NONSMOOTH REGULARIZATION IN ELECTROCARDIOGRAPHIC IMAGING

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Key words: electrocardiography, inverse problem, regularization, Tikhonov, total variation, simulation, finite element method

Electrocardiographic imaging is an inverse problem which consists of computing epicardial potential distributions from measurements of body surface potentials. Smoothing and attenuation of the electric field by the intervening body volume conductor render the problem ill posed, requiring regularization to stabilize the inverse solution. This paper reports a study on the efficacy of nonsmooth regularization in recovering steep potential gradients related to epicardial activation wave fronts by comparing total variation and Tikhonov solutions. Our preliminary results suggest that the inversion method applied to the electrocardiographic imaging problem should include the constraint on the total variation of the epicardial potential distribution.

1. INTRODUCTION

Electrocardiography (ECG) is a medical technique used to screen the course of the electrical cardiac cycle. The fluctuations of the electric potentials on the surface of the body, usually measured with a 12-lead system, allow the physician to detect and analyze certain cardiac arrhythmia. Electrocardiographic imaging, also called the inverse problem in ECG, is an emerging functional imaging modality developed as an extension to the standard ECG. Using denser electrode arrays and mathematical modeling, the electrocardiographic imaging aims at reconstructing epicardial potential distributions from measured body surface potentials (BSP). A straightforward way to find the solution to this problem is by minimizing (in the least-squares sense) the residual error between the model predicted BSP and the actual measured data. The main difficulty with this method resides in its ill-posed nature: small perturbations in the data induce wild oscillations in the reconstructed potential distribution. Due to the smoothing property of the intervening body volume conductor, the BSP are low-pass filtered images of the underlying electrical cardiac sources. To stabilize the inverse solution, one must compensate for this loss of information with additional knowledge, mathematically expressed as side constraints (or penalty terms) added to the standard least-squares cost function. Typical penalty terms include the Euclidian norms of either the solution (prevents the potential values to become too large) or of the derivative of the solution (forces the epicardial potential field to be smooth).
approach, known as regularization, provides a family of approximate solutions that depend on a positive parameter which controls the compromise between the stability of the solution and the fidelity to the data. Tikhonov regularization [1] was widely used in the ECG inverse problem [2], [3], [4], [5], [6], and validation of the reconstructed solutions was done either against directly measured epicardial potentials during open chest procedures [7], [8] or using human-shaped tanks filled with electrolytic solution [9]. Excellent reviews of the research conducted in electrocardiographic imaging can be found in [10], [11]. When applying Tikhonov regularization, the usage of $l_2$-based side constraints on either the amplitude or the derivative of the solution acts as a linear inverse filter. Therefore, in the attempt to filter out the high frequency noise, this scheme also cuts off the high frequency content in the solution, producing an undesirable smoothing effect on the steep transitions that are typical to the cardiac depolarization wave fronts. In contrast, the nonlinear total variation (TV) regularization [12] constrains the $l_1$ norm of the potential gradient, yielding inverse, edge-preserving and less blurring solutions. It was previously suggested that a TV-based inversion scheme might be more suitable for electrical imaging of the heart [13], [14]. This numerical study is aimed as assessing the efficacy of nonsmooth TV regularization vs. Tikhonov technique in recovering potential distributions around activation wave fronts.

2. METHODS

2.1. IMAGE-BASED MODEL CONSTRUCTION

A set of 60 serial cross-sectional CT scans were used to reconstruct the human thorax. The CT images were segmented to generate the following objects: skin, fat, bones, thoracic wall muscle, lungs, blood and heart muscle [15] (fig. 1). For each high resolution (256x256 pixels) segmented CT scan, points on the boundaries of the thorax and internal organs were automatically digitized and stored into special formatted ASCII data files. These files were then imported into I-DEAS software package (Structural Dynamics Research Corporation, Milford, OH, USA) and the spline interpolation routine was used to manually build the contours of the internal organs and thorax outline. The solid model of the thorax was then obtained by stretching non-planar, smooth surfaces on the cross-sectional profiles, and contains only the lungs, heart and thoracic block (fig. 2). The electrical conductivities used in this model are 0.07 S/m for the lungs, 0.15 S/m for the thorax block (averaged value) and 0.25 S/m for the heart muscle. The computational domain, discretized by using linear, 4-nodes tetrahedrons, consists of 15925 elements and 2825 nodes.

2.2. FORWARD PROBLEM

The forward problem in electrocardiography consists in computing the electric field inside the thorax produced by a potential distribution on the heart surface. If diffusion and propagation effects are neglected the cardiac source time-variation is instantaneously responded by the volume conductor. The electromagnetic field is then conveniently approximated as steady [16] and described by the following relations:
inside the volume conductor, $\Omega_{\text{thorax}}$:

$$\nabla \cdot (\sigma \nabla \phi) = 0$$  \hspace{1cm} (1)

on the internal boundary (epicardium), $S_{\text{epicardium}}$

$$\phi = \phi_0$$ \hspace{1cm} (2)

on the external boundary (thorax), $S_{\text{thorax}}$

$$\left(\sigma \nabla \phi\right) \cdot \mathbf{n} = 0$$ \hspace{1cm} (3)

on the interfaces between internal organs
\[ \phi' = \phi'' \quad \text{and} \quad (\sigma' \nabla \phi) \cdot \mathbf{n}' - (\sigma'' \nabla \phi) \cdot \mathbf{n}'' = 0 \] (4)

\( \sigma, \sigma' \quad \text{and} \quad \sigma'' \) are the electrical conductivities of the different regions, \( \phi \) and \( \phi_0 \) are electrical potentials, \( \mathbf{n} \) is the outward normal, and \( \mathbf{n}', \mathbf{n}'' \) are the normal vectors to the interfaces. In both forward and inverse ECG problem, the electric potential on the thorax is a linear function of the epicardial potential. Consequently, the solution to these problems may be thought of as finding the transfer relation that exists between the bioelectrical source (the epicardial potential) and the measurable electrical potential (the thoracic electrical potential), which is in fact the input-output relationship of the system under investigation. From this perspective, it is more convenient to utilize the integral form of the electrical field problem, namely the Fredholm integral equation of the first kind [17],

\[ \phi(P) = \int_{S_{\text{epicardium}}} K(P, Q) \phi(Q) dS_{\text{epicardium}}, \quad P \in S_{\text{thorax}} \quad \text{and} \quad Q \in S_{\text{epicardium}}, \] (5)

The kernel \( K(P, Q) \), sometimes called the transfer function, is the response to the impulse (Dirac-delta) function, and it completely characterizes the system. In the context of a numerical model, the electrical potential is a discrete rather than a continuous quantity and, consequently, the integral equation (5) becomes a matrix transfer relation

\[ \mathbf{y} = \mathbf{Cx} + \mathbf{e} \] (6)

In our finite element model \( \mathbf{y} \) (478×1) is the vector of torso potentials, \( \mathbf{x} \) (145×1) is the vector of epicardial potentials, \( \mathbf{e} \) (478×1) is the noise in the measurement, and \( \mathbf{C} \) (478×145) is the transfer matrix (acting as a smoothing operator). The object of the forward problem consists of finding the coefficients of the transfer matrix \( \mathbf{C} \). Eq. (1) was solved using the finite element method (as implemented by I-DEAS) as follows: a unit potential Dirichlet condition was applied successively to all nodes on the epicardium, whereas the other nodes were grounded (zero voltage) and the thorax surface was electrically insulated. This technique results in the forward transfer matrix epicardium → thorax surface. Each column in this matrix corresponds to the body surface potentials computed for a single activated node on the epicardium. Once the coefficients of \( \mathbf{C} \) are known, the epicardial potentials may be obtained from body surface potentials by numerical inversion schemes. The condition number of \( \mathbf{C} \) – defined as the largest to the smallest singular value of \( \mathbf{C} \) – was of order \( O(4) \).

2.3. INVERSE SOLUTIONS

The inverse solutions were found by solving the following penalized least squares problem:

\[ \mathbf{x}_c = \arg \min_{\mathbf{x}} \left\| \mathbf{Cx} - \mathbf{y} \right\|_2^2 + \alpha^2 \left\| \mathbf{Lx} \right\|_P^2 \] (7)

The first term of the cost function (7) represents the least-squares fit to the observed data, the second term is the side constraint, which captures prior information about the expected behavior of \( \mathbf{x} \), \( \alpha \) is a regularization parameter, and \( \mathbf{L} \) is a matrix operator. If \( \mathbf{L} = \mathbf{I} \) (the
identity matrix) and \( p = 2 \) then equation (7) yields the zero order Tikhonov regularization, with the solution given by the following normal equations
\[
x_{\text{Tikh}} = \left( C^T C + \alpha^2 I \right)^{-1} C^T y
\]
(8)

If \( L = D \) (discrete gradient operator) and \( p = 1 \), one obtains the TV penalty and the optimization problem (7) is nonlinear [18]:
\[
x_{TV} = \left( C^T C + \alpha^2 D^T W_x D \right)^{-1} C^T y
\]
(9)

where \( W_x \) is a weight, diagonal matrix given by
\[
W_x = \frac{1}{2} \text{diag} \left( \frac{1}{\sqrt{[Dx]^2 + \beta}} \right)
\]
(10)

To avoid the problems with nondifferentiability at the origin, the following approximation of the \( l_1 \) norm of the derivative was used [19]
\[
\|Dx\|_1 \approx \sum_{i=1}^{n} \sqrt{[Dx]^2 + \beta}
\]
(11)

where \( \beta > 0 \) is a small constant. Equation (9) was solved using the successive approximation method. For Tikhonov regularization, the optimal parameter \( \alpha \) was chosen using the L-curve method [20], which is based on a graphical plot on a logarithmic scale of \( \|x(\alpha)\|_2 \) vs. \( \|C(x(\alpha)) - y\|_2 \) for various values of \( \alpha \) (fig. 3).

Fig. 3 - The L-curve for choosing the regularization parameter \( \alpha \).
The vertical part of the $L$ curve corresponds to small values of $\alpha$ and underregularized inverse solutions (dominated by amplified noise). The horizontal part corresponds to large values of $\alpha$ for which the inverse solutions are overregularized (oversmoothed). The optimal value for $\alpha$ corresponds to the corner of the $L$ curve, which defines the transition between under- and overregularized solutions. In the case of TV regularization, the best reconstruction was selected based on visual inspection.

2.4. SYNTHETIC EPICARDIAL DISTRIBUTIONS

To test the quality of the inversion schemes, two synthetic voltage distributions were generated on the heart surface. A unit step epicardial distribution was created to model large potential gradients around activation wave fronts (fig. 4a). To simulate the cardiac depolarization wave, where the spatial variation of the potential is smooth, a dipolar source was created by assigning potentials of $\pm 1V$ and $-1V$ to two nodes spaced relatively farther apart (fig. 5a). The potential on the torso produced by these equivalent sources was computed (forward problem) and then altered with additive Gaussian noise for an SNR of 30dB.

3. RESULTS

3.1. STEP SOURCE

Figure 4 shows the distribution of body surface potentials produced by a unit step potential distribution on the heart surface. The optimal regularization parameter for Tikhonov inversion was $\alpha=0.046$ (fig. 3). For smaller values of $\alpha$ the solution is dominated by amplified noise, while for larger values of $\alpha$ the solution is oversmoothed. The best TV inverse solution was obtained for $\alpha=0.07$. The epicardial potential fields by the exact solution, least-squares (unregularized solution), TV and Tikhonov reconstructions are displayed in fig. 4c,d. The relative error (RE) in the inverse solutions with respect to the true solution and TV was 30%, with a correlation coefficient (CC) of 0.90, while Tikhonov regularization yielded inverse solution with RE=52% and CC=0.65.

3.2. DIPOLAR SOURCE

In this case, the optimal regularization parameters for Tikhonov and TV regularizations were 0.017 and 0.015 respectively. The exact and the inverse solutions are shown in fig. 5b,c. The relative error in the inverse solutions with respect to the true and TV was 57%, with CC=0.81, while Tikhonov regularization yielded inverse solution with RE=62% and CC=0.78.

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1 SNR(dB) = 10 \log_{10}[\text{Var}(Cx)/\text{Var}(\varepsilon)]$, where Var(\cdot) denotes the variance.
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Fig. 4 - Reconstruction of a step epicardial distribution (linear scale in volts): a) exact solution; b) least-squares solution; c) TV reconstruction; d) Tikhonov reconstruction.

Fig. 5 - Reconstruction of a dipolar epicardial distribution (linear scale in volts): a) exact solution; b) TV reconstruction; c) Tikhonov reconstruction.
4. DISCUSSION

The inverse problem of electrocardiography aims at becoming a meaningful diagnosis tool by noninvasively imaging the electrical activation patterns over the heart surface. The main difficulties in computing accurate and clinically relevant maps of cardiac depolarization reside in the ill-posed nature of the underlying continuous problem and the presence of noise in data. The standard least-squares approach leads to excessive noise amplification in the solution (fig. 5b) and quadratic Tikhonov regularization was extensively used in previous studies to stabilize the solution. In this study, we proposed the usage of the total variation constraint on the epicardial potential field as an alternative to the $l_2$-based penalty functionals. Other than quadratic TV-based regularization schemes may evidence large potential gradients by incorporating a solution-dependent weight factor in the inverse filter ($W$ in eq. 4). Compared with Tikhonov reconstruction, the TV inverse solution achieved better wave front preservation while suppressing the noise (fig. 5). The performances of these two regularization methods in recovering smooth potential fields produced by a dipolar source were similar when analyzed based on relative errors and correlation coefficients with respect to the true solution. By visual inspection of fig. 6 we note that Tikhonov regularization provides for a better localization of the maximum and minimum values of the epicardial potential field, while the TV-based solution overextends these two extremes.

One limitation of this study is the usage of a simplistic model of the human thorax. Ideally, such a model should bear the salient anatomical features that are expected to significantly influence the solution to the problem. Therefore, a primary issue in constructing the model used throughout this study was to decide which anatomical features are crucial in determining the body surface potentials within satisfactorily accuracy limits. In other words, the anatomical detail that has to be included in the model must be balanced against the ability to conveniently discretize the model and to obtaining an appropriate numerical solution to the forward problem. In our model the geometry of the heart was modified in such a way as to allow for a structured surface meshing and an easy computation of the discrete gradient operator. A more realistic heart representation would require unstructured finite element meshing and complex algorithms for finding a discrete version of the gradient operator [21]. Another simplification used in our modeling approach concerns the spatial distribution of the test epicardial potential fields. The cardiac electrical activity can be better represented using a bidomain representation of the heart [22], and by incorporating the dynamics of ionic channels [23]. However, this extension of the modeling approach requires a much denser finite element mesh of the heart to accurately simulate the excitation wave propagation. The mesh refinement will lead to an increase in the dimensionality of the solution. Too many nodes on the epicardium would increase the high frequency content of the input signal that will be outfiltered by the system, making the inverse solution much more sensitive to noise.
5. CONCLUSIONS

This computational study deals with the numerical inversion of ill-conditioned matrix operators found in the electrocardiographic imaging problem. Our preliminary results show that, although computationally more expensive than Tikhonov regularization, the TV approach is more effective in reconstructing large potential gradients characteristic to ventricular depolarization wave fronts. This finding suggests that the inversion method in electrocardiographic imaging should include the TV constraint on the epicardial potential distribution.

REFERENCES


