L}(Q_1^{MM})$ as first order.

Consistent: $C(\mathcal{L}(Q_1^{MM})) \neq L$ (forcing construction due to Jensen)

Theorems

- $C(\mathcal{L}(Q_0^{cf})) \neq L$, assuming $0^{\#}$.
- Proof depends on: if α is regular in L and cofinality of α is >ω, we can express this in $C(\mathcal{L}(Q_0^{cf}))$. But then α belongs to the set of canonical indiscernibles, i.e we can define $0^{\#}$ in $C(\mathcal{L}(Q_0^{cf}))$.

$C(\mathcal{L}(Q_0^{cf}))$ has inner models with large cardinals

- $C(\mathcal{L}(Q_0^{cf}))$ contains the Dodd-Jensen Core Model, same for the Härtig quantifier. (These versions of L "see" the ultrafilter which generates the iterated mouse. From this we get the original mouse. So all mice all present.)

- $C(\mathcal{L}(Q_0^{cf}))$ contains L^{μ}, if L^{μ} exists. (similar proof to the above)

Theorem

- If $V=C(\mathcal{L}(Q_0^{cf}))$ then continuum is at most ω_2, and there are no measurable cardinals.

Proof : $V=C(L(Q_0^{cf}))$ implies continuum is at most ω_2.

- Condensation argument.
- If *r* is a real, then *r* is in some $X \prec C(\mathcal{L}(Q_0^{cf}))$ such that X ``knows" about cofinality ω. Need witnesses both for cofinality ω and for cofinality greater than ω. In latter case we change the higher cofinalities to cofinality ω_1 by a chain argument.
- X then has cardinality ω_1.
- Then X isom. to L'_α for some α, $\alpha<\omega_2$
- Thus there are at most ω_2 reals.
- Consistently: exactly ω_2 reals.

$V=C(L(Q_0^{cf}))$ implies that there are no measurable cardinals.

- Suppose $i:V\rightarrow M$, κ first ordinal moved, M closed under κ-sequences.
- $(C(\mathcal{L}(Q_0^{cf}))^M=C(\mathcal{L}(Q_0^{cf}))$, since M and V have the same ω-cofinal ordinals (since they have the same ω-sequences).
- So M=V.
- $i:V\rightarrow V$, κ first ordinal moved
- Contradiction! (By Kunen.)
- This is like same proof for V=L.
- Smaller large cardinals are consistent with V=L, hence with $V=C(\mathcal{L}(Q_0^{cf}))$.

More theorems

- If there is a Woodin cardinal, then ω_1 is inaccessible in $C(\mathcal{L}(Q_0^{cf}))$. (Stationary tower forcing. Gives an embedding into a model which is closed under ω sequences, moving ω_1 to the Woodin cardinal. Then $(C(\mathcal{L}(Q_0^{cf}))^M = (C(\mathcal{L}(Q_{<\lambda}^{cf}))^V$.)

Generic absoluteness

Suppose there is a proper class of Woodin cardinals. Then:

- Truth in $C(\mathcal{L}(Q_\alpha^{cf}))$ is forcing absolute and independent of α. (Stationary tower forcing again.)
- Cardinals $>\omega_1$ are all indiscernible for $C(\mathcal{L}(Q_0^{cf}))$. (More STF.)
- **Is CH true in $C(\mathcal{L}(Q_\alpha^{cf}))$? This is forcing absolute and independent of α.**

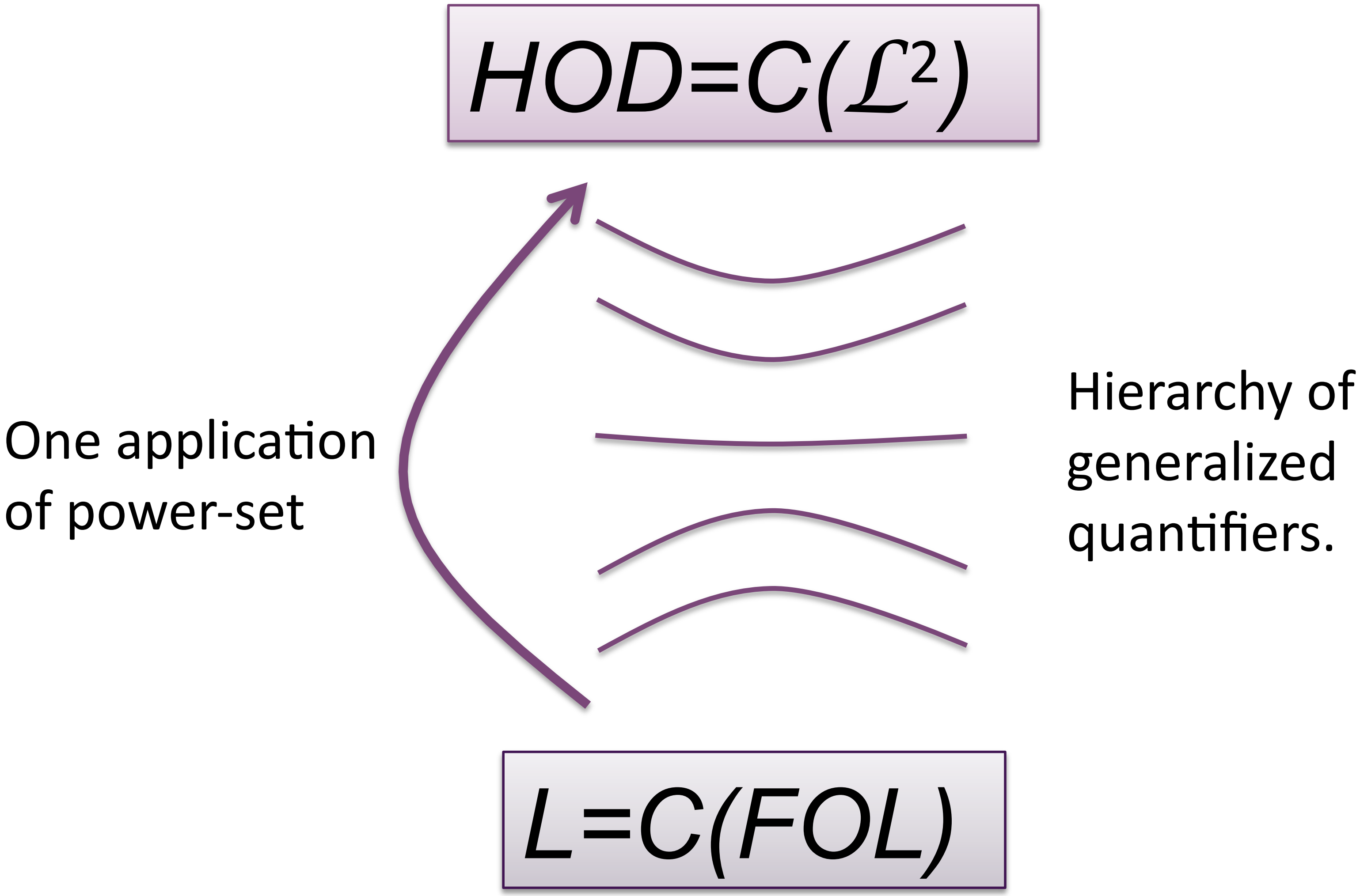
Avoiding HOD

- $C(L(Q_0^{cf})) \neq$ HOD if there are large cardinals.
- Depends upon Woodin's result: there is a sharp for the Chang model.
- Or: There is some countable sequence which is not in $C(L(Q_0^{cf}))$, but can be chosen to be in HOD.

- We have defined an equivalence relation on extensions of first order logic relative to an inner model construction.
- This seems to be a general method, applicable to various classes of logics relative to various structures.

Are we on the right track?

- We suggest that L is a relatively formalism independent notion of definability, as is HOD.
- We want models with large cardinals in them, or with inner models with large cardinals in them.
- We want to supplement L by an “essential” notion of definability, keeping generic asboluteness.
- $C(L(Q_0^{cf}))$ satisfies both, sharing the “good” properties of L and HOD but at the same time avoiding their “bad” properties.
- There may be other tests of formalism independence than varying the underlying logic.
- None have appeared on the horizon (to us).


$HOD = C(\mathcal{L}^2)$
One application of power-set
Hierarchy of generalized quantifiers.
$L = C(FOL)$

Semantic extensions of ZFC

- Replace FOL by an extension of it in the separation and replacement axioms of ZFC.
- Around $L(Q_0)$: exactly omega-standard models of ZFC. (Can define the standard natural numbers in the model.)
- Around $L(Q_0^{MM})$: exactly transitive models of ZFC. (Can define well-foundedness.)
- Between $L\omega_1\omega$ and $L\omega_1\omega_1$: exactly countably closed transitive models of ZFC.

Task: Reals

- Vary the logic in Kleene's Ramified Analytic Hierarchy (Kleene 1959).
- Any logical hierarchy seems amenable to this approach.

Part 3: The wider context: back to formalism freeness

- “Logic independence,” as evidenced in our (and Karp’s) project.
- Admitting a “mathematical” rather than logical treatment, as exemplified by constructibility given in terms of Gödel functions; Turing machines; AEC’s; Tarski/Birkhoff.
- Transcendence (even local transcendence) with respect to a logical hierarchy. (**In Princeton lecture**.)
- Formalism independence in the sense of stability under a class of presentations, viz computability. **(In Princeton lecture**.)

Anti-foundationalism

- “Expansive intuitionism”: the claim that the actual content of mathematics goes beyond any formalization (CM’s term for Poincare's reaction (or counterreaction) to formalism.)

Colin McLarty, “Poincare: Mathematics and Logic and Intuition”

Entanglement

- A truism of our Gödelian inheritance is that the syntax/semantics distinction is clearly defined. The view taken here is that that particular logical terrain has since turned out to be so intricate and fine-structured; so replete with delicate entanglements of syntax and semantics on the one hand, as well as with what appear to be purely semantic phenomena on the other, that it is nearly not a distinction anymore at all.

Syntax/Semantics

The terrain here is ever-shifting; the syntax/semantic distinction is not a tidy dichotomy but a vexed one, characterized by messiness, degrees of entanglement and degrees of formalism freeness.

Turing’s Influence

- The influence of the Turing analysis of computability on Gödel’s thinking was critical.

References

1. Kennedy, “On Formalism Freeness: Implementing Gödel's 1946 Princeton Bicentennial Lecture”, submitted.

2. Kennedy, Magidor, Vaananen, “Inner Models from Extended Logics”, preprint

Thank you!