# Introduction to Dynamical Modeling

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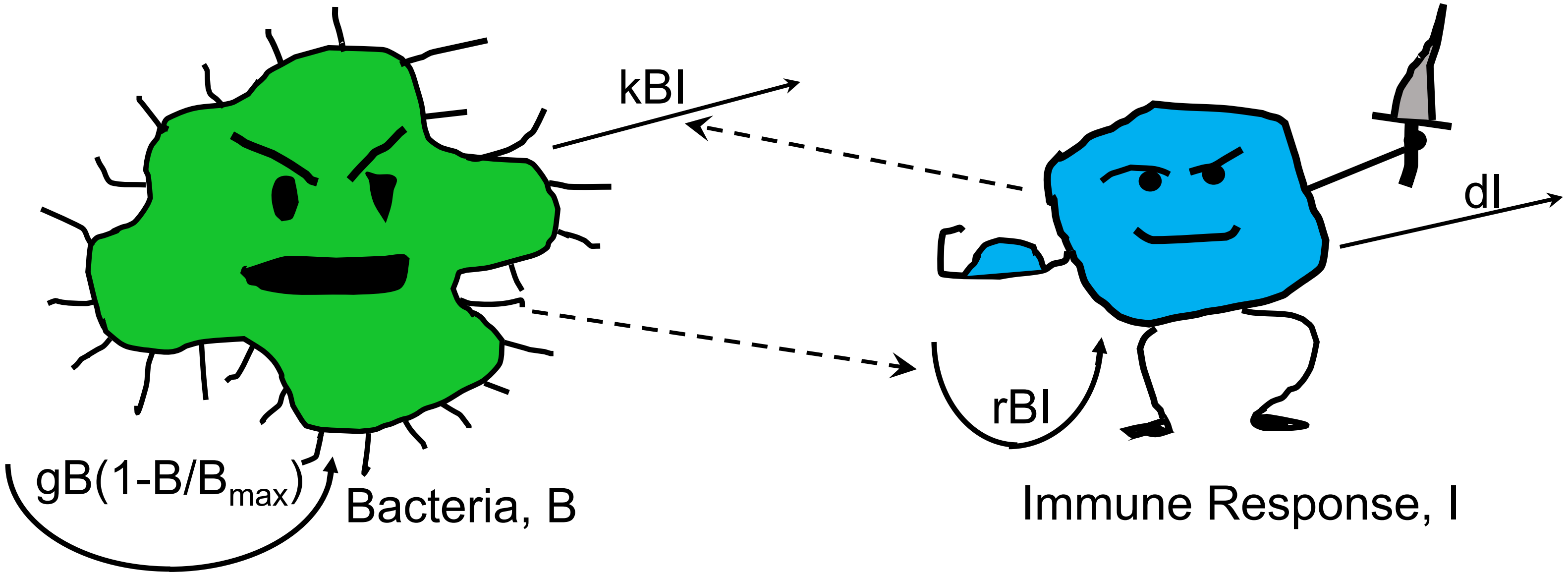
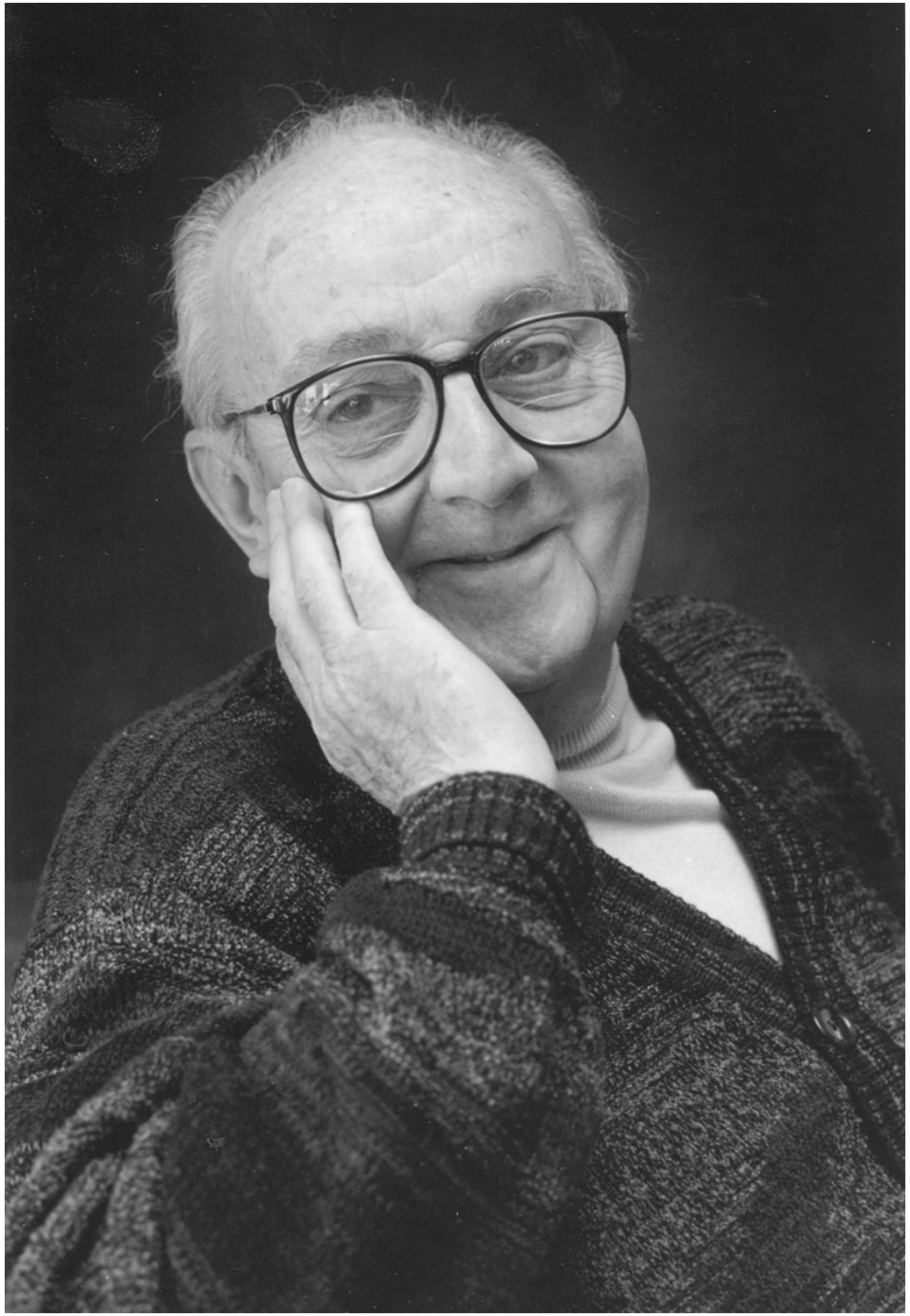




**“All models are wrong, but some are useful”.**

George Box, 1978

# Dynamical mechanistic models

- **Dynamical**: Tracking how things change in time
- **Mechanistic**: Having equations or computer rules that explicitly describe how things happen

Within dynamical models, two broad classes are distinguished:

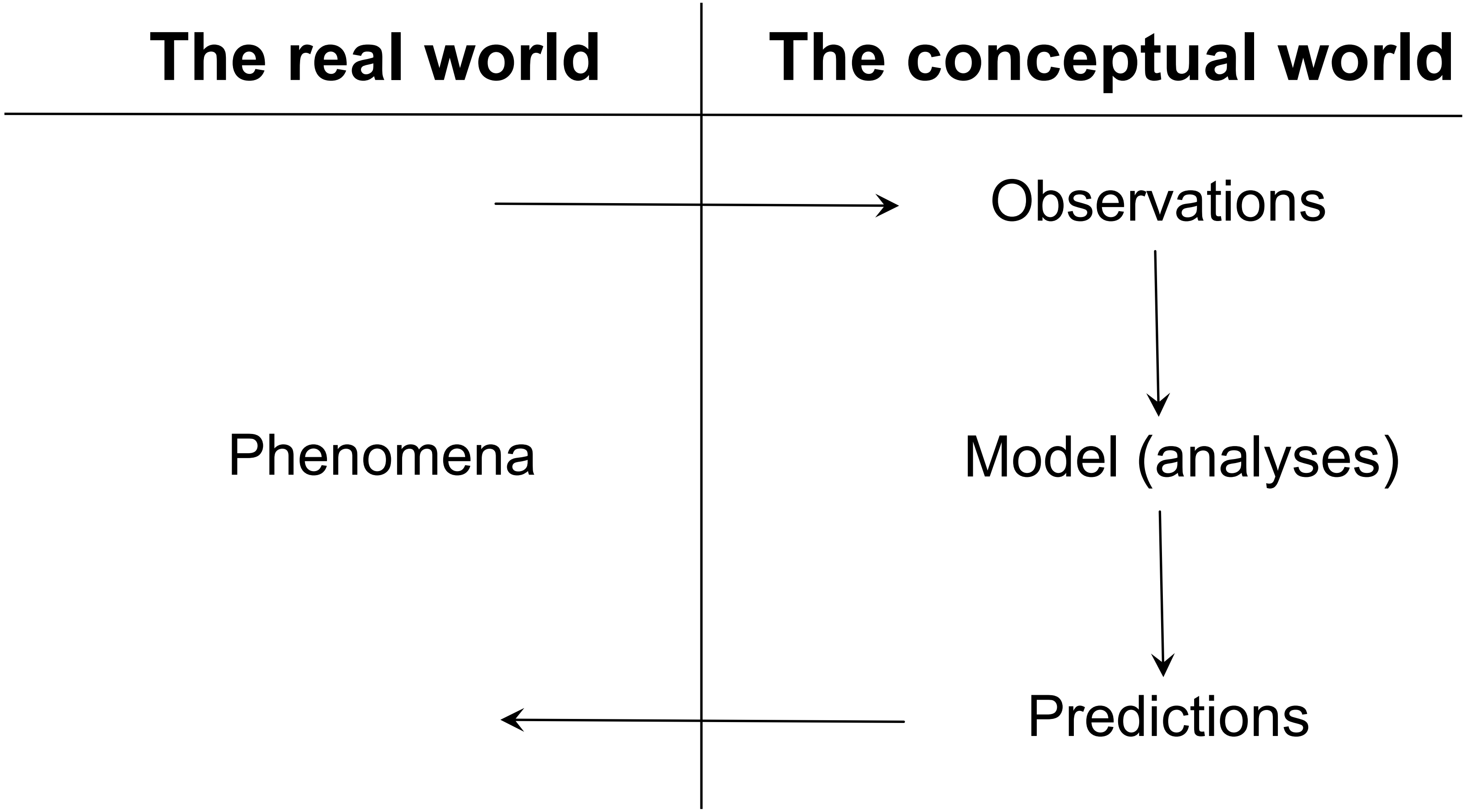
- **Deterministic**: Assumption that the future is entirely predicted (determined) by the model
- **Stochastic**: Assumption that random (stochastic) events affect the biological system, in which case a model can only predict the probability of various outcomes in the future

# Many types of Dynamical Models exist

Compartmental ↔ Agent-based
Discrete time ↔ Continuous time
Deterministic ↔ Stochastic
Homogeneous ↔ Spatial
Memory-less ↔ With memory
Small ↔ Big
Data-free ↔ With data

⟵ The models we'll be focusing on, formulated as Ordinary Differential Equations

# Modeling



**The real world** | **The conceptual world**

Observations

Phenomena

Model (analyses)

Predictions

*Dym and Ivey 1980*

# Malthusian growth model (1798)
# (a simple exponential growth model)

Thomas Robert Malthus (1766-1834)

## discrete-time bacterial growth model

birth

death

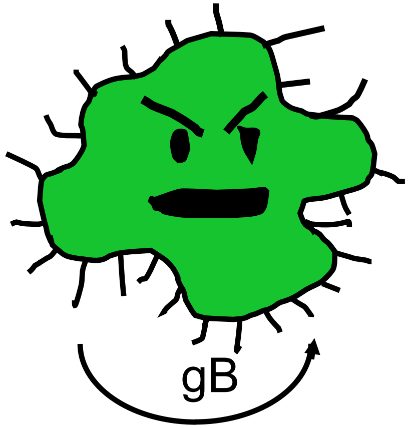
$$B_{t+\tau} = B_t + \underbrace{(bB_t - dB_t)}\tau = B_t + gB_t\tau$$

Total number of bacteria at the next time step

Total number of bacteria right now

term describes how change happens (the mechanisms)

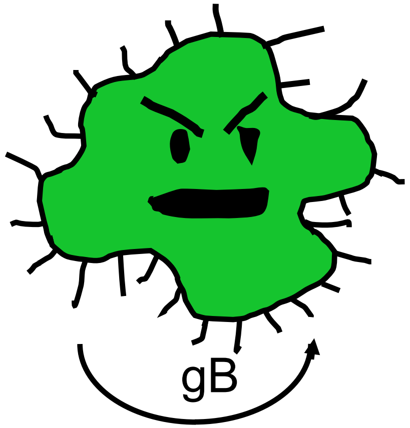
time step



Bacteria, B

# Discrete-time bacterial growth model

$$B_{t+\tau} = B_t + (\overset{\text{birth}}{b}B_t - \overset{\text{death}}{d}B_t)\tau = B_t + gB_t\tau$$



Bacteria, B

- Assume b=12/hour, d=2/hour, τ=1 hour.
- B at start (t=0) = 100
- What do we get after 1,2,3,4,… hours?

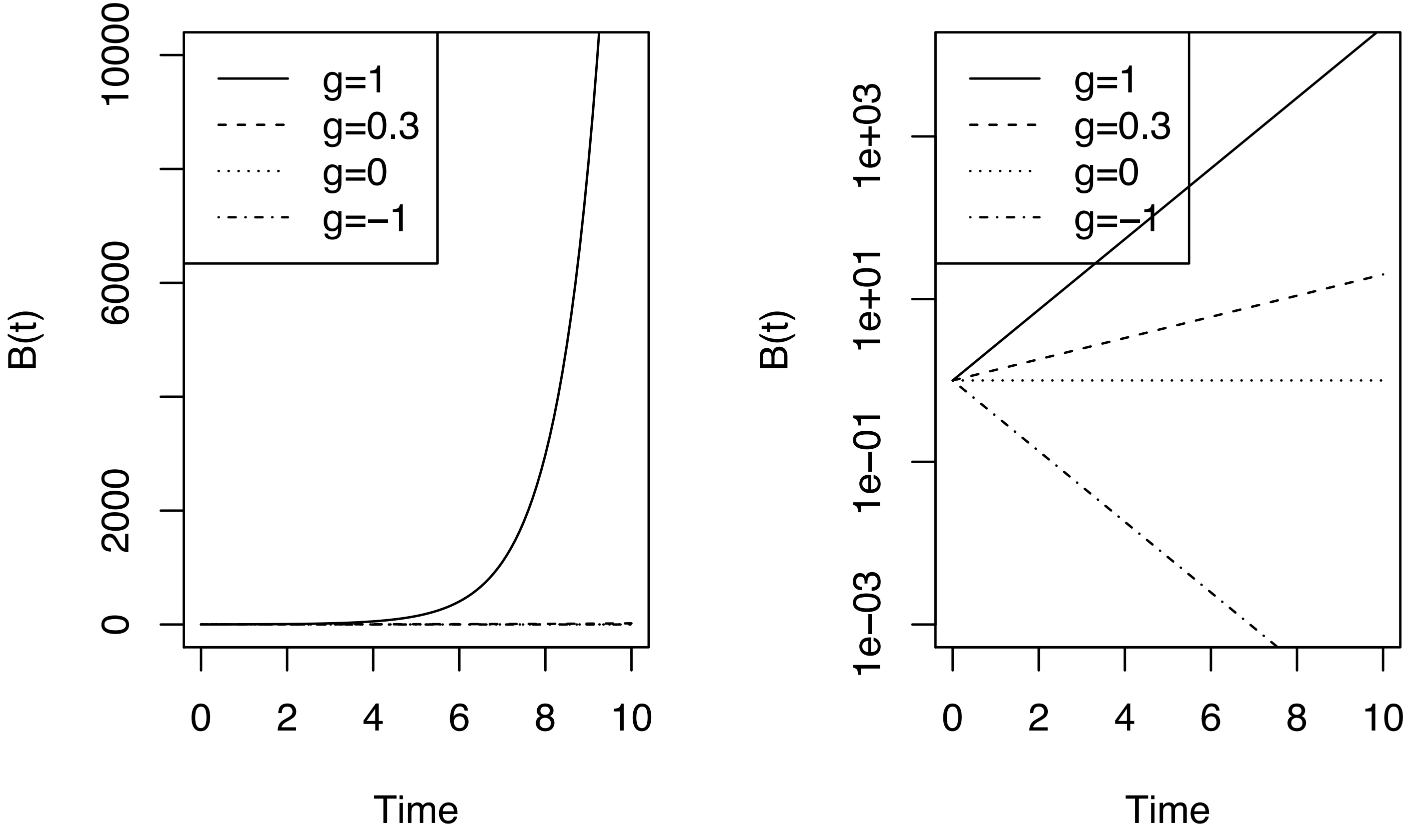
# Ordinary Differential Equations (ODE)

$$B_{t+\tau} = B_t + gB_t\tau \xrightarrow[\text{rewrite}]{} \frac{B_{t+\tau} - B_t}{\tau} = gB_t$$

$$\frac{B_{t+\tau} - B_t}{\tau} = gB_t \xrightarrow[\tau \longrightarrow 0]{} \frac{dB}{dt} = gB$$

Solution: $B(t) = B(0)e^{gt}$

# Malthusian growth model



$$\frac{dB}{dt} = gB$$

$$B(t) = B(0)e^{gt}$$

Notice:

1. Populations can change over many orders of magnitude. It is useful to plot populations on a log scale.
2. The relationship between the growth rate $g$ and the dynamics of population growth:
   - $g = 0$ corresponds to a stable population

   - $g > 0$ generates an exponentially increasing population (proliferation)
   - $g < 0$ generates an exponentially declining population (extinction)

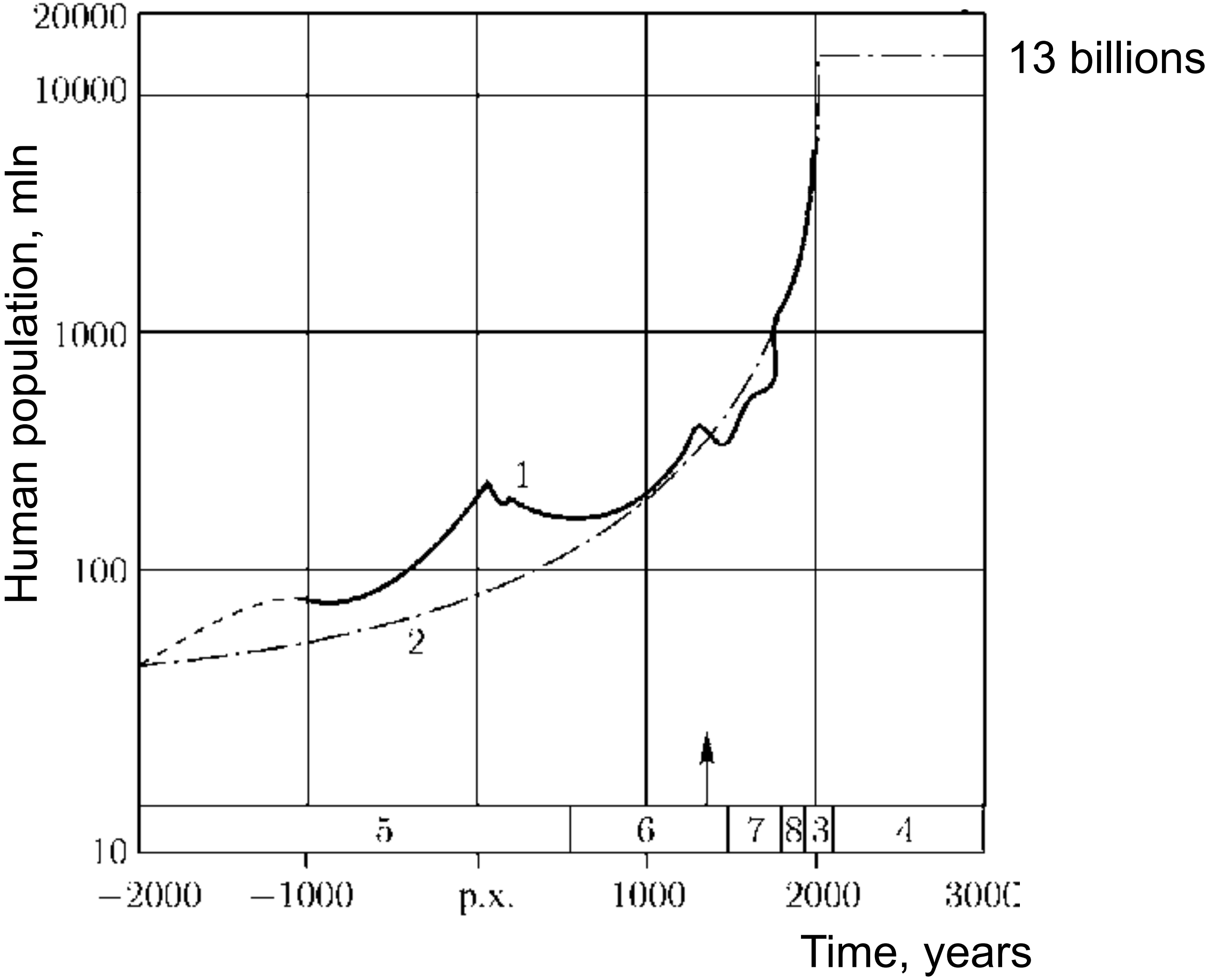
# Hyperbolic growth of the world population

$$\frac{dB}{dt} = gB^2$$

Solution:

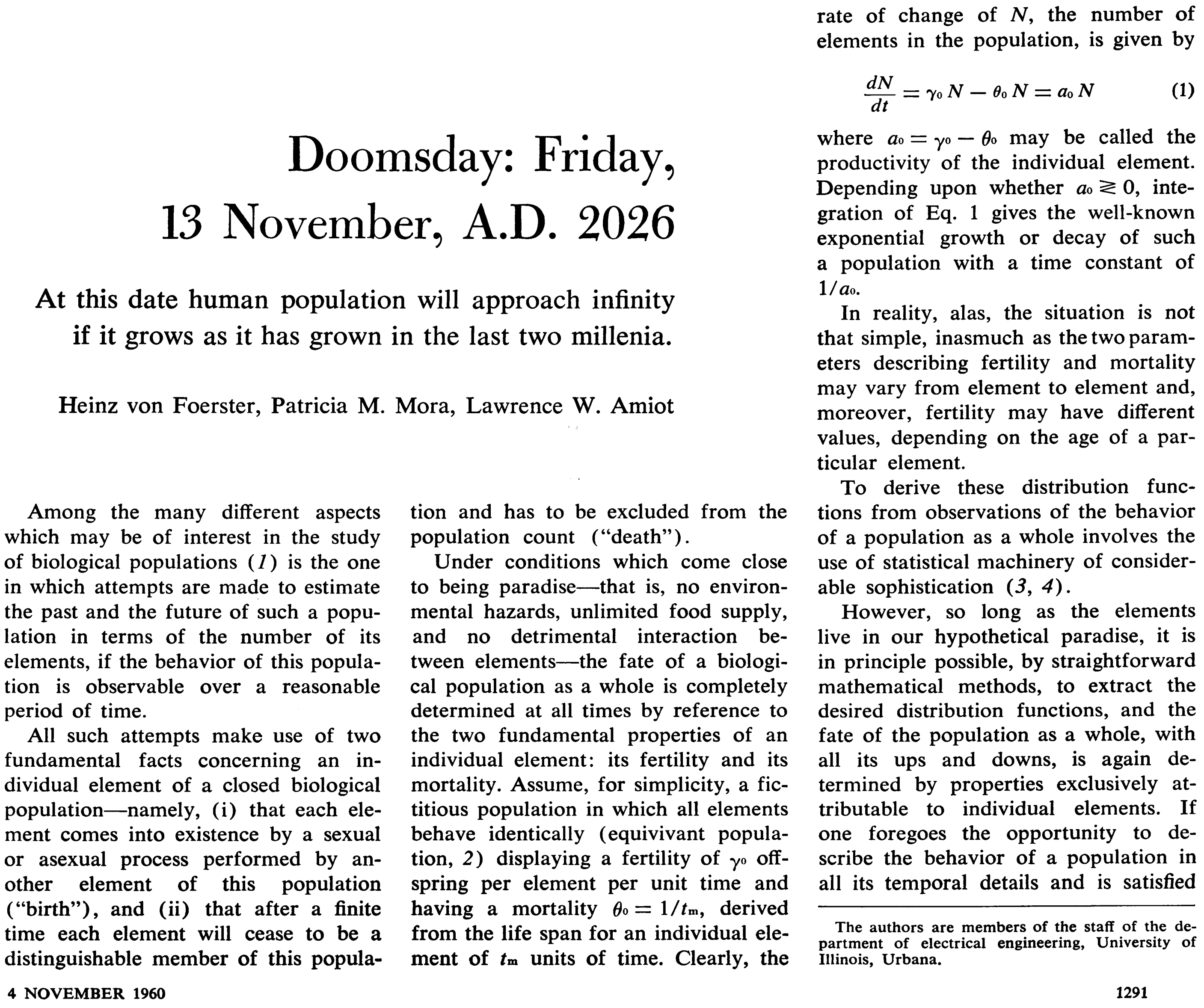
$$B(t) = \frac{1}{g(T_0 - t)}$$

$$T_0 = \frac{1}{gB(0)}$$



Data from Biraben, 1979

Heinz von Foerster (1911-2002)



# Doomsday: Friday, 13 November, A.D. 2026

## At this date human population will approach infinity if it grows as it has grown in the last two millenia.

Heinz von Foerster, Patricia M. Mora, Lawrence W. Amiot

Among the many different aspects which may be of interest in the study of biological populations (*1*) is the one in which attempts are made to estimate the past and the future of such a population in terms of the number of its elements, if the behavior of this population is observable over a reasonable period of time.

All such attempts make use of two fundamental facts concerning an individual element of a closed biological population—namely, (i) that each element comes into existence by a sexual or asexual process performed by another element of this population ("birth"), and (ii) that after a finite time each element will cease to be a distinguishable member of this population and has to be excluded from the population count ("death").

Under conditions which come close to being paradise—that is, no environmental hazards, unlimited food supply, and no detrimental interaction between elements—the fate of a biological population as a whole is completely determined at all times by reference to the two fundamental properties of an individual element: its fertility and its mortality. Assume, for simplicity, a fictitious population in which all elements behave identically (equivivant population, *2*) displaying a fertility of $\gamma_0$ offspring per element per unit time and having a mortality $\theta_0 = 1/t_m$, derived from the life span for an individual element of $t_m$ units of time. Clearly, the rate of change of $N$, the number of elements in the population, is given by

$$\frac{dN}{dt} = \gamma_0 N - \theta_0 N = a_0 N \qquad (1)$$

where $a_0 = \gamma_0 - \theta_0$ may be called the productivity of the individual element. Depending upon whether $a_0 \gtrless 0$, integration of Eq. 1 gives the well-known exponential growth or decay of such a population with a time constant of $1/a_0$.

In reality, alas, the situation is not that simple, inasmuch as the two parameters describing fertility and mortality may vary from element to element and, moreover, fertility may have different values, depending on the age of a particular element.

To derive these distribution functions from observations of the behavior of a population as a whole involves the use of statistical machinery of considerable sophistication (*3*, *4*).

However, so long as the elements live in our hypothetical paradise, it is in principle possible, by straightforward mathematical methods, to extract the desired distribution functions, and the fate of the population as a whole, with all its ups and downs, is again determined by properties exclusively attributable to individual elements. If one foregoes the opportunity to describe the behavior of a population in all its temporal details and is satisfied

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The authors are members of the staff of the department of electrical engineering, University of Illinois, Urbana.

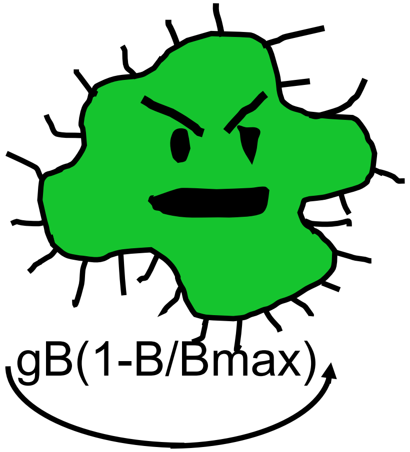
A 1960 November issue of Science

# Logistic growth model

P. F. VERHULST.

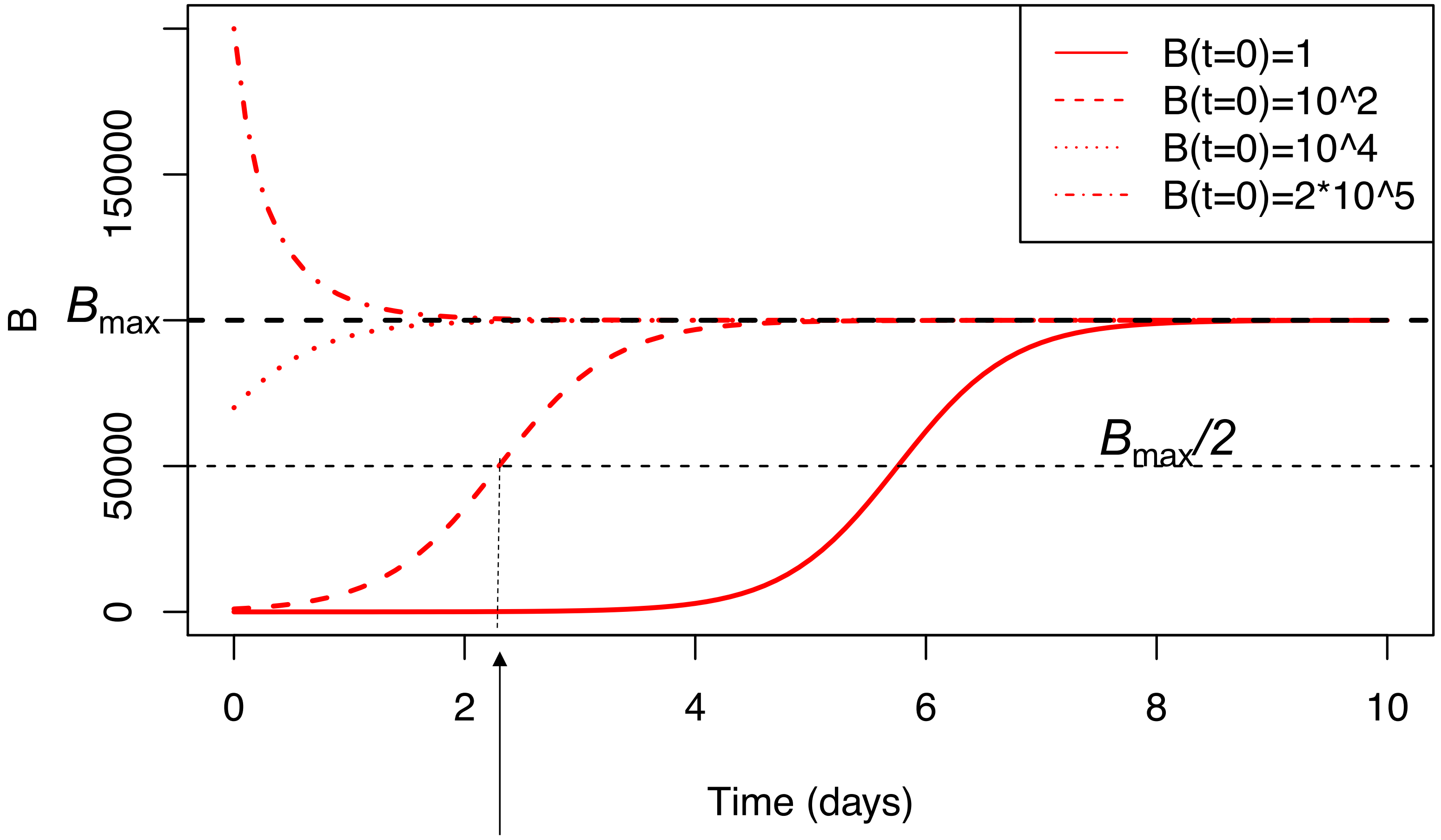
How could we implement saturating growth?

$$\frac{dB}{dt} = gB\boxed{\left(1 - \frac{B}{B_{max}}\right)}$$



Bacteria, B

# Logistic growth model



$$\frac{1}{g} ln(\frac{B_{max} - B(0)}{B(0)})$$

$$\frac{dB}{dt} = gB\left(1 - \frac{B}{B_{max}}\right)$$

$g$=2

$B_{\text{max}}=10^5$

# Logistic growth model

$$\frac{dB}{dt} = gB\left(1 - \frac{B}{B_{max}}\right) \xrightarrow[\text{rewrite}]{} \frac{dB}{dt} = gB - \delta B^2$$

$\delta$ is a coefficient for intraspecific competition

Members of the same species compete for limited essential resources, for example, for food, space, mates or any other resource which is required for survival or reproduction.

# Lotka–Volterra equations (predator–prey model)

**Alfred James Lotka** (1880 –1949)

**Vito Volterra** (1860 –1940)

prey $$\frac{dB}{dt} = gB - kBI$$

predator $$\frac{dI}{dt} = rBI - dI$$

1. Prey grows exponentially in the absence of the predator
2. Predator finds and eats prey (law of mass action)
3. The rate of increase in the predator population depends on how many prey are consumed.
4. Predators have a fixed death rate

# Lotka–Volterra equations (predator–prey model)

$$\text{prey} \qquad \frac{dB}{dt} = gB - kBI$$

$$\text{predator} \qquad \frac{dI}{dt} = rBI - dI$$

| Parameter | Description | Units |
|---|---|---|
| g | Represent the reproduction rate of the prey. The greater g is, the more rapidly the prey reproduces. | 1/time |
| k | Represents the death rate of the prey due to the presence of the predator. The greater k is, the greater the death of the prey is due to predation and the more effective the predator is killing the prey. | 1/time |
| r | Represents the reproduction rate of the predator. The greater r is, the more rapidly the predator reproduces and the more effectively the prey is able to nourish the predator. | 1/time |
| d | Represents the death rate of the predator. | 1/time |

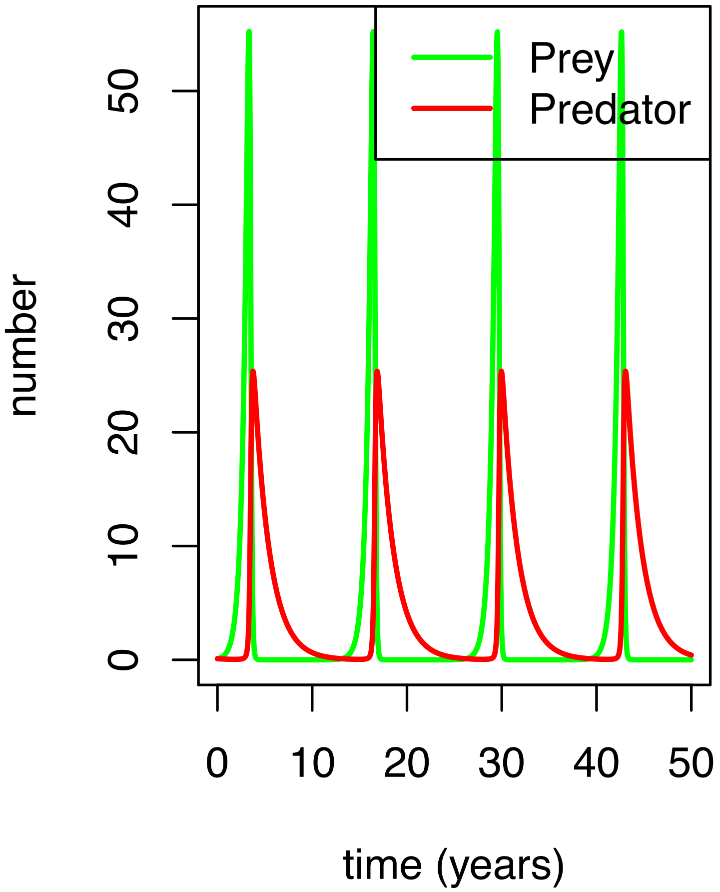
# Lotka–Volterra equations (predator–prey model)

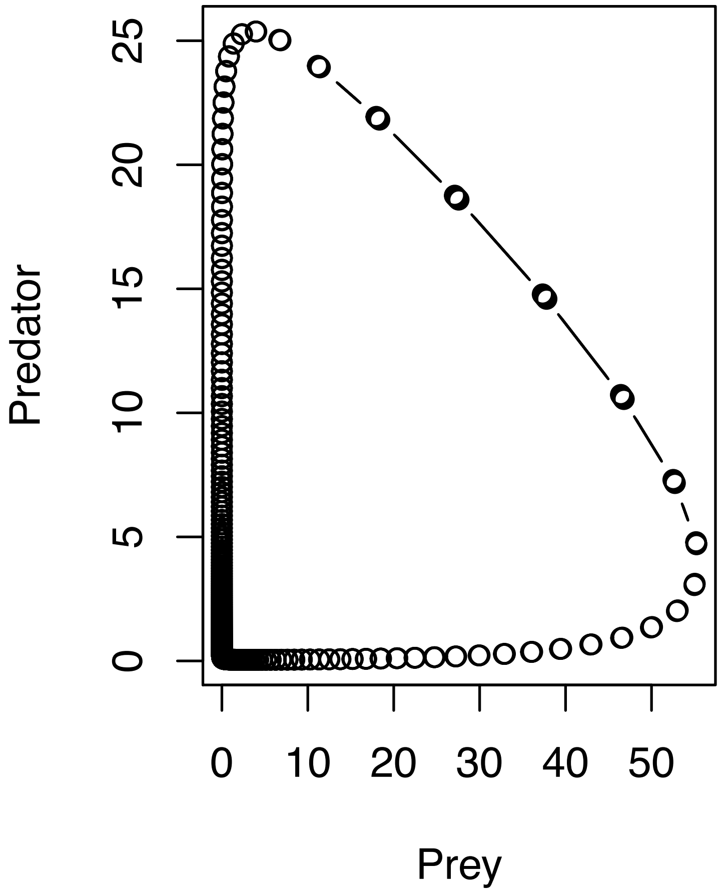
prey $$\frac{dB}{dt} = gB - kBI$$

predator $$\frac{dI}{dt} = rBI - dI$$

Steady state:

$$B^* = \frac{d}{r} \qquad I^* = \frac{g}{k}$$



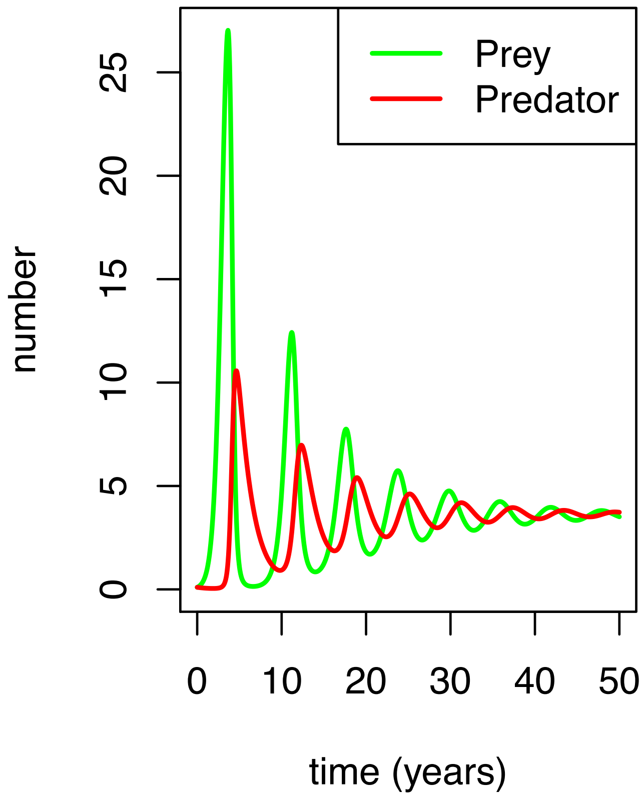


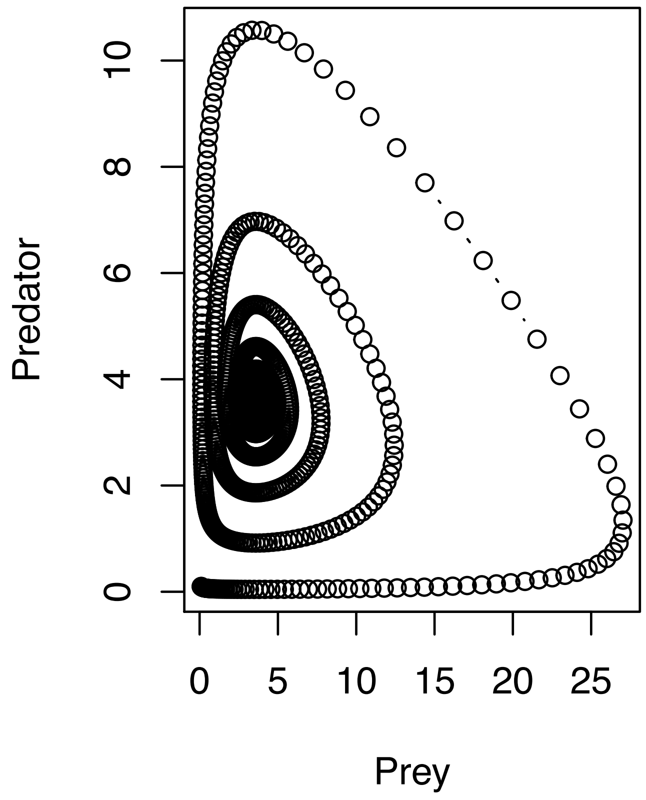
What is the effect of changing initial conditions?

# Addition of carrying capacity to the prey

prey $$\frac{dB}{dt} = gB(1 - \frac{B}{B_{max}}) - kBI$$

predator $$\frac{dI}{dt} = rBI - dI$$





# Thought experiment

Pest $$\frac{dB}{dt} = gB(1 - \frac{B}{B_{max}}) - kBI$$

Predator of pest $$\frac{dI}{dt} = rBI - dI$$

What will be the effect of pesticide on parameters?

- decrease the growth rate of the prey *g*
- increase the death rate of predator *d*

Consider a reduction in *g* and an increase in *d* by a factor *f*>1 and run the simulations

# Addition immune response to bacteria growth model

bacteria

$$\frac{dB}{dt} = gB\left(1 - \frac{B}{B_{max}}\right)$$

immune response

$$\frac{dI}{dt} = rBI - dI$$

What is missing?

# Addition immune response to bacteria growth model

bacteria $$\frac{dB}{dt} = gB(1 - \frac{B}{B_{max}}) - kBI$$

immune response $$\frac{dI}{dt} = rBI - dI$$