Proceedings of DETC'97
1997 ASME Design Engineering Technical Conferences
September 14-17, 1997, Sacramento, California

DETC97/VIB-3910

PERIODIC OPTIMAL CONTROL OF DAMPERS

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ABSTRACT

Optimal control of dampers has been proposed to mitigate vibration effects in mechanical systems. In many cases, systems are subject to periodic forcing and the goal is to maximize the energy dissipated by the damper. In contrast to prior work utilizing instantaneous or infinite-time-horizon optimization, this paper employs periodic optimal control to maximize the energy dissipated per cycle. For single degree of freedom systems in which the maximum allowable control effort is of the same order as the forcing magnitude, a state-dependent singular control law is shown to deliver maximum energy dissipation. Alternate control laws are proposed for situations when rattlespace requirements dictate damper displacements other than that of the singular solution.

INTRODUCTION

Dampers are used in a wide variety of applications to isolate structures and equipment from vibrations and to dissipate energy. Applications include flutter mitigation in turbine blades of aircraft engines and power plant generators (Sinha and Griffin, 1982), (Srinivasan and Cutts, 1984), vibration damping in large space structures (Ferri, 1987), and shock and vibration isolation for vehicles and equipment cradles (Lane et al., year), (Hrovat, 1993), (Karnopp and Trikha, 1969).

Dampers can be designed to be either passive (e.g., dashpots), active (e.g., motors) or semi-active (e.g., hydraulic cylinders with controllable orifice diameter). Semi-active dampers are the most appealing to designers because they deliver performance that rivals active dampers, while consuming only a fraction of the power required by them (Karnopp and Trikha, 1969).

In this paper, periodic optimal control is employed to maximize the energy dissipated by a damper. Since we are interested in establishing a benchmark for controller performance, no constraints are imposed on the damping force. For implementation, controller saturation as well as any relevant damper dynamics would have to be considered.

In the next section, control approaches which have been applied to dampers are reviewed. The following section presents the derivation of the singular controller which maximizes energy dissipation according to the system parameters and the periodic forcing. A penalty on control effort is then introduced to obtain controllers for a range of damper displacement amplitudes (rattlespace). Finally, numerical results are presented followed by conclusions.

CONTROL OF DAMPERS

Several control approaches have been pursued to maximize damper energy dissipation. These include Lyapunov's direct method, sliding mode control and LQR theory. The first method entails optimizing energy dissipation in an instantaneous sense by choosing the control which maximizes the derivative of a Lyapunov energy function. Semi-active controllers of this type have been developed for use with electrorheological (ER) fluid dampers (McClamroch et al., 1994) as well as friction dampers (Dupont et al., 1997). Sliding mode control, on the other hand, was successfully employed by Wang and co-workers (1994) to improve the performance of ER dampers.

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Another control approach that has received considerable attention is LQR theory. Ferri and co-workers (1992) have applied this technique to friction dampers. The cost function used was an infinite time integral of a weighted sum of system energy and control effort. Numerical simulations indicated a marked improvement in energy dissipation over a simple feedback controller given by $F_N(t) = k|\dot{x}|$, where $\dot{x}$ is the relative velocity and $F_N(t)$ is the normal force at a friction interface. In the vibration isolation of automobiles, Hrovat (1993) proposed LQR controllers using cost functions composed of mean-square rattlespace and a metric of ride discomfort. Similarly, Karnopp and Trikha (1969) have proposed the use of LQR theory in enhancing shock and vibration isolation in an aircraft landing/taxiing on a runway.

In many situations, the vibrational excitation is periodic. Instantaneous optimization approaches do not take into account system forcing and so the results tend to be suboptimal. Similarly, those who have applied LQR theory have considered only transient response or stochastic excitation. In contrast, the periodic controllers developed here maximize steady-state dissipation according to the particular forcing.

PERIODIC OPTIMAL CONTROL

In this section, a standard variational approach is employed to derive the optimal damping force for systems with periodic excitation. Given the forcing, it is expected that the system trajectory will be periodic. It is interesting to note, however, that the classical applications of periodic optimal control are systems which possess closed-loop equilibrium points. For example, Horn and Lin (1967) showed that in chemical reactor operation, periodic control laws improve performance in comparison to steady-state optimal controllers. Similar strategies were employed in the analysis of fuel-efficient cruise trajectories for aircraft, wherein the standard optimal controllers (LQR etc.) were replaced by periodic controls (Speyer and Evans, 1984).

Controller Design

Consider maximizing the energy dissipated by a control force, u on the system shown in Figure 1. A state variable representation of this system is given by

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = \frac{1}{m}\left(F_{ext}(t) - c\,x_2 - k\,x_1 - u\right) \tag{2}$$

where $F_{ext}(t)$ is a known external periodic force.

Since this is a nonautonomous system, these equations

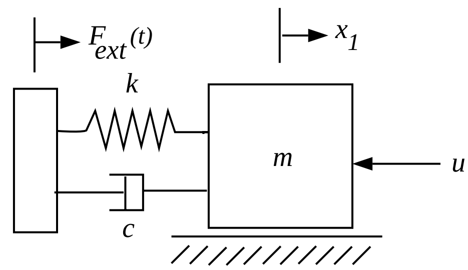


Figure 1. BLOCK MODEL FOR STUDYING FORCED VIBRATIONS.

can be redefined as

$$\dot{x}_1 = x_2 \tag{3}$$
$$\dot{x}_2 = \frac{1}{m}\left(F_{ext}(x_3) - c\,x_2 - k\,x_1 - u\right) \tag{4}$$
$$\dot{x}_3 = 1 \tag{5}$$

where $x_3 = t$.

Let the cost function be given by

$$J(u) = \min_{u\,\in\,\Omega} \frac{1}{\tau}\int_{t_0}^{t_0+\tau} \left[-u\,x_2 + \frac{\epsilon}{2}\,u^2\right]\,dt \tag{6}$$

where $\tau \in T \triangleq (0,\infty)$ is the time period of the system and t_0 is the initial time. Ω is the set of all admissible values for u, in which the maximum value of u is of the same order as the forcing magnitude and $u(t)$ is piecewise continuous.

An optimal controller with no constraints on the control effort expended ($\epsilon = 0$) will first be developed. It is assumed that the state variables as well as the time period are free variables. Furthermore, periodicity of the state variables, x_1 and x_2 is treated as an input to the problem. Penalty on control effort ($\epsilon \neq 0$) will be introduced in the subsequent sections.

With $\epsilon = 0$, the Hamiltonian can be written as

$$H\left(x,\lambda,u,x_3,\lambda_3\right) = -u\,x_2 + \lambda_1\,x_2 + \frac{\lambda_2}{m}\,\left(F_{ext}(x_3) - c\,x_2 - k\,x_1 - u\right) + \lambda_3 \tag{7}$$

Necessary conditions for optimality indicate that

$$\dot{\lambda}_1 = \frac{k}{m}\,\lambda_2 \tag{8}$$
$$\dot{\lambda}_2 = -\lambda_1 + \frac{c}{m}\,\lambda_2 + u \tag{9}$$

$$\dot{\lambda}_3 = -\frac{1}{m}\,\frac{\partial F_{ext}(x_3)}{\partial x_3}\,\lambda_2 \tag{10}$$

$$x_2 + \frac{\lambda_2}{m} = 0 \tag{11}$$

$$x_1(t_0) = x_1(t_0+\tau) \tag{12}$$

$$x_2(t_0) = x_2(t_0+\tau) \tag{13}$$

$$\lambda_1(t_0) = \lambda_1(t_0+\tau) \tag{14}$$

$$\lambda_2(t_0) = \lambda_2(t_0+\tau) \tag{15}$$

$$H(x,\lambda,u,x_3,\lambda_3) = K \tag{16}$$

$$H(t_0+\tau) = J(u) \tag{17}$$

along the optimal trajectory. Here, K is a constant to be determined.

Since the Hamiltonian is linear in u, $H_u = 0$ does not give us an expression for the optimal controller. Instead, it defines a singular arc, given by (11). This equation implies that

$$\dot{x}_2 + \frac{\dot{\lambda}_2}{m} = 0 \tag{18}$$

From equations (8)–(11) and (18), the expression for u can be determined to be[2]

$$u = F_{ext}(x_3) - c\,x_2 - k\,x_1 - \frac{m}{2\,c}\,\frac{\partial F_{ext}(x_3)}{\partial x_3} \tag{19}$$

With $x_3(t) = t$, and u as defined above, the system dynamics, (3)–(5) can be solved to give

$$x_1(t) = x_1(t_0) + \left[-\frac{1}{2\,c}\,F_{ext}(t_0) + x_2(t_0)\right] t + \frac{1}{2\,c}\int_{t_0}^{t} F_{ext}(\sigma)d\sigma \tag{20}$$

$$x_2(t) = x_2(t_0) + \frac{1}{2\,c}\left[F_{ext}(t) - F_{ext}(t_0)\right] \tag{21}$$

Note that equations (19)–(21) hold for any periodic input. Since any such input can be written as a Fourier series, we can let $F_{ext}(t) = A\,\sin(\bar{\omega}\,t + \phi)$ in our analysis, without any loss of generality. The optimal period of the system for this case is the same as the forcing period.

The initial conditions, $x_1(t_0)$ and $x_2(t_0)$, in the above equations can now be chosen such that the periodicity conditions, (12) and (13), are satisfied. With $F_{ext}(t)$ as defined above and $\phi = 0$,these initial values can be shown to be

$$x_1(t_0) = -\frac{A}{2\,c\,\bar{\omega}}\cos(\bar{\omega}\,t_0) \tag{22}$$

$$x_2(t_0) = \frac{A}{2\,c}\sin(\bar{\omega}\,t_0) \tag{23}$$

Since we now know x_1, x_2, and u along the singular arc, the transversality conditions of (16) and (17) give the average rate of energy dissipation,

$$J(u) = K = \frac{A^2}{8\,c} \tag{24}$$

for $\bar{\omega} = 2\,\pi/\tau$ and $t_0 = 0$.

Finally, since the Hamiltonian (7) is linear in u, we employ the generalized Legendre-Clebsch condition, $(\ddot{H}_u)_u \le 0$ as a weak local sufficiency condition. This is satisfied along the singular arc,

$$\frac{\partial}{\partial u}\,\frac{d^2 H_u}{dt^2} = -\frac{2\,c}{m^2} < 0 \tag{25}$$

Numerical Evaluation of Singular Control. To facilitate the comparison of controller performance due to variations in parameter values, let the ratio between the forcing frequency and the undamped natural frequency of the system be given by $\beta = \bar{\omega}/\omega_n$, where $\omega_n = \sqrt{k/m}$. Also, let the damping ratio of the system be given by $\zeta = c/(2\,m\,\omega_n)$. The state trajectories, optimal control force, net energy dissipated by the controller, and control effort spent can now be rewritten in terms of these quantities as

$$x_1(t) = -\frac{A}{4\,\zeta\,\beta\,m\,\omega_n^2}\,\cos(\bar{\omega}\,t) \tag{26}$$

$$x_2(t) = \frac{A}{4\,\zeta\,\omega_n\,m}\,\sin(\bar{\omega}\,t) \tag{27}$$

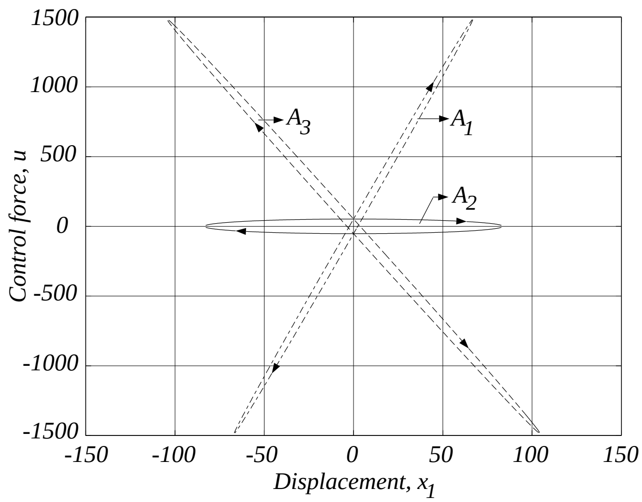
$$u(t) = \frac{A}{2}\,\sin(\bar{\omega}\,t) + (1-\beta^2)\,\frac{A}{4\,\zeta\,\beta}\,\cos(\bar{\omega}\,t) \tag{28}$$

$$E = \int_0^{\tau} u x_2\,dt = \frac{A^2}{16\,\zeta\,\omega_n\,m}\,\tau = K\,\tau \tag{29}$$

$$U = \int_0^{\tau} u^2(\sigma)\,d\sigma = \frac{A^2}{8}\,\left[1 + \frac{(1-\beta^2)^2}{4\,\zeta^2\,\beta^2}\right]\,\tau \tag{30}$$

where E is the energy dissipated per cycle and U is the control effort expended per cycle. All the results described in this section were obtained with parameters chosen as follows: $A = 105$, $m = 1$, $c = 0.1$, $\bar{\omega} = 2\,\pi$. k was chosen to give a desired value of β.

[2] Since this control law is valid only along the singular arc, it will be referred to as the "singular controller" from here onward in this paper.



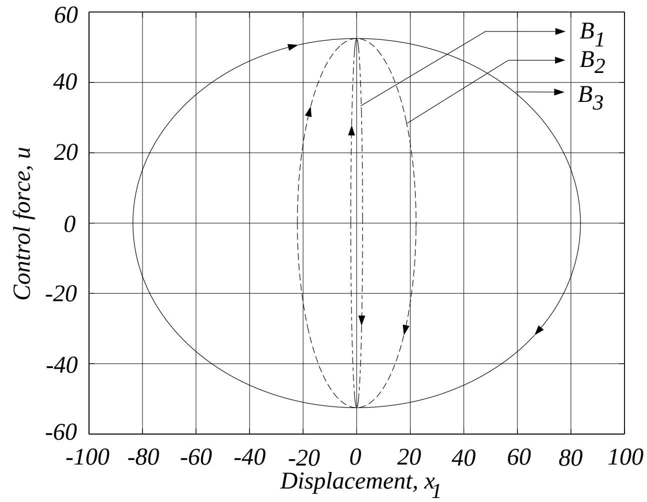


Figure 2. CONTROL FORCE VERSUS DISPLACEMENT FOR $\beta = 1.25$ (A_1), 1 (A_2), 0.8 (A_3) ALONG SINGULAR ARC TRAJECTORY. ζ IS HELD CONSTANT AT 0.00796.

Figure 3. CONTROL FORCE VERSUS DISPLACEMENT FOR $\zeta = 0.3$ (B_1), 0.03 (B_2), 0.00796 (B_3) ALONG SINGULAR ARC TRAJECTORY. β IS HELD CONSTANT AT 1.

Figure 2 depicts control force versus displacement along the singular arc, for $\beta = 0.8$, 1.0, and 1.25, while holding ζ constant ($= 0.00796$). The area inside this curve represents the amount of energy dissipated per cycle by the controller—equal to E. It can be shown that the optimal control force corresponds to a passive system.[3] From (27) and (28), the optimal control impedance is

$$\begin{aligned} c^* &= \frac{u(t)}{x_2(t)} \\ &= 2\,\zeta\,\omega_n\,m + (1-\beta^2)\,\frac{\omega_n\,m}{\beta}\,\cot(\bar{\omega}\,t) \\ &= c + (1-\beta^2)\,\frac{\omega_n\,m}{\beta}\,\cot(\bar{\omega}\,t) \end{aligned} \tag{31}$$

From this equation it can be observed that the optimal control force is viscous in nature for $\beta = 1$. Otherwise, the corresponding passive system would store and release energy during each period. For example, a passive implementation of c^* could be comprised of a spring ($\beta > 1$) or mass ($\beta < 1$) in parallel with a viscous damper.

While not apparent from this figure, (29) indicates that the amount of energy dissipated per cycle, E, depends on τ. Therefore, as β increases ($\bar{\omega}$ is increased while holding ω_n constant), the amount of energy dissipated per cycle decreases. Thus for the areas in Figure 2, $A_1 < A_2 < A_3$ corresponding to $\beta = 1.25$, 1.0, and 0.8, respectively. The average rate of energy dissipation, E/τ, however, remains constant. It can also be observed from this figure and equation (26) that the peak-to-peak amplitude of oscillation decreases with increase in the value of β.

Figure 3 represents optimal control force versus displacement for $\zeta = 0.00796$, 0.03, and 0.3, with β and ω_n held constant. It is clear from this figure, as well as from (26) and (29), that the peak-to-peak amplitude of oscillation besides the average rate of energy dissipation, E/τ decrease with increasing ζ. In other words, as the amount of internal damping in a system increases, the scope for improvement in energy dissipation through an external damping force decreases.

The ratio between amount of control effort expended per cycle and time period of the system, U/τ, for different values of β and ζ is presented in Figure 4. This figure indicates that the minimum control effort is expended when $\beta = 1$, independent of the value of ζ. For $\beta \neq 1$, however, the control effort decreases with increasing ζ.

Since E/τ remains the same for any value of β, and it increases as ζ decreases, a key factor in determining operating conditions under which the singular optimal controller will be most effective is U/τ. It can, therefore, be deduced from figure 4 that an external damper will be most effective

[3] A system is said to be passive if, for all time, the power entering the system is greater than or equal to the rate of change of energy stored in the system.

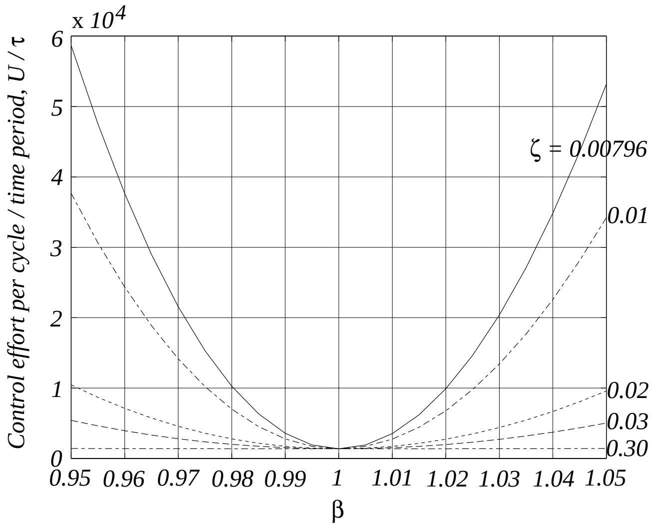


Figure 4. AVERAGE CONTROL EFFORT EXPENDED VERSUS β FOR DIFFERENT VALUES OF ζ.

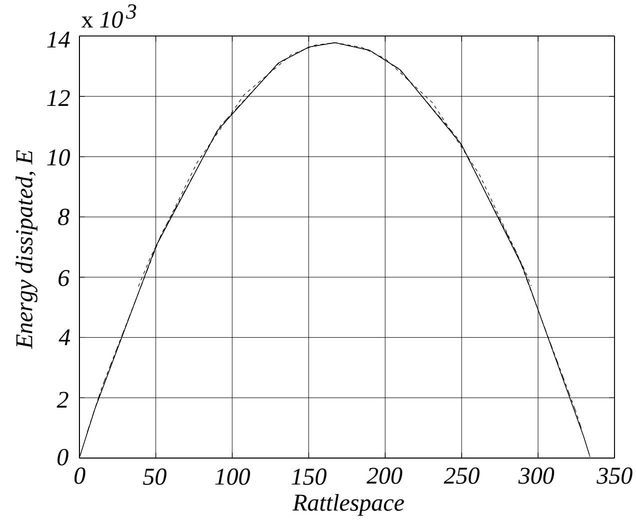


Figure 5. ENERGY DISSIPATED PER CYCLE VERSUS RATTLESPACE (x_1 PEAK-TO-PEAK) FOR $\epsilon = 0.01\ (\ldots), 0.1\ (-\cdot-), 1\ (--), 100(—)$.

when $\beta = 1$ and ζ is very small. Of course most systems for which energy dissipation is of import have very low internal damping (directly related to ζ) and operate near resonance ($\beta = 1$).

Rattlespace Constraints. In many practical systems, the rattlespace[4] may be smaller than what is commanded by the singular controller described above. To determine an optimal control force for systems with rattlespace constraints, we redefine our problem as follows: For a given value of $x_1(t_0)$ and $x_2(t_0)$, determine the maximum amount of energy that can be dissipated by an external control force, while ensuring periodicity about $x_1(t_0)$ and $x_2(t_0)$. Since the choice of initial values for x_i obviates the necessary condition on periodicity of λ_i, a numerical solution entails solving for λ_i such that x_1 and x_2 are periodic.

To solve this problem, a penalty on control effort is introduced ($\epsilon \neq 0$ in equation (6)). The optimal control force for this case can be determined to be

$$u_\epsilon = \left(x_2 + \frac{\lambda_2}{m} \right) / \epsilon \qquad (32)$$

Numerical analysis was used to compare the performance of the constrained rattlespace controller with that of the singular controller. Four different values of ϵ were considered (0.01, 0.1, 1, and 100).

Figure 5 depicts energy dissipated versus rattlespace, for $\beta = 1$, $\zeta = 0.00796$ and the four values of ϵ. It is evident from this figure that there exists a unique value of displacement amplitude (≈ 167) where the maximum energy is dissipated. This amplitude is equal to that of the singular arc amplitude (refer to the peak-to-peak value of x_1 in Figure 2). In fact, at this amplitude, the trajectories are those of the singular arc, independent of ϵ.

It is also clear from this figure that the amount of energy dissipated by the constrained rattlespace controller is practically independent of the value of ϵ. Figure 6, however, indicates that control effort expended for $\beta = 1$ and $\zeta = 0.00796$ is highly dependent on ϵ at rattlespace amplitudes other than that of the singular arc. From Figures 5 and 6, it is clear that the control trajectory obtained for large ϵ should be used in any implementation.

CONCLUSIONS

In this paper, periodic optimal controllers were designed to maximize the energy dissipated by a damper. For single degree of freedom systems, a singular control law was shown to deliver the maximum energy dissipation. The singular controller can be implemented as a passive system. Except when forced at resonance, however, the damping

[4]Rattlespace is defined as the permissible peak-to-peak displacement of a system.

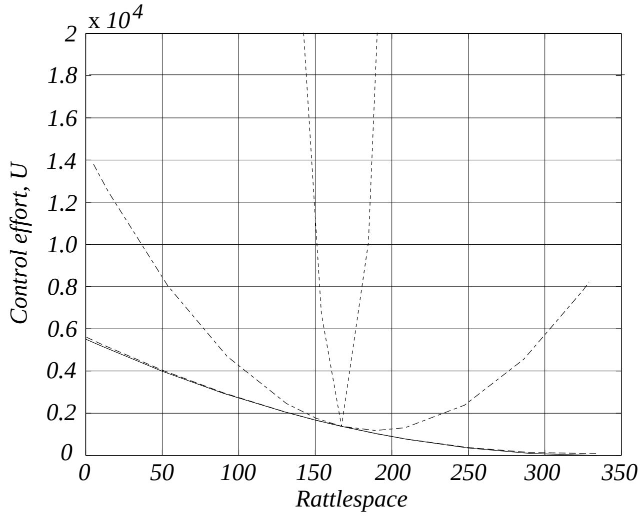


Figure 6. CONTROL EFFORT EXPENDED VERSUS RATTLESPACE, (x_1 PEAK-TO-PEAK) FOR $\epsilon = 0.01\ (\ldots), 0.1\ (-\cdot-), 1\ (--), 100(—)$.

system would include energy storage elements.

Constrained optimal controllers were proposed for systems with rattlespace less than what is commanded by the singular controller. The controller performance indicates that the energy dissipated is virtually independent of any penalty imposed on the control effort. The control trajectories obtained for the largest penalty can, therefore, be used to deliver maximum energy dissipation for the allowed rattlespace, while expending the least control effort.

The periodic forcing acting on the system was assumed to be composed of one frequency. Since any periodic input can be written as a Fourier series, the results obtained in this paper can be appropriately extended to such inputs.

ACKNOWLEDGEMENTS

The authors wish to thank Prof. David Castañon for his valuable suggestions at various stages of this project. Our thanks are also due to the National Science Foundation for partly supporting this work through Dynamic Systems and Control Program under grant MSS-9302190.

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