The Vertical Hand Span: Nonstandard Units, Expressions, and Symbols in the Classroom

LYNNE GODFREY
The Cambridge Algebra Project and Cambridge Public Schools

MARY CATHERINE O’CONNOR
Boston University

Student creation of nonstandard units of measurement, and symbols for such units, is a pedagogical technique intended in part to promote students' understanding of mathematics as communication (NCTM, 1989). This paper examines a conflict that arose during such an activity in a sixth-grade classroom. In our view, the students' disagreement reflects the tension among several conflicting requirements of language and symbol use in mathematical communication. Iconic, transparent symbols exist alongside those that are arbitrary and opaque. Technical terms depart from their everyday linguistic counterparts in unpredictable ways. Most importantly, students must recognize and honor conventional, historically grounded uses of expressions and symbols, while also using and actively interpreting innovative and contextually determined uses. We explore the implications of these complexities for student learning and teaching.

INTRODUCTION

Student-generated data and representations are an important part of many experience-based curricula in mathematics and science, and are assumed to confer many cognitive benefits. When students create their own idiosyncratic units of measurement, for example, we expect that the process of measuring will become new and productively problematic to them, as they attempt to fit their chosen unit to the dimensions of the objects to be measured (Hirstein, Lamb, & Osborne, 1978; Wilson & Osborne, 1988). Students confront the process of subdividing a unit of measure, and may discover problems of measurement error and standardization through active participation in solving a problem with their...
own chosen unit. In general, we expect children to develop a deeper, "first-person" understanding of measurement through such activities, an understanding that will serve them in learning to use the standard systems of measurement encountered in school science.

Similar benefits are expected when students develop their own symbols and forms of notation for the dimensions or operations they are working on. Such activities are thought to promote metacognitive awareness of the properties of symbolic systems that students encounter in math and science. It is often hoped as well that such experiences will help students lose the antipathy they may have for formal notations. When students become originators and creators—participants in the activity of creating symbols for operations or relationships—they may come to see symbols and notations as practitioners see them: as useful tools for thinking and communicating.

Student-generated units and symbols are not meant as a substitute for learning standard measures, symbols, and notations, but rather as preparation for them (Wilson & Osborne, 1988). It is not easy to enunciate students into a view of mathematics as communication (NCTM, 1989); much of pedagogical tradition works against that view. Through creating and using nonstandard, student-generated symbols and measures, it is hoped that students will more readily deal with traditional mathematical texts and language as real communication. These activities are thus intended to provide a way into a community that creates new measures and symbols as they are needed, flexibly applying the means that already exist and adding to those tools when necessary. To what extent such activities successfully foster these ends has not been well studied.

In our experience, classroom activities in which students create their own symbols and units turn out to be more complex than might be expected for reasons that are not usually considered in theoretical or practical discussions of such activities. In this brief account we will show that students engaged in this activity are becoming familiar with some subtle and important aspects of mathematical communication. We will also discuss, however, an interesting tension that emerges, observable in students' varying attitudes toward the activity itself—the creation and use of nonstandard measuring units, and the use of nonstandard symbols and linguistic expressions to denote them. Although teachers and researchers have discussed the cognitive aspects of student-generated symbols, units, and data, rarely do they discuss the difficulties involved in bringing students together to collectively adopt new conventions—conventions that differ from the standard. We present our story, and then discuss some issues and dilemmas that underlie episodes of this kind.

BACKGROUND

The Algebra Project Transition Curriculum

The Algebra Project transition curriculum for the middle school (Moses, 1992; Moses, Kamii, Swap, & Howard, 1989) contains numerous stages at which
students create their own symbols, and several points at which students develop their own units of measure. Some of this work is done individually, but students always discuss their choices and creations, first with their team, then with the larger group.

When the episode began, Lynne had been teaching math in this sixth-grade class since the beginning of the year. This session took place in March of the school year. On this day, 16 sixth-grade students were working in teams of 3 or 4. They had been working on an Algebra Project transition curriculum unit where they were comparing the heights of two of their team members. Their instructions were: “Compare the heights of two team members. Do not measure their heights using standard measuring units.”

We should first contextualize these events as part of the 5-step curricular process, a central part of the Algebra Project curriculum. In this cycle of exploration, students first take part in some physical activity, chosen to present a complex experience that must then be “mathematized.” After depicting aspects of their experience in a drawing or model, the students discuss it in “everyday language”—the informal variety of language they use in nontechnical contexts of everyday life. At this time they are introduced to specific features of the experience that will form the basis for their mathematization of the complex event. For example, through discussion, students come to agree that a trip on the mass transit system consists of a starting station, finishing station, direction, and number of stops. The group members reach agreement on the features of the event and on the definition of those features—for example, does the number of stops in a trip segment include the station at which the trip began or does one begin counting at the first stop after the starting station? Their everyday descriptions of the event are first cast as informal, intuitive “observation statements.” Gradually these are cast into more “regimented” language (Moses et al., 1989)—a variety that uses linguistic expressions and symbols to denote precise, explicit, and stable mathematical and scientific meanings. This is the next step in the process of creating a symbolic representation of selected aspects of the original experience.

The comparing heights activity taking place during this class session was part of a larger enterprise. As part of the Algebra Project transition curriculum, students eventually come to understand subtraction of integers in terms of comparison (Moses et al., 1989). In order to get to the point where they can use formal, “regimented” language about comparison to express statements about subtraction of integers, they first go through several activities in which their own, informal language about comparison is gradually analyzed in terms of features of a comparison event. Then their observation statements about comparison are changed into a more regimented statement that includes symbolic representations of the features and dimensions in the activity at hand.

In the comparing heights activity, student small groups had chosen nonstandard measures, and had agreed on what to call them. They had agreed on which team members would be compared, and on who would be measuring. They had
also discussed “height” as a central feature of the situation, and had created symbols for (a) the feature “height,” (b) the standard of comparison and the object of comparison (here two students’ heights), (c) the operation of comparison, (d) the outcome of the comparison (either “less than” or “greater than”) and (e) the nonstandard unit of measure they had created. Finally, they had carried out the measurement, and had recorded their results on chart paper.

At the end of this activity Lynne asked teams to share the symbols they had created and to present their measurement situation in everyday language (e.g., “Kadeem is 2 [nonstandard units of measure] taller than Daria”) and in a more regimented sentence (e.g., “Kadeem’s height, compared to Daria’s height, is two [nonstandard units of measure] greater.”) Each team was given chart paper to record their symbolic representations and key. The KND team had volunteered to report out first in the class discussion.

The KND Team
The KND team had formerly been the KLND team, but they had recently lost one of their team members who had transferred to another school. The team was in the process of redefining their roles as team members, deciding how they would organize their duties and the work. The person who had left was “L.” She had been the organizer for their team. The other team members are N, D, and K: Noel, a very thoughtful, sensitive, poetic kind of fellow who put great care into his math writing and the construction of his symbols; Daria, a petite, sometimes angry, sometimes funny girl struggling to keep up with the inconsistencies of her home life and her school life; and Kadeem, a sparkling, mischievous, always involved student who would usually forget his pencil or his homework.

The three team members had come a long way in their transition. They were no longer fighting over whose description should be read or whose symbols should be used or who should do the presentation. In previous team discussions each one thought his way or her way was the only way to do it and their team spent long periods of time locked in verbal battle. Now, as a team, they talked through the pros and cons of each symbol, they gave rational explanations, and they conceded where they could see the benefit for the team.

KND PRESENTS TO THE CLASS

KND’s First Draft
After 2 days of working through to consensus, the team agreed to use a combination of Noel’s and Kadeem’s symbols. They agreed to compare Noel’s and Daria’s heights, and Kadeem would do the reporting out for the team with Noel and Daria at hand to help with class questions. Their first chart looked like Figure 1.

During the class discussion, students scrutinized the KND team chart. As was the normal practice, they asked questions about things that they didn’t understand, or that seemed interesting or problematic. Oftentimes, during this group
How much shorter is Daria than Noel?
Answer in people talk
$\frac{1}{3}$ of a handspan

Feliks Height
Daria's height compared to Noel's height is $\frac{1}{3}$ handspan.

Symbolic Representation

Figure 1. How much shorter is Daria than Noel? (First draft).
review and editing process, the student team that has created the symbolic representation comes face to face with other students’ perceptions about the transparency or opaqueness of its representational choices.

Students immediately had questions about the symbols in the expression “Noel’s height.” KND’s symbol for “height” was an upright arrow with four little tail feathers. In each expression where it occurred, there was a symbol standing for the expression that would pick out the actual locus of the comparison—“Daria’s” and “Noel’s.” These were pictographs of the person whose height was being measured. Within their symbolic representation, the KND team had depicted Noel standing in the first part of the equation. The second time they symbolized the expression “Noel’s height,” however, they had depicted Noel sitting (see Figure 2). It turned out that the KND team had not left quite enough room to draw the legs and so had opted to depict Noel sitting to avoid the graphic infelicity of having his legs hang down into the lines of text.

Although both depictions of Noel stood for the same relation (the relation linking the quantity being compared, a person’s height, with a specific locus of the quantity, i.e., Noel), the KND team quickly learned that the audience was not willing to unquestioningly accept them as the same symbol. Although both the standing and the sitting pictograms were equally good (or equally poor) depictions of Noel, they were sufficiently different from each other that people questioned whether they counted as instances of the same symbol.

In everyday language and in mathematical communication, many expressions (in fact an infinite number) can be used to denote the same entity. So why were these students objecting to KND’s use of two slightly different symbols to stand for the same relation? They were behaving according to two assumptions that pervade normal language use: Each word, symbol, or expression has a unique meaning, and a speaker’s choice of word, symbol, or expression indicates an intentional selection from the possible array of meanings. If a speaker uses Expression A to refer to an object, and then later on uses Expression B to refer to the same object, the listener will assume that there is some additional intended communicative significance to the choice of Expression B, because Expression A was also available for that purpose.

Through their objections, the students were indicating their belief that as readers they had a right to expect a particular meaning to be denoted by only one symbol. They interpreted KND’s use of a different symbol as an indication that some additional meaning was intended, whether or not the same referent was picked out. Was this “seated Noel” symbol in fact standing for something different? If so, what? The KND team conceded. Kadeem promised the class members that KND would make some changes to their key so it would be consistent with their structured language and symbolic representation. The next day, when they brought in the key, they had incorporated both symbols—standing Noel and seated Noel—as synonyms in the key.
KND's Second Draft
The second day they came back with their original chart, and a taped-on addition that looked like Figure 3. At this point, there were a few questions about an inconsistency between a symbol in the actual chart and the same symbol in the key. In the key, the two-headed arrow was missing over the two heads on their “compared to” symbol. This inconsistency was essentially treated as a typo, and KND quickly agreed to remedy it.

Nonstandard Measures and the Kadeem Hand Span
Previous to the KND team report, there had been some discussion about standard and nonstandard measuring units. First, Andrew, Daniel, and Jose’s team had reported that they had used cinder blocks for measuring and comparing the heights of two of their team members. After a discussion of the difficulties inherent in such a nonstandard measure, the group self-named “the A Team” (Elliot, Brandon, and Justin) had reported their results, presenting to the class a mathematical sentence that said “Justin’s height compared to Elliot’s height was 2 inches taller.”
asked a question that appeared to be about the symbol, but turned out to have a wider significance. He questioned Kadeem: Which part was the hand span? Kadeem showed what he meant, using his hand. He indicated the length of his hand from the bottom of his palm to the tip of his longest finger.

This caused a small uproar among some of the students. Elliot and several others began to describe what they thought the expression hand span meant, showing a spread-fingered hand and indicating the width of their hands. Elliot said “hand span is like wing span—from tip to tip.” Elliot told Kadeem he couldn’t do what he had done—he could not make up a new meaning for a word like hand span. Kadeem then turned to Lynne and said he had thought it was okay in this class to create new meanings for words.

All eyes turned to Lynne. She asked the class: “Do you all think this is an okay thing to do in this class? Is it okay to create new definitions for old words?” Several students said yes, and gave examples from previous lessons where they thought the class had constructed its own definitions. They recalled times in which the class had discussed the interpretation and use of the words template, equivalent, equal, length, and width.

Even so, Daniel suggested, Kadeem should call his symbol something other than hand span. Another student offered a possible substitute—how about calling it “hand length”? The other team members seemed to want to agree with this quickly, but Kadeem insisted he meant “hand span.” Robin said that hand span was horizontal and in their symbol they showed a vertical line. Another student, attempting to incorporate all positions, suggested calling it a “vertical hand span.”

Elliot continued to insist they not use the phrase hand span at all. Lynne asked him to explain why he felt so strongly about this. He replied “Because everybody knows that when you say hand span you mean across the hand, the width, and if
Class members objected after the A Team’s presentation: Inches were definitely not nonstandard units. Lynne and the class looked back at the instructions for the lesson, which clearly said “do not use standard measuring units.” The A Team answered that they had not actually measured each other using standard measuring units, because they had not carried out the measurement themselves. Instead, they had decided to use previous knowledge: The gym teacher had measured them the day before. In their view, the precision of the gym teacher’s measurement made it preferable. After that, however, they acknowledged that inches were certainly standard units, and did not meet the spirit of the instructions. It was suggested they redo the comparison using some nonstandard measure.

The topic then turned to KND’s second draft, specifically the unit KND had chosen: a “Kadeem hand span.” First, one student asked a question about KND’s symbol for their unit of measure: a six-fingered hand (see Figure 4). They wanted to know why it had six fingers—did this mean the team members had six fingers on their hand? Kadeem said no, it did not mean he had six fingers on his hand, but it was a symbol, and they could make the symbol for “half a Kadeem hand span” with six fingers if they wanted to.

Tina asked a question about the little arrow next to KND’s symbol for hand span. Kadeem said this arrow meant “half” of a Kadeem hand span. Then Elliot
they [KND] use it to mean the length, people will get really confused and nobody will understand their symbols.”

Several class members, including Kadeem and Elliott, then discussed whether you could combine the two terms, vertical and hand span, to describe the symbol that the KND team wanted to use. Could the phrase vertical hand span be used to describe what they had done with their nonstandard unit of measure? During this part of the class discussion most students seemed to agree that it was legitimate to combine the terms to create a new meaning. In support of the new compound term, several students gave other examples of times when they had agreed as a class to use a particular expression or definition that did not follow the conventional meaning or use. In fact, one example referred to something Elliot’s team had done. His team had decided to use the symbols > and < to mean “taller than” and “shorter than,” respectively. One student pointed out that this was different than the conventional meanings of “greater than” and “less than.” Elliot maintained that in that instance it had been ok, but in this instance it was not because the meaning for “hand span” had already been determined and the class could not change it.

At the end of the class period, Lynne summed up the discussion by saying that there were several points and suggestions that came up in the discussion:

1. There is a conventional understanding of the term “hand span” as described by Elliot et al.
2. The class made two suggestions for words to describe KND’s symbol: “hand length” and “vertical hand span”

With this information on the table, Lynne asked KND to make a decision on their team symbol and to redo the key for “half a Kadeem hand span” and to report out their decision to the class the next day. On the way down to lunch she followed up with Elliot to see what his thoughts were. He again described the meaning of “vertical” and described the conventional meaning of “hand span” and said he was sure it was not okay to combine the two terms.

The next day, the KND team announced that it had decided to leave the symbol the way it was, but to describe it as a “vertical hand span” in the key. That seemed to be the last word on the issue, and Elliot did not pursue it. We are left to think about the implications of the episode.

**DISCUSSION**

Though it is just one example, this episode is not atypical of the kinds of discussion that occur on a regular basis in this classroom. The eventual integration of the standard definition and the student’s creation is an issue inherent in the very nature of the curricular process and the methods used in this curriculum. We would venture to say that such episodes probably occur in other classrooms...
where student invention and construction of units and symbols are considered useful and productive activities. What is the significance of conflicts like the one described here, and how should they be handled?

Constructivism in most of its instantiations asserts that we cannot successfully impose on students a version of reality that they have not actively participated in constructing. They may know about those unfamiliar realms as a result of traditional teaching, but they will not be able to use the tools found there with any degree of power or persuasiveness. Thus when we provide students with experiences that are intended to enable them to construct the relevant understandings, it is our responsibility to try and understand what is at stake for them in these activities (Cobb, Yackel, & Wood, 1992; Fischbein, 1990). How do they understand what we are asking them to do? If we pursue this question in the current episode, some fundamental issues become visible, even in a seemingly simple conflict about word usage and symbol choice such as this one.

Many observers would deny that this episode has any general significance. They would argue that Elliot is right, the meaning of “hand span” is given, and should not be tampered with. Elliot has linguistic authority on his side in the form of a codification of the history of English—the dictionary. Moreover, they would add, the teacher has an obligation to ensure that Kadeem knows that he cannot simply use any word in any way he wishes. Kadeem needs to be cognizant of the conventional, community-based nature of word meaning. In order for speakers to understand each other, they must adhere to conventional meanings, or as Elliot says, “people will get really confused and nobody will understand their symbols.”

We would like to argue that such a response is incomplete. It is based on an overly simple view of what is going on in this episode, and in mathematical language use in general, and thus misses the most important pedagogical implications of the episode. In fact, we are fairly certain that Kadeem knows the power of conventional word meaning. Every day, he and all the other students in the class use thousands of words that each have a conventional meaning, meanings that he would not think of expanding in a nonstandard direction. We are sure that he and the other students, like all social beings who depend on conventional communicative practices, have a well-developed sense of the necessity for shared conventions. In our view, the episode does not evidence a streak of linguistic irresponsibility residing within Kadeem.

Rather, we think the conflict itself and its significance for mathematics teaching resides in two important issues. One is the inherent complexity of contextually situated semiotic behavior (communication using both language and symbols). The other is the sometimes conflicting goals we hold for students concerning the use of language and symbols in mathematical behavior in the classroom. In this episode, several such goals come explicitly into conflict. Any curriculum that takes seriously the injunction to teach mathematics as communication will encounter such conflicts. Thus our study, if it can be shown to have
any generality, is worth examining as a story about children learning to use language and symbols according to the requirements of specific, constraining forms of complex activity.

Goals for Student Use of Symbolic and Linguistic Expressions
There are many ways in which linguistic and symbolic elements are a resource for thinking and learning in mathematical domains. Perhaps most obviously, everyday language functions as a crucial representational resource for mathematical thinking in that it allows us to represent problem situations in a variety of ways, prompting us to think about different approaches to problem exploration and solution. But we limit our focus here to only one subarea of language use in the mathematics classroom: the relationship between everyday use of linguistic expressions and symbols, and technical use of these elements within the mathematical register. In this regard, there are several different sorts of requirements or practices in which students must become adept.

Mathematical Communication as Use of Traditional Conventions. First, we want students to be able to use the terms of the mathematics register in ways that honor their traditional meaning. In other words, we want students to realize that terms like relative frequency or variable or oblique angle or the symbol \( \pi \) have a precise meaning that is shared in the mathematics community. We want them to realize that when they use these terms, if they use them in ways that are consistent with the meanings the community has given them, they will be understood—that there is a payoff to adhering to conventional norms when using linguistic expressions and symbols.

In this kind of communication, history is present in the use of language and symbols. There is an authoritative source in the background of such communication, a source in terms of which the meaning of expressions must be calculated. The student is in the role of novice and must first discover and then adhere to the meanings and usage conditions given by the authoritative community of experts. The values inherent in this scenario are reinforced throughout the process of schooling—authoritative texts are the default basis for interpretation and action in many school subjects and across most educational settings, as has been well documented.

Mathematical Communication as Context-Specific and Innovative. The previous scenario contrasts with another one, highly valued within current consensus documents of the mathematics education community (NCTM, 1989; NRC, 1989), in which students are apprentices in the process of creating mathematical meanings. In order to do this, they must fully appreciate the fact that mathematical symbols and terms are the creation of human communicators within particular contexts. In this scenario, their job as learners is twofold. They must become adept at figuring out what another communicator means by a particular expres-
sion. Is the intended meaning a completely conventional one, or a meaning specified for the time being, an innovation intended only for use in that particular context? At the same time students must become adept at expressing the meanings they want to convey to others in ways that will be clear and precise and that will fit the needs of the context. (This task is parallel to requirements of human language use in everyday settings. See, e.g., Clark, 1992).

This appreciation for mathematics as flexible communication generally does not evolve very far within a standard curriculum. An anecdote, no doubt familiar in character to many readers, will illustrate what we mean. One of us recently tutored a high school student who was about to take the college entrance exam, the SATs. One question in the practice test booklet went roughly as follows:

The operation * is defined for any numbers x and y as \( x * y = (x + y) \). Which is greater, \( 3 * 4 \) or \( 4 * 3 \)?

Although she had passed high school algebra, this question made absolutely no sense to her at all. After stating that she didn’t know what they were talking about, she asked “why is that asterisk there?” She did not recognize the first sentence as a definition of the meaning of the asterisk symbol. She read the words, but she did not recognize the problem statement as a message from another communicator about how a symbol was to be interpreted within their (admittedly somewhat distant) interaction.

In our view, the background presuppositions involved in making sense of problems like this one include the following: (a) the understanding that a speaker, any speaker, can decide on a meaning for a symbol, any symbol; (b) the understanding that the meaning can and must be expressed clearly so that others can use the symbol to communicate; (c) the understanding that further uses of that symbol must be consistent with that meaning—for so long as the symbol is taken to stand for that meaning; and (d) the understanding that there is an essential arbitrariness to the relation between the symbol and the meaning. In other words, one should not expect to find hints about the meaning of the symbol by looking at the actual symbol itself. One must rely on the stipulations of one’s fellow communicator. All of these understandings are entailed in the SAT practice problem. If you want to play that game, you must already know and accept at least these four preconditions.

To some extent, what is true of symbols is also true of linguistic expressions that function as technical terms. As Pimm (1987) observed, there are many elements of everyday language that have been adopted into the mathematics register. Yet the mappings between everyday meanings of the terms and their technical meanings are not always straightforward. Learners inevitably try at first to use everyday language understandings to discern the meanings of new terms they encounter in the specialized registers of math and science, but often this strategy backfires. (Pimm provides an interesting discussion of learners’ attempts
to capture the mathematical meaning of terms like diagonal and limit in this way.) As is true with symbols, the relationship between technical terms in the mathematical register and their everyday counterparts is unpredictable: although a technical term may preserve some aspect of the meaning of its everyday counterpart, a student can never be sure which aspects will be preserved. Thus the relationship is in many cases essentially arbitrary, for all practical purposes.

Partly for this reason, many students have long since given up on trying to figure out what a new technical term means. They've been wrong too many times, trying to apply the meaning of an expression in everyday language to a situation in which the same expression has a technical meaning. Other students, much fewer in number, seem to delight in acquiring new technical terms by carefully sorting out the similarities and differences between the technical and everyday meanings. In the Algebra Project curriculum, by demystifying the process of encountering or creating a new term or symbol, the teacher attempts to foster in students an active stance toward making sense out of the language and symbols they encounter.

Thus any real sense of mathematics as communication entails an understanding of the conditions of choosing and specifying one's communicative tools within a particular context. It also entails a valuing of oneself and others as communicators, not simply as workers blindly following rules and conventions. How do we imbue students with these values? Most school settings do not support students coming to view themselves and their peers as communicators in the field. Thus it is often difficult to get students to actively use language and symbols in ways that will provide them with a first-person point of view on mathematical communication. Our use of activities in which students create their own units of measure, and name and symbolize them, is part of an effort to do just this.

**Elliot's Interpretation**

Let us return to the episode related earlier, and view it again in terms of these perspectives on language use. Elliot appears to be responding to Kadeem's choice of label as though it were the linguistic equivalent of stealing, or at least trespassing. Elliot had no objection to Kadeem's actual choice of his hand as a nonstandard measure, but he did object to the name Kadeem chose for it. In Elliot's view, Kadeem did not have the right to appropriate a word for which there was already a designated meaning—a meaning that conflicted in a fundamental way with the actual measurement procedure it was now proposed to denote. In other words, the most salient directional feature of the traditional meaning—across the hand—was literally perpendicular to the most salient directional feature of Kadeem's proposed meaning—along the length of the hand. In proposing this new meaning, Kadeem was squarely coming into conflict with the authoritative source on word meanings. For Elliot, this event fell into the bailiwick of the historically prior linguistic authorities.
We note here that the compound term *hand span* in fact does not occur in the Oxford English Dictionary (OED) nor the American Heritage Dictionary (AHD; 1981), nor is it mentioned in any of the many discussions of anatomically derived measures in Klein’s history of human metrological enterprises (Klein, 1974). We do find “hand,” an archaic measurement term differing from Elliot’s description of hand span, which garners only four sentences in Klein’s (1974) 700-page tome:

Another “manual” unit survives today in restricted use: the *hand*, sometimes in the past called the palm, but different in magnitude from the old Roman palmus (2.9 inches). The hand was the transverse length (breadth) of the four extended fingers. By a statute of the reign of Henry VIII, the hand was defined as a length of 4 inches. It is used today in measuring the height of a horse from ground to top of shoulder. The horse-height hand of 4 inches seems substantially to exceed the width of the four fingers of most human hands—even of most palms measured across the knuckles. (p. 57)

However, the word “span” is found in the dictionaries, and here is where we can see the origins of Elliot’s position. The sense of “span” as a measure unit goes back to Old English, and the OED mentions that it is “the distance from the tip of the thumb to the tip of the little finger, or sometimes to the tip of the forefinger, when the hand is fully extended; the space equivalent to this taken as a measure of length, averaging nine inches.” This meaning is listed as archaic in the current American Heritage Dictionary. However, the hand remains as part of the conceptual schema for two senses of the modern transitive verb “span,” which is given as follows: “1. To measure by, or as if by, the fully extended hand. 2. To encircle with the hand or hands, in or as if in measuring.” In everyday usage, the most common sense of the verb is probably the neutral “to extend across” as in “to span the globe.” Its most common use as a noun still carries a trace of its measurement past, although its meaning as an actual measure unit has been bleached out: “The extent or measure of space between two points or extremities, as of a bridge or roof; breadth” (AHD, 1981, p. 1237).

Whether or not there is still a conventional measure unit authoritatively called “the hand span,” Elliot’s position can be stated as follows: There is enough linguistic history behind the terms *hand* and *span* that Kadeem’s usage of these is not allowable. Most important, that linguistic history is relevant to our communication at this moment. It is liable to interfere with effective communication right now.

The concern with effective communication is in fact a value that is explicitly fostered in Lynne’s classroom. We can see it in the group discussions of KND’s inconsistent use of the “Noel” symbol and incomplete rendering of the “compared to” symbol. This issue of audience accessibility is a topic that frequently recurs when the class designs its own symbols. Symbols do not have to be
pictographs—semilateral depictions of what they stand for—still, iconicity in symbols (transparent similarity between what the symbol stands for and the symbol itself) is viewed as sometimes useful for communicative or mnemonic purposes. Lynne often poses a question such as “How likely is it that a person outside our class could understand your symbol?” Students often discuss who will have access to a symbol, who must have access, what audiences are irrelevant for design purposes, how they remember what a symbol means, and so on.

Thus Elliot’s point could fairly be seen as posing a linguistic version of the iconicity question: How arbitrary or how motivated does a linguistic sign have to be in order to ensure that it is communicatively adequate? Elliot would maintain, we can assume, that a new use of a linguistic expression or graphic symbol that already exists is acceptable only if it extends the meaning in a predictable way. Thus use of the > sign to mean “taller than” is a modest extension, interpretable in context. However, because the traditional meaning of (some version of) the expression “hand span” encodes the width of the hand, a new definition that uses the length of the hand will be confusing and is hence out of bounds. It is the linguistic equivalent of putting a picture of a woman on the door of the men’s room.

Kadeem’s View

In Elliot’s view, this situation is governed by linguistic norms that presuppose historically based disciplinary convention as the ultimate authority. We might guess that in Kadeem’s view, on the other hand, and that of the KND team, the assigned activity was one in which students were supposed to take on a communicative authority they did not usually have. In this view, the most central aspect of the situation is that they were involved in creating symbols and meaningful expressions. They even got to create some that were partially arbitrary, thus requiring the audience to work a bit to understand their choices and conventions.

In the episode already reported, we can see that KND’s choices of symbols were fairly transparent: Once we know what they mean, it is easy to remember the intended interpretation for the comparison operator, the objects of comparison, and the hand span unit symbols. The iconicity that characterizes these symbols is, however, balanced by arbitrariness. Part of what a good mathematical communicator knows is that a symbol has no obligation to be iconic. It may be purely arbitrary with no hint of its motivation. It is this balance between iconicity and arbitrariness that Kadeem achieves as he steadfastly hangs onto his six-fingered hand symbol: The symbol denotes a unit of measurement. The unit of measurement is based on Kadeem’s hand. Yet the symbol is allowed to have six fingers, whereas Kadeem’s actual hand has only five. In this way, he asserts that his symbol, like other symbols, may depart from the concrete reality on which it is based. Kadeem ventures even further into this semiotically powerful role when he insists that his use of the term hand span be allowed to hold within the realm of his team’s work.
Whether or not Kadeem knew the original meaning of hand span before being informed of it by Elliot and others, he clearly did not feel compelled to submit to the authoritative source: its traditional meaning. Is this just contrariness, or perhaps a product of some underlying personal conflict with Elliot? Lynne's experience in the classroom would suggest that the answer is no. In our view, however, the personal side of this is not as important as the fact that their conflict symbolizes two different sets of values and conventions concerning language use and mathematics learning. As we stated earlier, both of these sets of values and practices have an important role to play in students' apprenticeship into the higher levels of mathematics and science.

Perhaps more important, we see the conflict as arising out of two different evaluations of the ongoing classroom activity. Recall that Elliot's group tacitly rebelled against the assignment in the first place: The A Team did not use a nonstandard unit of measure, but instead wrote their formal observation statement using inches as their unit of measure. Although not openly rebellious, their action indicated that in some sense they rejected the very premises of the assignment. Kadeem's team, on the other hand, embraced the opportunity to find, personalize, and symbolize a unit of measure intimately connected to their own idiosyncratic situation. Arbitrariness being the right of the symbol maker, they took it on fully.

**IMPLICATIONS AND CONCLUSIONS**

We have suggested that during these episodes of presenting and defending invented measures, symbols, and labels, students may come to be aware of several important aspects of mathematical communication. These include aspects that are often in productive tension with each other, including the following:

- The communicative and mnemonic value of iconicity;
- The communicator's right to arbitrariness, and the concomitant responsibility of the receiver to actively check the intended significance of the symbol, expression, etc;
- The pervasive importance of historically grounded conventions;
- The need for consistency in use of a term or symbol, across a period of time, with respect to a particular, locally determined context.

In some sense, the conflict we have described arises from Elliot's and Kadeem's differing interpretations of the context: the assigned classroom activity and its meaning in relation to the larger scheme of things.

What are the implications of these two perspectives for Elliot and Kadeem as mathematics learners? First, there is a disturbing implication to each position, a potential danger so to speak. One might worry that Kadeem will not take the standard communicative tradition in mathematics seriously enough. Perhaps his
idiosyncratic attempts will somehow weaken his interest in the standard conventions. In our view, this danger is mitigated by the fact that the very process of becoming a mathematical communicator will tend to exert its own enculturating force: As long as Kadeem is taken seriously as a mathematical communicator, he will want to successfully communicate and thus will adhere to conventions when necessary, just as he does in everyday language.

In Elliot’s case, one might have two worries. First, if Elliot does not fully participate in and experience important facets of the creation of units, symbols, and labels at this point in his mathematical career, due to an enthusiastic alignment with disciplinary tradition, will he be able to summon up the resources to do so when more challenging problems require it? In Elliot’s case, it is likely that this will not be a problem. He is an able student, and does not seem to be objecting to the activity on the basis of any difficulty with using and interpreting symbols. Perhaps more troubling is the second worry: Do Elliot’s various objections signal a difficulty with the larger notion of mathematics as communication? Will Elliot be able to accept other students, people like Kadeem, as legitimate mathematical communicators, and thus, contributors to the tradition? There is no obvious answer. And it is here that the one of the major challenges for the classroom teacher can be seen.

The Algebra Project transition curriculum reflects the expectation that such classroom interactions about symbols, language, and meaning will create in students a deeper ability to engage with mathematical and scientific language, to see it as meaningful in the context of authentic investigations. Just as the mappings from experience to symbols may be more or less arbitrary, or more or less iconic, so the mappings from everyday language to technical language may be more or less arbitrary, more or less transparent. To become flexible and fearless users of symbols and mathematical language—learners unafraid of formalization—students must develop ways of grappling with just this unpredictability between different arenas of language use. Through these experiences it is hoped that students will come to see the writers of textbooks and tests as communicators, too—communicators that use language and symbols in ways that are sometimes conventional and sometimes innovative. Thus texts that have heretofore been opaque or mysterious may be approached by students as actual attempts to communicate, attempts that must be interpreted through active and reciprocal attempts.

No less important, however, is our hope that students will come to see each other as legitimate communicators. This goal will be no less difficult to accomplish. Students differ in what they bring to this task, both in terms of their readiness to participate in the exploration of linguistic mappings, and in their tacit beliefs about the nature of the activity at hand, and its relationship to “real” mathematics. The challenge for teachers thus includes not just the obligation to provide experiences in which students can become practiced at mathematics as
communication, but also the obligation to help students fully take part in the
variety of traditions implied by that perspective.

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