Swelling of Elastic Materials
Fluids Deforming Solids

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4U Summer School on Complex Motion in Fluids – Denmark (2015)
Fluids & Elasticity

**Flow** through porous medium
- Darcy’s Law

**Elastic** deformation of medium
- Biot’s Poroelasticity

**Swelling**
- Polymers

**Fluids Deforming Solids**
- Surface Tension – Elastocapillarity
- Swelling & Growth
- Maxwell Stresses
Swelling a Sponge
Pine Cones

Tree-bound pine cones:
Hydrated & closed, protecting seeds

Fallen pine cones:
Dried out & opened, releasing seeds

Articular Cartilage

Shape change caused by ion concentration.

Residual strain at physiological conditions: **3-15%**

Tensile prestress in cartilage protective against frequent compresses forces.

Lichens in the Rain

Swelling & Growth

Materials Science

Swelling of a sponge.

Mechanics

An almond leaf which was attacked by *Taphrina Deformans*.

Flow in Porous Media

Navier-Stokes Equations (momentum conservation)

Inertial acceleration

\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \mu \nabla^2 u + f \]

Forces

- pressure
- diffusion
- body forces

Continuity Equation (mass conservation)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \]
Fluid Behavior

Inertial Forces

Trailing airplane vortices

Viscous Forces

Coiling honey

Reynolds Number: inertial/viscous
**Reynolds Number**: inertial/viscous

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}
\]

**Inertial acceleration**

**Forces**

**Steady inertial forcing** due to the convective derivative:
- Time dependence arises from \( U_0 \)

**Linear unsteady term** sets the inertial time scale to establish steady flows:

Time scale estimated by balancing unsteady inertial force density with viscous force density

\[
f_u \sim \frac{\rho U_0}{\tau_i} \quad f_v \sim \frac{\mu U_0}{L_0^2}
\]

Time required for a vorticity to diffuse a distance \( L_0 \), with a diffusivity \( \nu = \frac{\mu}{\rho}, \tau_i \approx 10 \text{ms} \)


Airplane: [http://eis.bris.ac.uk/~glhmm/gfd/Airplane-ChrisWillcox.jpg](http://eis.bris.ac.uk/~glhmm/gfd/Airplane-ChrisWillcox.jpg)

**Fluid Behavior**

**Reynolds Number:** inertial/viscous

Fluid element accelerating around curve.
- During a turn time:
  \[ \tau_0 \sim \frac{w}{U_0} \]
- Loss of momentum density:
  \[ \rho U_0 \]
- By exerting an inertial centrifugal force density:
  \[ f_i \sim \frac{\rho U_0}{\tau_0} = \frac{\rho U_0^2}{w} \]

Fluid element in a channel of contracting length.
- By mass conservation, velocity increases as:
  \[ u \sim U_0 (1 + \frac{z}{l}) \]
- Gain momentum at a rate:
  \[ f_i \sim \rho \frac{du}{dt} = \rho U_0 \frac{du}{dz} \sim \frac{\rho U_0^2}{l} \]
**Fluid Behavior**

### Reynolds Number: inertial/viscous

**Inertial Forces**

\[ f_i \sim \rho U_0 \tau_0 = \rho U_0^2 / w \]

**Viscous Forces**

- Viscous force densities result from gradients in viscous stress:

\[ p_x = 0 \quad p_z = \rho U_0 \]

\[ f_i \sim \rho \frac{d}{dt} U_0 \frac{dU_0}{dz} \sim \frac{\rho U_0^2}{l} \]

\[ p_i = \rho U_0 \]

\[ f_i \sim \rho U_0 L_0^2 / \mu \]

**Ratio of these two force densities is the Reynolds number:**

\[ \frac{f_i}{f_v} = \frac{\rho U_0 L_0}{\mu} \equiv R \]

---


Airplane: http://eis.bris.ac.uk/~gihmm/gfd/Airplane-ChrisWillcox.jpg

Honey: http://www.honeyassociation.com/webimages/honey-dipper.jpg
**Fluid Behavior**

**Reynolds Number:** inertial/viscous  \[ \frac{f_i}{f_v} = \frac{\rho U_0 L_0}{\mu} \equiv \mathcal{R} \]

Estimation of Reynolds numbers for common microfluidic devices.

- Typical fluid – water
  - Viscosity: 1.025 cP @ 25°C
  - Density: 1 g/mL
- Typical channel dimensions
  - Radius/height (smaller than width): 1 – 100 µm
- Typical velocities
  - Average velocity: 1 µm/s – 1 cm/s

**Typical Reynolds number:**

\[ \mathcal{R} \sim \mathcal{O}(10^{-6}) - \mathcal{O}(10^{1}) \]

Low Reynolds number: viscous forces > inertial forces

- Flows are **linear**.
- Nonlinear terms in Navier-Stokes disappear
  - Linear, predictable **Stokes flow**

---


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Fluid Behavior

**Reynolds Number:** inertial/viscous
\[
\frac{f_i}{f_v} = \frac{\rho U_0 L_0}{\mu} \equiv \mathcal{R}
\]

**Typical Reynolds number:**
\[
\mathcal{R} = \frac{\rho U_0 d_{30}}{\mu} \leq 1
\]

Low Reynolds number: viscous forces > inertial forces
- Flows are **linear**.
- Nonlinear terms in Navier-Stokes disappear
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Stokes Equations (momentum & mass conservation)

\[ 0 = \nabla \cdot \mathbf{u}_\beta \]

\[ 0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{u}_\beta \]

...but, in order to proceed, we need to prescribe BC’s on each grain...

**Stokes Equations** (momentum & mass conservation)

\[
0 = \nabla \cdot \mathbf{u}_\beta \\
0 = -\nabla p_\beta + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \mathbf{u}_\beta
\]

...but, in order to proceed, we need to prescribe BC’s on each grain...

**Darcy-Brinkman-Stokes Equations**

Averaging over pressures and velocities.

\[
0 = -\nabla \langle p_\beta \rangle + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{u}_\beta \rangle - \mu_\beta k_{ij}^{-1} \cdot \langle \mathbf{u}_\beta \rangle
\]

Void space: \( \langle \phi_\beta \rangle = \frac{1}{V} \int_{V_\beta} dV \)  

\[ \text{Viscous friction} \]

Viscous Drag

Darcy-Brinkman-Stokes Equations

Averaging over pressures and velocities.

\[ 0 = -\nabla \langle p_\beta \rangle + \rho_\beta \mathbf{g} + \mu_\beta \nabla^2 \langle \mathbf{u}_\beta \rangle - \mu_\beta k^{-1}_{ij} \cdot \langle \mathbf{u}_\beta \rangle \]

Often negligible

Viscous friction

Darcy’s law (volumetric flux, isotropic medium)

\[ \mathbf{q} = -\frac{k}{\mu} (\nabla \langle p_\beta \rangle - \rho_\beta \mathbf{g}) \]
Figure 1.1: Water-level fluctuations due to a passing train. An approaching train compresses the aquifer, which quickly raises the pore pressure in the affected region. Fluid then flows away from the high-pressure region. As the train departs, the aquifer expands, thereby quickly reducing the pore pressure in the affected region. Fluid again flows in response to the pressure differences, but this time it builds up the pore pressure. The approximately equal and opposite behaviors demonstrate that the aquifer is elastic (Domenico and Schwartz, 1998, p. 65; Jacob, 1940).
Coupled problem:

Pore pressure has time dependence, as does poroelastic stresses/strains.

Poroelasticity:

Cannot solve fluid flow problem independent of stress field.

- Stress changes in fluid-saturated porous media typically produce significant changes in pore pressure.

Increment in total work associated with strain increment and fluid content.

\[ dW = \sigma_{ij} \, d\varepsilon_{ij} + p \, d\zeta \]
**Biot Poroelasticity**

**Time dependence** work is related to the fluid flux through Darcy’s law.

\[
\alpha \frac{\partial \varepsilon}{\partial t} + S \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla^2 p
\]

Squeeze the soil – how much water comes out? Pressurize the water – how much water will go in the soil?

Compression of the medium (e.g. soil) includes compression of pore fluid and particles plus the fluid expelled from an element by flow.

**Resistance** of medium defined by bulk and shear moduli.

\[
\left( K + \frac{4}{3} G \right) \nabla^2 \varepsilon = \alpha \nabla^2 p
\]

Stress caused by (1.) hydrostatic pressure of water filling pores, and (2.) average stress in porous network. **Stresses** in the soil **carried** in part by the fluid and in part by solid.
Biot Poroelasticity

Swelling Spheres

Polymers & Swelling

Polymer Chains

Good Solvent

Small Molecules

Catalyst, Energy

Bad Solvent

Swelling
Free Energy

Gibbs Free Energy of Dilution

\[ \Delta G = \Delta H - T \Delta S \]

Heat \quad Entropy

Equilibrium Swelling \quad \Delta G = 0

At constant pressure

Determination via Osmotic Pressure

Excess pressure required to keep mixed phase in equilibrium with the pure liquid.

\[ \Pi = -\frac{RT}{V} \ln \left( \frac{p}{p_0} \right) \]

\( p \): Vapor pressure of liquid in equilibrium with mixture.

\( p_0 \): Saturation pressure
Free Energy

Gibbs Free Energy of Dilution

Enthalpy ~ Internal Energy  Entropy of dilution (Boltzmann)

Flory-Huggins Equation
Free, long polymer chains

$$\Delta G = RT \left[ \ln (1 - \nu_2) + \nu_2 + \chi \nu_2^2 \right]$$

Flory-Huggins Chi parameter: dimensionless, polymer/fluid interactions.

Good solvents: \( \chi \sim 0.1 - 0.5 \)
Flory-Rehner Equation

Crosslinked polymer networks

- The \textbf{entropy} change caused by \textbf{mixing} of polymer and solvent.
- The \textbf{entropy} change caused by \textbf{reduction} in numbers of possible chain \textbf{conformations} on swelling.
- The \textbf{heat of mixing} of polymer and solvent, which may be positive, negative, or zero.

Equilibrium swelling of a crosslinked network:

\[
\ln \left(1 - v_2\right) + v_2 + \chi v_2^2 + \frac{\rho V_s}{M_c} v_2^{1/3} = 0
\]

Approximate equilibrium stretch:

\[
\lambda_{eq} \approx \left(\frac{RT}{V_s} \frac{1/2 - \chi}{G}\right)^{1/5}
\]
Table 1. Solubility Parameters, Swelling Ratios, and Dipole Moments of Various Solvents Used in Organic Synthesis

<table>
<thead>
<tr>
<th>solvent</th>
<th>$\delta^a$</th>
<th>$S^b$</th>
<th>$\mu$ (D)</th>
<th>ref$^c$</th>
<th>rank$^d$</th>
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<tbody>
<tr>
<td>perfluorobutylamine</td>
<td>2.6</td>
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<td>0.0</td>
<td>10</td>
<td>32</td>
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<td>10</td>
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<td>10</td>
<td>18</td>
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<tr>
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<td>1.35</td>
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<td>1.35</td>
<td>0.0</td>
<td>10</td>
<td>38</td>
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</table>

$^a\delta$ in units of cal$^{1/2}$ cm$^{-3/2}$. $^bS$ denotes the swelling ratio that was measured experimentally; $S = D/D_0$, where $D$ is the length of PDMS in the solvent and $D_0$ is the length of the dry PDMS. $^c$References refer to literature values of $\delta$ and $\mu$. $^d$Rank refers to the order of the solvent in decreasing swelling ability (see Figure 1).
Swelling a Disk

\( \lambda \) is the initial swelling ratio.

Swelling a Disk

Swelling Dynamics

Linearized, similar to poroelasticity

\[
\left( K + \frac{4}{3} G \right) \nabla \nabla \cdot u + G \nabla^2 u = \nabla^2 p
\]

Incompressibility & Darcy’s law

\[
\nabla \cdot u = k \nabla^2 p
\]

Volume change (e.g. sphere)

\[
\alpha(x, t) = \nabla \cdot u
\]

Satisfied by diffusion relation:

\[
\frac{\partial \alpha}{\partial t} = D \nabla^2 \alpha \quad D \equiv \left( K + \frac{4}{3} G \right) k
\]

Swelling of Elastic Materials
Fluids Deforming Solids

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Geometric Non-linearities:

- **Buckling**
- **Wrinkling**
- **Folding**
- **Creasing**
- **Snapping**

How do objects change shape?
How do you “grow” a structure into a desired shape?
Thin Structures

Bending vs. Stretching

\[ U_m \sim E h \varepsilon_{\alpha\beta}^2 \]
Energy in Compression $\sim$ thickness

\[ U_b \sim E h^3 \kappa^2 \]
Energy in Bending $\sim$ thickness$^3$

Thin structures deform by **bending** & avoid **stretching**

A still photo is a kind of lie

PETER DOIG
Swelling Dynamics

Diffusive-like Dynamics

Beam Bending

Dynamics of bending: Diffusion of temperature

• Thermal diffusion through the beam thickness.

• Shape obtained by minimizing the bending moment in the beam.

• Beam curvature as temperature diffuses.

Beam curvature as solvent diffuses:
\[
\frac{\kappa_1 h}{\varepsilon_m (1 + \nu)} = 1.33e^{-\frac{\pi^2 t / \tau}{4}} - 0.77e^{-\frac{9\pi^2 t / \tau}{4}} + \ldots
\]

Poroelastic time scale:
\[
\tau_p \approx \frac{\mu h^2}{k E}
\]

\(\kappa_1\) = Principal curvature
\(h\) = Thickness
\(\varepsilon_m\) = Maximum strain
\(\nu\) = Poisson’s ratio

\( \delta_s \) = Solubility parameter  
\( \mu \) = Solvent polarity  
\( \varepsilon_{eq} \) = Strain at equilibrium

Nonlinear Swelling

Nonlinear Swelling

What happens when you swell a thicker beam?
Mechanical Instability

Mechanical Instability

**Bending and Buckling**


Bending and Buckling

Bending vs. Swelling

Can the fluid bend the structure?

**Bending**

\[ U_b = \frac{B}{2} \int_0^L \theta'(s)^2 \, ds \sim Eh^3 \]

**Swelling**

\[ U_s = \int_{V_f} \sigma \varepsilon_{eq} \, dV_f \sim E \varepsilon_{eq}^2 V_f \]

Length scale:

\[ l_{es} \sim (\varepsilon_{eq}^2 V_f)^{1/3} \]

---

Thin structures bend...

Thick structures stay flat, while their surface creases...

“Elastoswelling” length \( \ell_{es} \sim (\varepsilon_{eq}^2 V_f)^{1/3} \)
Deformation Transition


\[ \ell_{es} \sim (\varepsilon_{eq}^2 V_f)^{1/3} \]

<table>
<thead>
<tr>
<th>Material</th>
<th>( \delta_s ) (cm(^{1/2}))</th>
<th>( \mu ) (D)</th>
<th>( \varepsilon_{eq} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PDMS</td>
<td>7.3</td>
<td>0.6-0.9</td>
<td>–</td>
</tr>
<tr>
<td>Diisopropylamine</td>
<td>7.3</td>
<td>1.2</td>
<td>1.13</td>
</tr>
<tr>
<td>Triethylamine</td>
<td>7.5</td>
<td>0.7</td>
<td>0.58</td>
</tr>
<tr>
<td>Hexanes</td>
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<td>Toluene</td>
<td>8.9</td>
<td>0.4</td>
<td>0.31</td>
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<tr>
<td>Ethyl acetate</td>
<td>9.0</td>
<td>1.8</td>
<td>0.18</td>
</tr>
</tbody>
</table>
Controlling Shape

Microfluidic Swelling
Controlling Shape

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10 mm
What about wetting?
Fluid Behavior

Viscous Forces

Coiling honey

Interfacial Forces

Wetting of water on a textured surface

Capillary Number: viscous/interfacial

Fluid Behavior

Viscous Forces

Capillary Number: \( \text{viscous/interfacial} \)

Interfacial Forces

Surface tension acts to reduce the interfacial area.
\[ \sigma_c \sim \frac{\gamma}{R} \]

Viscous stresses act to extend and drag the interface downstream.
\[ \sigma_v \sim \frac{\mu U_0}{h} \]

Characteristic droplet size:
\[ R \sim \frac{\gamma}{\mu U_0 h} = \frac{h}{C} \]

Capillary number:
\[ C \equiv \frac{\mu U_0}{\gamma} \]

**Fluid Behavior**

**Viscous Forces**

**Capillary Number:** viscous/interfacial

Large **surface-to-volume** ratios in microfluidic devices
- Makes surface effects increasingly important.
- Important when free fluid surfaces are present.

**Interfacial Forces**

Surface tensions can exert significant stress
- Result in free surface deformations.
- Can drive fluid motion.

**Capillary forces tend to draw fluid into wetting microchannels**
- Occurs when **solid-liquid** interfacial **energy** is **lower** than the **solid-gas** interfacial **energy**.

---

Capillary Rise

Classical Problem:

- Noted as early as 15th by Leonardo da Vinci.
- Attributed to circulation in plants in 17th century by Montanari.

\[
U = 2\pi \gamma \cos \theta_e r z_m + \frac{1}{2} \rho g \pi r^2 z_m^2
\]

\[
\frac{dU}{dz_m} = 0
\]

\[
\ell_{cg} \sim \frac{\gamma \cos \theta_e}{\rho g d_0}
\]

Balance: Surface Tension & Gravity

Elastocapillarity

Fluid-structure interaction:

- Droplet bends and folds the sheet.
- Droplet is minimizing the amount of its surface in contact with air.
- Liquid-air surface area is minimized at the expense of bending the sheet.

Fluid-structure interaction:

Elastic energy of a plate – bending:
\[ U_e = \frac{1}{2} \iint_P dx\, dy \, \int_{-h/2}^{h/2} \, dz \left( \sigma_{\alpha\beta} \varepsilon_{\alpha\beta} \right) \]

Relation between in-plane strain to out-of-plane bending:
\[ \varepsilon_{\alpha\beta}(x) = z \frac{d^2 w}{dx^2} = \frac{z}{R} \]

Bending energy:
\[ U_b = \frac{1}{2} \iint_P dx\, dy \, \frac{Eh^3}{12} \left( \frac{1}{R} \right)^2 \]

\[ U_b \sim Eh^3 \]

Elastocapillarity

Fluid-structure interaction:

Bending energy:

\[ U_b \sim Eh^3 \]

Surface energy:

\[ U_\gamma \sim \gamma L^2 \]

Elastocapillary length:

\[ \ell_{ec} \sim \sqrt{\frac{Eh^3}{\gamma}} \sim \sqrt{\frac{B}{\gamma}} \]

Elastocapillary bending of sheet:

\[ 7\ell_{ec} \leq L \leq 12\ell_{ec} \]

Elastocapillarity

Capillary rise between flexible fibers.

\[ B \frac{\partial^4 d}{\partial x^4} = \gamma \kappa_f \]

Beam bending

Laplace pressure

Curvature of meniscus

\[ \kappa_f = \frac{1}{d(z_m)} \cos \theta_e + \frac{2}{b} \cos \theta_e \]

Assume gap is much smaller than width: \( d_0 \ll b \)

Initial gap: \( d(z_m, t) = d_0 \)

Approximation of meniscus curvature:

\[ \kappa_f \approx \frac{1}{d_0} \cos \theta_e \]

References:

Capillary rise between flexible fibers.

\[ B \frac{\partial^4 d}{\partial x^4} = \gamma \kappa_f \]

Beam bending

Laplace pressure

Approximation: \( \kappa_f \approx \frac{1}{d_0} \cos \theta_e \)

Scaling: \( B \frac{d_0}{\ell^4} \sim \frac{\gamma}{d_0} \cos \theta_e \)

\[ \ell_{ec} \equiv \left( \frac{B d_0^2}{\gamma \cos \theta_e} \right)^{1/4} \]

Balance: Bending & Surface Tension
Capillary rise between flexible fibers.

**Potential energies:**

Elastic energy

$$U_e \sim \frac{Bwh_0^2}{\ell^3}$$

Gravitational potential energy

$$U_g \sim \rho gwh_0\ell^2$$

Surface energy

$$U_c \sim w\gamma$$

---


Capillary rise between flexible fibers.

**Characteristic length scales:**

**Elastocapillary length**

\[ \ell_{ec} = \frac{U_e}{U_c} = \left( \frac{Bh_0^2}{\gamma} \right)^{1/4} \]

**Capillary gravity length**

\[ \ell_{cg} = \frac{U_c}{U_g} = \frac{\gamma}{\rho gh_0} \]

**Elastogravity length**

\[ \ell_{eg} = \frac{U_e}{U_g} = \left( \frac{Bh_0}{\rho g} \right)^{1/5} \]

Capillary rise between flexible fibers.

**Dimensionless parameters**

Bond number:

$$B = \frac{l}{l_{cg}}$$

Elastocapillary number

$$E = \left(\frac{l}{l_{ec}}\right)^4$$

Elastogravity number

$$G = BE = \left(\frac{l}{l_{eg}}\right)^5$$


Elastocapillarity

Capillarity & Swelling

Solid: Polyvinylsiloxane
Fluid: Silicone Oil (5 cSt)

20x faster than real time

\[ E \approx 1 \text{ MPa (PVS)} \]
\[ L = 20 \text{ mm} \]
\[ d \approx 2 \text{ mm} \]
\[ h \approx 0.5 \text{ mm} \]

Capillarity & Swelling

![Diagram showing capillarity and swelling with images and a graph illustrating the relationship between height and time.]

- **a.** Images of a slender structure in different stages of swelling.
- **b.** Graph with the following key points and annotations:
  - $L - \sqrt{\frac{d_0}{\ell_{ec}}}$
  - $z_j$
  - $\sim \sqrt{t}$
  - Meniscus
  - Curvature

**Equation:**

$$L - \sqrt{\frac{d_0}{\ell_{ec}}}$$

**Graph Notes:**
- The graph shows the growth of the meniscus and curvature over time.
- The y-axis represents the height ($z_m$) in millimeters, ranging from $10^{-1}$ to $10^1$.
- The x-axis represents time ($t$) in seconds, ranging from $10^{-2}$ to $10^3$.
- The graph includes data points for both the meniscus and curvature, with the meniscus marked by circles and the curvature marked by triangles.

**Legend:**
- Meniscus
- Curvature

**Inference:**
- The swelling process exhibits a relationship that can be described by the equation $L - \sqrt{\frac{d_0}{\ell_{ec}}}$.
- The height of the meniscus and curvature grow in proportion to the square root of time, as indicated by the annotation $\sim \sqrt{t}$.

**Analysis:**
- The images and graph together illustrate the physical phenomena of capillarity and swelling, showing how slender structures behave under swelling conditions.

**Conclusion:**
- The study likely explores the mechanics of slender structures, focusing on how they expand or swell, potentially under the influence of fluid pressure or other external forces.
1. Elastocapillary rise between flexible fibers.

At short times, elastocapillary rise dominates the deformation.

2. Swelling-induced bending.

Bending is constrained by surface tension, as the beam bends with a lower curvature than a free swelling beam.

3. Bending dominates surface tension.

Separation occurs as the “natural” curvature of the beam exceeds the fluids ability to confine it.


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   Separation occurs as the “natural” curvature of the beam exceeds the fluid’s ability to confine it.

Swelling & Peeling

Baobab Flowering
What about geometry?
Gauss Curvature

\[ \kappa_1 = R_1^{-1} \]
\[ \kappa_2 = R_2^{-1} \]

\[ K = \kappa_1 \kappa_2 = \diamond^4[w, w,] = 0 \]
\[ K = \kappa_1 \kappa_2 = \diamond^4[w, w,] > 0 \]
\[ K = \kappa_1 \kappa_2 = \diamond^4[w, w,] < 0 \]

Developable

\[ K = 0 \]
Dancing Disks

Axisymmetric Disks

Diffusive dynamics:

\[ \psi(\bar{x}_3, \tau) = 1 + \sum_{n=0}^{\infty} A(n) \sin[\lambda(n)\bar{x}_3] e^{-\lambda(n)^2 \tau} \]

Bending dynamics:

\[ \bar{\kappa}_\varepsilon(\tau) = \frac{d}{S} \int_{-1/2}^{1/2} \psi(\bar{x}_3, \tau) \bar{x}_3 \, d\bar{x}_3 \]

\[ d/S = 12\varepsilon_{eq}L^2/h^2 \]

Dynamics: Twisting

\[ a/b = 1 \]

\[ a/b = 1.4 \]
How do thin structures grow?

Permanent shape change and mass increase.

- Radial growth
- Through-thickness growth
Shaping elastic sheets by prescribing non-Euclidean metrics

- Prepare gels that undergo nonuniform shrinkage.
- Buckling thin films based on chosen metrics.

**Goal:** Use swelling to predictably & permanently morph plates into shells

---

**Stretching Dominated**
Will bend as much as possible while minimizing stretching.

**Stretching Energy of the Plate** (incompressible)
\[ \mathcal{U}_s \approx h \int_A E \left[ \text{tr}^2 \left( \alpha - \bar{\alpha} \right) + \text{tr} \left( \alpha - \bar{\alpha} \right)^2 \right] \sqrt{\alpha} \, dA \]

**Stretching Energy** (Assume all strains zero, except: \( a_{\theta\theta} - \bar{a}_{\theta\theta} \))
\[ \mathcal{U}_s \approx Eh \int_0^R \frac{(a_{\theta\theta} - r^2)^2}{r^3} \, dr + Eh \int_{R_e/\alpha}^{R} \frac{(a_{\theta\theta} - \alpha^2 r^2)^2}{\alpha^2 r^3} \, dr \]

**First Fundamental Form**
\[ ds^2 = d\rho^2 + a_{\theta\theta}(\rho) \, d\theta^2 \]

**Gaussian curvature**
\[ -\partial_\rho \rho \sqrt{a_{\theta\theta}} / \sqrt{a_{\theta\theta}} \]

**Minimize Stretching Energy** (Constant K metric)
\[ a_{\theta\theta}(\rho) = \left( \sin \left( \sqrt{K} \rho \right) / \sqrt{K} \right)^2 \]

Geometric Composite

**Minimize Stretching Energy** (Constant $K$ metric)

$$a_{\theta\theta}(\rho) = \left(\sin\left(\sqrt{K}\rho\right)/\sqrt{K}\right)^2$$

**Taylor Expand** $a_{\theta\theta}(\rho)$  
(Assume: $|K| < \alpha^2/R_e^2$)

$$a_{\theta\theta}(\rho) = \rho^2 - \frac{K}{3} \rho^4 + \mathcal{O}(\rho^5)$$

*Flat metric*  
*Kind of non-Euclidean metric*

**Experiments:** Mechanical Strain

The elastomer contains free, uncrosslinked polymer chains.
Residual Swelling

Residual Swelling

Stretching Energy

\[ U_s \approx \int_0^R \frac{(a_{\theta\theta} - \alpha^{-2}r^2)^2}{\alpha^{-2}r^3} \, dr + \frac{E_a}{E_d} \int_R^{R_e} \frac{(a_{\theta\theta} - \alpha^2r^2)^2}{\alpha^2r^3} \, dr \]

Modulus difference

Stretching ratio

- Depends on chemical and material properties.
- Should vary with \( R/R_e \)
- Will depend on concentration gradient of free chains.

ansatz

- from conservation of mass & proportional to mass uptake in annulus.

\[ \alpha = 1 + \eta(c_d - c_a) \left( \frac{R}{R_e} \right)^2 \left[ 1 - \left( \frac{R}{R_e} \right)^2 \right] \]

Approximate Analytical Solution

- Taylor expand about \((\alpha - 1)\)
- \(\overline{E} = E_a/E_d\) and \(\overline{R} = R/R_e\)

\[
KR_e^2 \approx 96(1 - \alpha_{\text{max}})\overline{E}\overline{R}^3 \frac{(1 - \overline{R}^2)(1 - \overline{R}^3)}{\overline{R}^6 (1 - \overline{E}) + \overline{E}}
\]

2D analog to Timoshenko’s model for thermal beam bending of bimetallic strips.
Swelling Dynamics

Growing Sheets

Consider a thin structure with a growing top layer.

- Caused by *swelling, growth, heating*, etc.

Elastic energy density depends on material properties & metric tensors

\[
\bar{U} = \int [(1 - \nu)|\mathbf{a} - \mathbf{\bar{a}}|^2 + \nu \text{tr}^2(\mathbf{a} - \mathbf{\bar{a}})\sqrt{\mathbf{a}}] \, dA + \frac{h^2}{3} \int [(1 - \nu)|\mathbf{b} - \mathbf{\bar{b}}|^2 + \nu \text{tr}^2(\mathbf{b} - \mathbf{\bar{b}})\sqrt{\mathbf{a}}] \, dA
\]

\[
\text{Stretching Energy} \quad \text{Bending Energy}
\]

Lateral distances ($\Lambda_0^2$) and curvatures ($\kappa_0$) that make the sheet stress free.

Metric tensor

\[
\mathbf{\bar{a}} = \Lambda_0^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{\bar{b}} = \kappa_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Isometric Limit

Numerical Simulations using COMSOL Multiphysics

- Finite, incompatible tridimensional elasticity with a Neo-Hookean incompressible material.
- Distortions used to simulate prestretch.
- Top layer subjected to distortion field:
  \[ F_o = \lambda (e_1 \otimes e_1 + e_2 \otimes e_2) + e_3 \otimes e_3 \]

Residual Swelling Experiments

- PVS Bilayer (Total thickness ~ 600um)
- Axisymmetric, circular plate.
- Pink (top) source of swelling for green (bot).
- Real time = 75 minutes (Video: 640x RT)

Isometric Limit

Ellipse

Rectangle

Cross

Leaf

Isometric Limit
In the **isometric limit**, the stretching energy is zero.

- i.e. \( a = \bar{a} \)

Curvature tensor (Cartesian)
- Second fundamental form:
  \[
  L du^2 + 2 M du dv + N dv^2
  \]

Minimize bending energy
- Constrain the mid-surface to be flat
- Impose Lagrange multiplier enforcing
  \[
  \Lambda_0^{-4} \left( LN - M^2 \right) = 0
  \]

\[ \text{Gaussian Curvature} \]

Minimization yields:
\[
L + N = \kappa_0 (1 + \nu) \quad \text{and} \quad K = 0
\]
In the limit of **large stretching**, the sheet adopts an **isometry**.

For **small stretching**, the sheet is initially spherical curved.
- Bifurcation from spherical to cylindrical.

Classical problem (limited to circular and elliptical disks)
- Stoney formula relating stress to curvature.
- Strain mismatch work (Hyer, Freund, Seffen, etc.)
Assuming a metric with constant $K$ (Gauss normal coords)

$$\overline{U}_s = \frac{1}{9} \lambda_0^6 K^2 \int_A r^4 \, dA$$

**Stretching Energy**

Assuming metric is axisymmetric.

Assume $K$ is homogenous.

$$a_{rr} = \bar{a}_{rr}, \quad a_{r\theta} = 0$$

$$a_{\theta\theta}(\rho) = \rho^2 - \frac{K}{3} \rho^4 + \mathcal{O}(\rho^5)$$

**Shape factor:**

$$S \equiv \left( \frac{2}{9} \frac{1}{A} \int_A r^4 \, dA \right)^{1/4}$$

**Structural Slenderness:**

$$\gamma = h/S$$

---

Energetic cost to continue bending into a spherical cap:

$$\overline{U}_{bb} = \frac{1}{9} L^4 A_o^{-2} \int r^4 dA + h^2 A A^2 (L - \kappa_o)^2$$

Energy to bend as a cylinder:

$$\overline{U}_{ab} = \frac{1}{4} h^2 A A_o^{-2} \kappa_o^2$$

Energy balance:

$$\frac{1}{2} \overline{L}_b^4 + \gamma^4 \left( \overline{L}_b^2 - 2 \overline{L}_b \overline{\kappa}_{ob} + \frac{3}{4} \overline{\kappa}_{ob}^2 \right) = 0$$

Bifurcation curvature:

$$\overline{\kappa}_{ob} = \frac{2\sqrt{2}}{33/4 \gamma^2}$$
Shell Growth

...with M. Trejo, J. Bico, and B. Roman.
Swelling Structures

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