PAY FOR PERCENTILE

GADI BARLEVY
FEDERAL RESERVE BANK OF CHICAGO

DEREK NEAL
UNIVERSITY OF CHICAGO AND NBER

PRELIMINARY

Date: May, 2009.
We thank Fernando Alvarez, Roger Myerson, Kevin Murphy, and Phil Reny for helpful discussions. Neal thanks the Searle Freedom Trust for research support.
ABSTRACT

We propose an incentive pay scheme for educators that links educator compensation to the ranks of their students within appropriately defined comparison sets. Because this scheme employs only ordinal information, it allows education authorities to employ new assessments at each testing date without ever having to equate various assessment forms. Thus, our scheme mitigates incentives for teachers to teach to a particular assessment form while eliminating any means of influencing reward pay by corrupting the equating process or the scales used to report assessment results. We argue that the assessments and procedures that make up performance pay systems should be separate and distinct from those used to track student progress over time. Effective incentive provision does not require the use of a reference assessment scale or procedures for equating the scales of various assessments. Further, the use of incentive systems that require equating may well introduce pressures that corrupt the equating procedures and compromise our understanding of how student achievement is changing over time.
1. Introduction

In modern economies, most wealth is held in the form of human capital, and publicly funded schools play a key role in creating this wealth. Thus, reform proposals that seek to enhance the efficiency of schools are an omnipresent feature of debates concerning public policy and societal welfare. In recent decades, education policy makers have increasingly designed these reform efforts around measures of school output rather than school inputs. Although scholars and policy makers still debate the benefits of smaller classes, improved teacher preparation, or improved school facilities, few are willing to measure school quality using only measures of school inputs. During the 1990’s many states adopted accountability systems that dictated sanctions and remediation for the staff of schools based on how their students performed on standardized assessments. In 2001, the No Child Left Behind Act (NCLB) mandated that all states adopt such systems or risk losing federal funds, and more recently, several states and large districts have introduced incentive pay systems that link the salaries of individual teachers to the performance of their students.

We pose the provision of teacher incentives as a mechanism design problem. In our setting, an education authority possesses two sets of test scores for a population of students. The first set of scores provides information about student achievement at the beginning of a school year. The second set provides information about the achievement of the same population of students at the end of the school year.

We begin by noting that if the authority knows the mapping between the test score scale and the expected value of student skill, the authority can implement a piece rate incentive scheme based on the observed value of each student’s skills. However, in practice, education authorities may find it difficult to maintain precise knowledge of this mapping. In order to deter “teaching to the test” and related behaviors, an education authority that wants to implement a piece-rate pay for performance system must use a series of assessments that are comparable in terms of subject matter but different in terms of specific item content and form. This step can minimize teacher effort distortion, but the authority must equate the various assessments before it can translate scores into skill measures. Equating is a complex task, and the presence of equating studies as part of the system opens the door to scenarios where certain teachers or groups of teachers contest their reward pay by challenging the validity of the equating methods. Further, because proper equating is complex and difficult to verify, the equating process is a target for corruption.\footnote{A significant literature on state level proficiency rates under NCLB suggests that political pressures have compromised the meaning of proficiency cutoff scores in numerous states. States can inflate their proficiency rates by making exams easier while holding scoring procedures constant or by introducing a completely new assessment and then producing a crosswalk between the old and new assessment scale that effectively lowers the proficiency threshold. See Cronin et al (2007)}

Given these observations, we turn our attention to mechanisms that require authorities to make incentive payments based only on the ordinal information contained in assessment results, without any knowledge of how the fall and spring assessments are scaled. Systems that are ordinal in this sense can employ new assessment forms at each point in time without any requirement that these assessments be equated. Thus, these systems mitigate two
types of activities that the existing literature identifies as means of corrupting educational performance measures. The use of new assessment forms reduces incentives to coach students regarding the particular form of an assessment. Further, the absence of equating studies removes any opportunity to manipulate reward pay by manipulating the relative difficulty of various assessments or how different assessment results are scaled.

We describe a system called “Pay for Percentile,” that works as follows. For each student in a school system, first form a comparison set of students against which the student will be compared. Assumptions concerning the nature of instruction dictate exactly how to define this comparison set, but the general idea is to form a set that contains all other students in the system who begin the school year at the same level of baseline achievement in a comparable classroom setting. At the end of the year, give a cumulative assessment to all students. Then assign each student a percentile score based on his end of year rank among the students in his comparison set. For each teacher, sum these within-peer percentile scores over all the students she teaches and denote this sum as a percentile performance index. Then, pay each teacher a common base salary plus a bonus that is proportional to her percentile performance index. We demonstrate that this system can elicit efficient effort from all teachers in all classrooms to all students.

The linear relationship between bonus pay and our index does not imply that percentile units are a natural or desirable scale for human capital. Rather, within comparison set percentiles tell us what fraction of head-to-head contests teachers win when competing against other teachers who educate similar students. For example, a student with a within-comparison set percentile score of .5 performed as well or better than half of his peers. Thus, in our scheme, his teacher gets credit for beating half of the teachers who taught similar students. Our scheme involves a linear relationship between total bonus pay and the fraction of contests won because all of the contests share an important symmetry. Each pits a student against a peer who has the same expected spring achievement when both receive the same instruction and tutoring from their teachers.

Our work builds on the insights of Lazear and Rosen (1981), who demonstrate that tournaments can elicit efficient effort from workers even when firms are only able to rank the performance of their workers. In their model, workers make one effort choice and compete in one contest. In our model, teachers make multiple effort choices, and these choices simultaneously affect the outcomes of many contests. We find that a common prize for winning each tournament can still induce efficient effort even though teachers make instructional effort choices that affect all their students simultaneously and some students gain more from instruction than others.

Within our framework, it is natural to think of teachers as workers who perform complex jobs that require multiple tasks because each teacher must devote effort to general classroom instruction as well as one-on-one tutoring for each student. Our results offer new insight for the design of incentives in this setting. If employers can form an accurate ordinal ranking of worker performance on each task that defines their job, the percentiles from these rankings are a set of performance indices that may provide a basis for efficient incentive pay.
Our results suggest that incentive systems should be built around measures of relative performance among teachers who teach similar students. Further, our results highlight defects in existing systems that arise because policy makers have paid insufficient attention to the interpretation of assessment scales and the difficulties that arise when cardinal pay for performance systems require them to equate scores from different exam forms. Our pay for percentile system avoids these problems because it can be implemented even if the education authority does not know how to equate scores on different assessments.

Many engaged in current education policy debates implicitly argue that proper equating of different exam forms is fundamental to sound education policy because education authorities must be able to document the evolution of the distribution of student achievement over time. However, our results demonstrate that education authorities should treat the provision of incentives and the documenting of student progress as separate tasks. Equating studies are not necessary for incentive provision, and the equating process is more likely to be corrupted when high stakes are attached to the exams in question.

Because our scheme allows education authorities to use new assessment instruments at each point in time without the need to equate successive assessments, it addresses frequently voiced concerns about teaching to the test and cheating as responses to assessment based incentive programs. However, because our scheme still links teacher rewards and the ordinal outcomes of assessments, one may worry that it too invites teachers to engage in activities that inflate assessment results relative to student subject mastery. Here, we make the strong assumption that, if the authority gives a new assessment at each point in time, the only way teachers can directly affect the rank of their students is by teaching. Our aim is to address the design of optimal performance pay systems in settings where at least the ordinal information in assessments cannot be contaminated by actions that the education authority cannot observe. We do not address the design of a process for generating new assessments that provide no incentive for effort distortion, but we readily stipulate that this design challenge is a daunting one.

2. Basic Model

Here, we describe our basic model and our key result concerning multiple, simultaneous tournaments. In the following section, we derive our pay for percentile system as a limiting result within this framework. Assume there are $J$ classrooms, indexed by $j \in 1, ..., J$. Each classroom has one teacher, so $j$ also indexes teachers.

Each classroom has $N$ students, indexed by $i \in 1, ..., N$. Let $a_{ij}$ denote the initial human capital of the $i$-th student in the $j$-th class. Students within each class are ordered from least to most able, i.e.

$$a_{1j} \leq a_{2j} \leq \cdots \leq a_{Nj}$$

For now, we assume all classes are identical, i.e. $a_{ij} = a_i$ for all $j \in \{1, ..., J\}$ Later, we show how our approach can be adapted to a setting where classroom composition differs among teachers.

--

2See Holmstrom and Milgrom’s (1991) for a precise treatment of this hidden action problem.
Teachers can undertake two types of efforts to help students acquire additional human capital. They can teach the class as a whole or tutor individual students. The tutoring instruction is student-specific, and any effort spent on teaching student \( i \) will not directly affect any student \( i' \neq i \). Classroom teaching benefits all students in the class. Examples include tasks like lecturing or planning assignments.

Let \( e_{ij} \) denote the effort teacher \( j \) spends on individual instruction of student \( i \), and \( t_j \) denote the effort she spends on classroom teaching. The human capital of a student at the end of the period, denoted \( a'_{ij} \), depends on his initial ability \( a_{ij} \), the efforts of his teacher \( e_{ij} \) and \( t_j \), and a shock \( \varepsilon_{ij} \) that does not depend on teacher effort, e.g. random disruptions to the student’s life at home. We assume the production of human capital is separable between the student’s initial human capital and all other factors and takes the form

\[
a'_{ij} = g(a_{ij}) + t_j + \alpha e_{ij} + \varepsilon_{ij}
\]

where \( g(\cdot) \) is an increasing function and \( \alpha > 0 \) measures the relative productivity of classroom teaching versus individual instruction. The shock \( \varepsilon_{ij} \) is pairwise independent for any pair \( (i,j) \). Let \( F(x) \equiv \Pr(\varepsilon_{ij} \leq x) \). We assume there is an associated density distribution \( f(x) = \frac{dF(x)}{dx} \) that is unimodal, symmetric around 0, and differentiable.

Let \( X_j \) denote teacher \( j \)'s expected income. Then her utility is assumed to be

\[
U_j = X_j - C(e_{1j}, \ldots, e_{Nj}, t_j)
\]

where \( C(\cdot) \) denotes the teacher’s cost of effort. We assume \( C(\cdot) \) is increasing in all of its arguments and is strictly convex. We further assume it is symmetric with respect to individual effort, i.e. let \( e_j \) be any vector of tutoring efforts \( (e_{1j}, \ldots, e_{Nj}) \) for teacher \( j \), and let \( e_j' \) be any permutation of \( e_j \), then

\[
C(e_j, t_j) = C(e_j', t_j)
\]

We also impose the usual boundary conditions on marginal costs. The lower and upper limits of the marginal costs with respect to each dimension of effort are 0 and \( \infty \) respectively. These conditions ensure the optimal plan will be interior. Although we do not make it explicit, \( C(\cdot) \) also depends on \( N \). Optimal effort decisions will vary with class size, but the tradeoffs between scales economies and congestion externalities at the center of this issue have been explored by others.\(^3\) Our present goal is to analyze the optimal provision of incentives given a fixed class size, \( N \), and here, we suppress reference to \( N \) in the cost function.

2.1. **Social Optimum.** Assume that each teacher has an outside option equal to \( U_0 \) and consider the objective function of an omniscient social planner:

\(^3\)See Lazear (2001).
Here, \( R \) is the social value of a unit of \( a' \). Because we have normalized units of time so that \( \frac{\partial a'_{ij}}{\partial t_j} = 1 \), \( R \) is also the gross social return per student when one unit of teacher time is effectively devoted to classroom instruction. Since \( C(\cdot) \) is strictly convex, the first-order conditions are necessary and sufficient for an optimum. Since all teachers share the same cost of effort, the optimal allocation will dictate the same effort levels in all classrooms, i.e. \( e_{ij} = e_i \) and \( t_j = t \) for all \( j \). Hence, the optimal effort levels dictated by the social planner, \( e_1, ..., e_N \) and \( t \), will solve the following system of equations:

\[
\frac{\partial C(e_j, t_j)}{\partial e_i} = R\alpha \quad \text{for } i = 1, ..., N
\]

\[
\frac{\partial C(e_j, t_j)}{\partial t_j} = RN
\]

In this framework, the cost and returns associated with devoting instruction time to a student are not a function of the student’s baseline characteristics. Thus, the social optimum dictates not only homogeneous effort vectors among classrooms but also homogeneous tutoring effort among students in the same classroom.\(^4\)

### 3. Relative Performance Pay Using Piece Rates

Now consider the effort elicitation problem faced by an education authority that supervises our \( J \) teachers. For now, assume that this authority knows everything about the technology of human capital production, but cannot easily monitor teachers to observe their effort decisions \( e_{ij} \) or \( t_j \). Instead, the authority observes test scores that are perfect indicators of ability, \( s = m(a) \) and \( s' = m(a') \), where \( m(a) \) is a strictly monotonic function.

We begin by assuming that the authority knows both \( m(a) \) and \( g(a) \), i.e. it knows how to invert the psychometric scale \( s \) and recover \( a \), and it knows how the natural rate of human capital growth varies with the baseline skill level of students. In this case, the authority can form the expected value of each teacher \( j \)'s contribution to student \( i \), which we denote by

\[
V_{ij} = a'_{ij} - g(a_{ij}) = m^{-1}(s'_{ij}) - g(m^{-1}(s_{ij}))
\]

\(^4\)The proof of this claim is straightforward given that \( \alpha \) is the same for all students. In section 4, we turn to a variation of the model in which students of different baseline achievement levels receive different benefits from receiving instruction, and here, the social optimum will involve different tutoring levels for students of different achievement levels.
Given \( V_{ij} \), the education authority can elicit efficient effort from teachers by paying piece rates per student equal to \( RV_{ij} \). In practice though, education authorities do not necessarily know either \( g(a) \) or \( m(a) \). We begin by considering a setting where the authority knows \( m(a) \) but does not know \( g(a) \) with certainty. In the next section, we will consider the case where the authority knows neither \( m(a) \) or \( g(a) \) with certainty.

We shall now argue that if the authority knows \( m(a) \) but does not know \( g(a) \) with certainty, it can continue to elicit the optimal effort level from all teachers. To show this, we shall construct a scheme in which each teacher’s pay depends only on test scores at the end of the period, i.e. \( s'_{ij} \). In particular, suppose each teacher receives a base pay \( X_0 \) and then a bonus that is constructed as follows. First, for each student \( i \), we construct a comparison group comprised of all students with the same initial test score at the beginning of the period. Under our assumptions, this set comprises the \( i \)-th student in each classrooms. We then define \( \pi'_i \) as the average achievement for this group at the end of the period, i.e.

\[
\pi'_i = \sum_{j=1}^{J} a'_{ij}
\]

The bonus we pay each teacher \( j \) for the performance of her student at baseline achievement level \( i \) is then \( R(a'_{ij} - \pi'_i) \). Here, we pay teachers for how their students perform relative to comparable students rather than for how their students perform relative to their own initial achievement level. Thus, we do not need to know the natural rate of human capital growth, \( g(\cdot) \), for any level of baseline achievement.

As long as base pay, \( NX_0 \), is at least \( C(e^*, t^*) + U_0 \), one can show that teachers will be willing to participate and will all choose the optimal levels of effort. Intuitively, since the unobserved component of the natural rate of human capital growth is the same for all students in our comparison group, we can use relative performance to figure out the value added of each teacher.\(^5\) We show in the next section that similar reasoning allows us to overcome the problem of not being able to observe the mapping \( m(\cdot) \) between scores and ability because the scale will be common across all students who take the same test.

4. Tournaments

Now, consider a case where the education authority does not create and administer the assessments but must hire a testing agency to provide \( s \) and \( s' \), the vector of fall and spring test scores. Because previous psychometric research has established that exam scores become inflated when school systems administer the same form of the same high stakes exam multiple times,\(^6\) the education authority will instruct its testing agency to administer exams in the fall and spring that cover the same topics but contain different questions and

---

\(^5\) The idea of using relative performance to account for common unobserved components is described in more detail in Holmstrom (1982).

\(^6\) See Koretz (2002).
then place the results from these exams on a common scale such that \( s = m(a) \) and \( s' = m(a') \).

Note that in order to implement the piece rate scheme we describe above, the authority must announce the salary schedule in terms of \( s \) and \( s' \). However, once the authority announces how it will invert \( s' = m(a') \), teachers face a strong incentive to lobby the testing agency to secretly administer or scale assessments in a manner that weakens effort incentives and allows teachers to collect base salaries that reflect efficient effort levels while providing less effort.

For example, if teachers can secretly pressure the agency to correctly equate the fall and spring scores but then report spring scores equal to the correct spring scores divided by some constant greater than one, all teachers will optimally choose less than efficient effort, but their base pay will remain constant and their expected bonus pay will still be zero. A similar result can be achieved by manipulating the content of the spring exam in a way that compresses scores while claiming that the fall and spring exams are comparable and should be scored the same way.

If the authority knew that teachers were engaging in this lobbying activity and also knew all the ways that the testing agency could manipulate \( s' \), the authority might be able to mitigate these concerns by choosing a rule for converting the vectors \( s' \) and \( s \) into measures of relative performance that accounted for the anticipated corruption of assessment results. However, it seems reasonable to assume that, at any point in time, education authorities only know that there is some probability that the results reported by the testing agency have been compromised. In this setting, the authority will not always be able to elicit efficient effort by paying piece rates tied to student human capital levels. Thus, in the next section, we consider incentive schemes in an environment where the authority is required to make performance payments that are invariant to the scale used to report assessment results. Such schemes can be implemented without equating and therefore without the corruption activities that equating invites.

We contend that the literature on the implementation of NCLB contains much suggestive evidence that education authorities should benefit from using incentive schemes that employ different assessments at each point in time and do not require that assessments be equated over time. Reports on NCLB charge that several states have inflated the growth in their reported proficiency rates by making assessments easier without making appropriate adjustments to how exams are scored or by introducing new assessments and equating the scales between the old and new assessments in ways that appear generous to the most...
recent cohorts of students. The incentives for this type of scale corruption are particularly easy to understand. However, we expect numerous performance pay or accountability systems that tie rewards and punishments to the meaning of a particular psychometric scale to face similar corruption pressures. Thus, we consider an environment where the authority instructs the testing agency to administer new forms of the assessment each period and must restrict its attention to incentive schemes that are scale invariant, i.e. schemes that rely only on ordinal information and are thus implemented without regard to the scaling of various assessment results. Here, the authority trusts that assessment results correctly order students according to their human capital levels at a point in time, but the authority ignores the cardinal content of the assessment results.

Consider the following compensation scheme: Each teacher receives a base pay of $X_0$ per student. Further, each teacher knows that she is competing against one other teacher and that the results of this contest will determine her bonus. Any teacher $j$ does not know who her opponent will be when she makes her effort choice. She knows only that her opponent will be randomly chosen from the set of other teachers in the system and that her opponent will be facing the same compensation scheme that she faces. At the end of the year when teacher $j$ is matched with some other teacher $j'$, teacher $j$ will receive a bonus $(X_1 - X_0)$ for each student $i$ whose human capital is higher than the corresponding student in teacher $j'$’s class, i.e. if $a_{ij}' \geq a_{ij''}$. The total compensation for teacher $j$ is then

$$NX_0 + (X_1 - X_0) \sum_{i=1}^{N} \mathbb{I}(a_{ij}' \geq a_{ij''})$$

where $\mathbb{I}(A)$ is an indicator that equals 1 if event $A$ is true and 0 otherwise. We note here that teacher $j$’s compensation is not affected by the scale used to report assessment results. Because ordinal comparisons determine all payoffs, teacher behavior and teacher welfare are invariant to any re-scaling of the assessment results that preserves ordering.

We assume that each student in teacher $j$’s class is compared with the comparable student in the class of teacher $j'$. However, this is not essential. More generally, the teacher against whom teacher $j$ is compared can vary across students, e.g. we can let the identity of $j'$ be a function of $i$, but this would be notationally more cumbersome.

For each $i \in \{1, ..., N\}$, let us define a new variable $\nu_i = \varepsilon_{ij} - \varepsilon_{ij''}$ as the difference in the shock terms for students in the two classes whose initial human capital is $a_i$. Let $H(x) \equiv \Pr(\nu_i \leq x)$ denote the distribution of $\nu_i$. We define $h(x) = dH(x)/dx$, and note that given our assumptions about $F(.)$, $H(.)$ is also unimodal, mean zero, and symmetric.

---

8See Peterson and Hess (2006) and Cronin et al (2007). In 2006, the state of Illinois saw dramatic increases in proficiency rates that were coincident with the introduction of a new assessment. See ISBE (2006).

9Further, even if we set aside corruption concerns, the type of relative value-added scheme we describe above may be politically unsustainable if teachers believe that it is too difficult to implement it correctly. While proficiency count systems like NCLB require only that the scales for two different assessments are correctly equated at one skill level, the proficiency level, correct implementation of relative performance systems that are scale dependent requires that scales be properly equated at every level of skill because, beginning at any skill level, the meaning of all potential score gains and losses must remain constant over different forms of the assessment.
Since the initial ability of the students who are compared to each other is identical, the maximization problem for teacher $j$ is

$$\max_{e_{j}^{d}, t_{j}} NX_{0} + (X_{1} - X_{0}) \sum_{i=1}^{N} H(\alpha(e_{ij} - e_{ij'}) + t_{j} - t_{j'}) - C(e_{j}, t_{j}) - U_{0}$$

The first order conditions for each teacher are given by

$$(3) \quad \frac{\partial C(e_{j}, t_{j})}{\partial e_{ij}} = \alpha h(\alpha(e_{ij} - e_{ij'}) + t_{j} - t_{j'})(X_{1} - X_{0}) \text{ for } i = 1, 2, ..., N$$

$$(4) \quad \frac{\partial C(e_{j}, t_{j})}{\partial t_{j}} = \sum_{i=1}^{N} h(\alpha(e_{ij} - e_{ij'}) + t_{j} - t_{j'})(X_{1} - X_{0})$$

Consider setting the bonus $X_{1} - X_{0} = R/h(0)$ and suppose both teachers $j$ and $j'$ choose the same effort levels, i.e. $e_{ij} = e_{ij'}$ for all $i$. Then (3) and (4) become

$$\frac{\partial C(e_{j}, t_{j})}{\partial e_{i}} = R\alpha \quad \text{for } i = 1, ..., N$$

$$\frac{\partial C(e_{j}, t_{j})}{\partial t_{j}} = RN$$

Recall that these are the first order conditions for the planner’s problem, and thus, the socially optimal effort levels $e_{1} = ... = e_{N} = e^{*}$ and $t = t^{*}$ solve these first order conditions. Nonetheless, the fact that these levels satisfy teacher $j$’s first order conditions is not enough to show that they are in fact optimal responses to the effort decisions of the other teacher. Since $H(\cdot)$ is neither strictly convex nor strictly concave everywhere, we have not shown that $e_{ij} = e^{*}$ and $t_{j} = t^{*}$ is a global best response to itself. Appendix A provides proofs for the following two propositions that summarize our main results for two teacher contests:

**Proposition 1:** Let $\sigma$ denote the variance of $\varepsilon_{ij}$. There exists $\sigma$ such that $\forall \sigma > \sigma$, both teachers choosing the socially optimal effort levels, $e_{1} = ... = e_{N} = e^{*}$ and $t = t^{*}$ solve these first order conditions. Nonetheless, the fact that these levels satisfy teacher $j$’s first order conditions is not enough to show that they are in fact optimal responses to the effort decisions of the other teacher. Since $H(\cdot)$ is neither strictly convex nor strictly concave everywhere, we have not shown that $e_{ij} = e^{*}$ and $t_{j} = t^{*}$ is a global best response to itself. Appendix A provides proofs for the following two propositions that summarize our main results for two teacher contests:

$\sigma$ is needed to rule out cases where, given that the other teacher is choosing $e_{1} = ... = e_{N} = e^{*}$ and $t = t^{*}$, teacher $j$’s expected gain from responding with $(e^{*}, t^{*})$ as opposed to some lower effort level does not cover the incremental cost. If $\sigma$ is too small, some effort level less than $(e^{*}, t^{*})$ may be the best response when the opposing teacher chooses $(e^{*}, t^{*})$. However, as $\sigma$ increases, the change in expected prize winnings associated with moving from this lower effort level to the socially optimal levels increases and at some level of $\sigma$ these extra expected winnings will cover the associated change in total cost. Thus, if the element of chance in these contests is important enough,

\[\text{Lazear and Rosen (1981) require a similar condition for existence in their single task, two person game.}\]
a pure strategy Nash equilibrium exists which involves both teachers choosing the socially optimal effort vectors, \((e^*, t^*)\), and Proposition 2 adds that this equilibrium is unique.

**Proposition 2:** In the two teacher contest described here, whenever a pure strategy Nash equilibrium exists, it involves both teachers choosing the socially optimal effort levels, \(e_1 = \ldots = e_N = e^*\) and \(t = t^*\).

Taken together, our propositions imply that our tournament scheme can elicit efficient effort from teachers who compete against each other in seeded competitions. Thus, the efficiency properties of two person contests involving a single dimension of effort carry over to two person contests involving an \(N+1\) dimensional effort choice.

Finally, to ensure that teachers are willing to participate in this scheme, we need to make sure that

\[
NX_0 + \frac{RN}{2h(0)} - C(e^*, t^*) \geq U_0
\]

Given this constraint, the following compensation scheme minimizes the cost of providing efficient incentives

\[
X_0 = \frac{U_0 + C(e^*, t^*)}{N} - \frac{R}{2h(0)}
\]

\[
X_1 = \frac{U_0 + C(e^*, t^*)}{N} + \frac{R}{2h(0)}
\]

Note that the authority needs only four pieces of information to implement this contest scheme. The authority needs to know each student’s teacher, the ranks implied by \(s\) and \(s'\), and the ratio \(\frac{R}{h(0)}\). Recall that \(R\) is the gross social return per student generated by one effective unit of classroom instruction. If we stipulate that the authority knows what effective instruction is worth to society but simply cannot observe whether or not effective instruction is being provided, \(h(0)\) is the key piece of information that the authority requires.

Here, \(h(0)\) is the rate at which the probability that a given teacher wins one of our contests changes if this teacher deviates from \((e^*, t^*)\) by a small amount. It will be difficult for any authority to learn \(h(0)\) precisely, but one can imagine experiments that could provide considerable information about \(h(0)\). The key observation is that, for many different prize levels, there exists a symmetric Nash equilibrium among teachers in pure strategies. Thus, given our tournament mechanism and some initial choice for the prize structure, suppose the authority selected a random sample of students from the entire student population and then invited these students to a weekend review class during the final month of the school year. Let \(\Delta t\) be the length of the review session. If the identity of the students is kept confidential, such an experiment will not change the effort choices of individual teachers. However, given any symmetric equilibrium, the ex post probability that a particular student who received such tutoring will score better than the corresponding student he is competing against on behalf of his teacher should increase by \(\Delta p \approx \frac{h(0) + h(\Delta t)}{2} \Delta t\). Our
production technology implicitly normalizes the units of $\varepsilon$ so that shocks to achievement can be thought of in terms of additions to or deletions from the hours of quality instruction $t$ students receive. If we assume that the authority can perfectly monitor instruction quality during these experimental sessions and if we choose a $\Delta t$ that is a trivial intervention relative to the range of shocks, $\varepsilon$, that affect achievement during the year, the sample mean of $\frac{\Delta p}{\Delta t}$ provides a useful approximation for $h(0)$.\textsuperscript{11}

In this section, we have described two pay for performance systems that elicit efficient effort. In both systems, the authority must know $R$, and they both require that payments be made based on relative performance measures among sets of students who began the year with the same level of baseline achievement. The system based on ordinal contests among teachers does not require that scores be equated among various forms of the assessment and thus has the advantage of being robust against efforts to corrupt assessment scales and weaken performance incentives.

5. Pay for Percentile

However, contests against a single opponent may create different corruption concerns. In our setting, the opponent for teacher $j$ is announced at the end of the year after students are tested. Thus, some teachers may respond to this system by offering bribes to school officials in exchange for being paired with a teacher whose students performed poorly. If one tried to avoid these bribes by announcing the pairs of contestants at the beginning of the year, then one would worry about collusion on low effort levels among contestants. Given these considerations, we now turn to performance contests that involve large numbers of teachers competing anonymously against one another. We then turn to the limiting case in which all teachers compete against all other teachers.

Suppose that each teacher now competes against $K > 1$ teachers who also have $N$ students. To simplify notation, we assume again that the initial distribution of human capital is the same in all classes. Each teacher knows that $K$ other teachers will be drawn randomly from the entire population of teachers to serve as her contestants, but teachers make their effort choices without knowing whom they are competing against. In this setting, teacher $j$’s problem is

$$\max_{e_j, t_j} NX_0 + \sum_{k=1}^{K} \sum_{i=1}^{N} H(\alpha(e_{ij} - e_{ik}) + t_j - t_k)(X_1 - X_0) - C(e_j, t_j) - U_0$$

The first order conditions are given by

\textsuperscript{11}Recall that $R$ is the social value of a unit of effective instruction time. Thus, the prize $\frac{R}{h(0)}$ determined by this procedure will be the same regardless of the units used to measure instruction time, e.g. seconds, minutes, hours.
(5) \[ \frac{\partial C(e_j, t_j)}{\partial e_{ij}} = \sum_{k=1}^{K} \alpha h(\alpha(e_{ij} - e_{ik}) + t_j - t_k)(X_1 - X_0) \] for \( i = 1, \ldots, N \)

(6) \[ \frac{\partial C(e_j, t_j)}{\partial t_j} = \sum_{k=1}^{K} \sum_{i=1}^{N} h(\alpha(e_{ij} - e_{ik}) + t_j - t_k)(X_1 - X_0) \]

As before, suppose all teachers put in the same effort level, i.e. given any \( j \), \( t_j = t_k \) and \( e_{ij} = e_{ik} \) for all \( i = 1, \ldots, N \) and \( k = 1, \ldots, K \). In this case, the right-hand side of (5) and (6) reduce to \( \alpha Kh(0)(X_1 - X_0) \) and \( NKh(0)(X_1 - X_0) \), respectively. Thus, if we set \( X_1 - X_0 = \frac{R}{Kh(0)} \) and assume that all teachers choose the socially optimal effort levels, the first order conditions for each teacher are satisfied. Further, Proposition 1 extends trivially to contests among \( K > 2 \) teachers. Therefore, we know that given the prize, \( \frac{R}{Kh(0)} \), a pure strategy Nash equilibrium in which all \( K \) teachers choose the socially optimal levels of effort exists.

Next, assume that \( K = J \). This means that, among our population of teachers whose students share the same initial distribution of baseline achievement, each teacher competes against every other teacher. Now let \( K = J \to \infty \) and let \( A'_{ij} \) denote a terminal score chosen at random and uniformly from the set of all terminal scores \((a'_{i1}, \ldots, a'_{ij})\). Since the distribution \((a'_{i1}, \ldots, a'_{i,j-1}, a_{i,j+1}, \ldots, a'_{ij})\) converges to the distribution \((a'_{i1}, \ldots, a_{i,j-1}, a_{i,j+1}, \ldots, a_{ij})\) as \( K \to \infty \), it follows that

\[ \lim_{K \to \infty} \sum_{k=1}^{K} \frac{\Pr(a'_{ij} \geq A'_{ik})}{K} = \Pr(a'_{ij} \geq A'_{ij}) \]

and the teacher’s maximization problem reduces to

\[ \max_{e_j, t_j} N X_0 + \frac{R}{h(0)} \sum_{i=1}^{N} \Pr(a'_{ij} \geq A'_{ij}) - C(e_j, t_j) - U_0 \]

This pay for percentile scheme is the limiting case of our simultaneous contests scheme as the number of teachers grows large.

So far, we have assumed identical classes, but given our assumptions that teacher time produces the same increase in human capital for each student regardless of his initial human capital and that each teacher’s cost of effort is symmetric over students, the composition of each class does not matter for either the first best effort levels or the effort levels that teachers choose under our proposed scheme.

Suppose each class \( j \in 1, \ldots, J \) has \( N \) students, whose respective ability levels \( a_{ij} \) for \( i \in 1, \ldots, N \) satisfy

\[ a_{1j} \leq a_{2j} \leq \cdots \leq a_{Nj} \]

Rather than require that \( a_{ij} = a_i \) for all \( j \), we need only require that for each \( a_{ij} \), there exist at least \( K \) students in other classrooms \((i'j', t')\) such that \( a_{ij} = a_{i'j'} \). If \( K \) is large,
our pay for percentile scheme can be applied to any classroom by calculating the final assessment percentile score of each student $i$ within the set of all students who share $i$’s baseline achievement level.

This result holds because nothing that depends on $a_{ij}$ appears in the first-order conditions for the planner’s problem. The optimal allocation of effort $(e^*, t^*)$ is the same for all teachers regardless of the distribution of baseline achievement in their classrooms. Further, given the symmetry in our cost function, the cost associated with teacher $j$ choosing $(e^*, t^*)$ as a best response to all teachers of all other students choosing $(e^*, t^*)$ is not influenced by the distribution of baseline achievement in $j$’s class.

6. HETEROGENEOUS GAINS FROM INSTRUCTION

In this section, we turn to the case where the same amount of teacher instruction generates different expected gains in human capital depending on the baseline achievement levels of particular students. Here, the socially efficient vector of effort choices for teacher $j$ is a function of the distribution of baseline achievement in her class. In this setting, we cannot base comparison sets on the baseline achievement of individual students alone. If two teachers both have one student who begins at $a_i$, the marginal cost of tutoring effort for such a student, given that the teachers are providing efficient effort to other students in their classes, depends on the distribution of baseline achievement among his peers.

Properly seeded tournaments pair contestants who face the same total and marginal costs of effort. Thus, for our scheme to work in this more general case, we require a stronger assumption concerning the availability of comparison sets. Above, we assume that for each student, there exist $K$ other students with the same initial achievement level. We now require that for each classroom, there exist $K$ other classes with the same composition of initial achievement level. For each $j$, there exist at least $K$ other $j'$ such that $a_{ij} = a_{ij'}$ for all $i \in \{1, ..., N\}$. Given such comparison sets, our pay for percentile scheme can direct efficient effort to all students and classrooms even in the presence of heterogeneous returns to instruction.

Consider the following generalization of our human capital production technology:

(7) \[ a'_{ij} = g(a_{ij}) + \gamma(a_{ij})t_j + \alpha(a_{ij})e_{ij} + \varepsilon_{ij} \]

Here, $i$ does not refer points in the baseline achievement distribution for a class but rather points in the population distribution of baseline achievement. We normalize $\gamma(a_{ij}) = 1$. Now, $R$ is the gross social return to giving one unit of effective instruction to students at the lowest level of baseline achievement. We maintain our assumption that the cost of spending time teaching students does not depend on their identity, but now the return to teacher effort differs with initial baseline achievements. Further, our assumption that the $\varepsilon_{ij}$ are pairwise identically distributed has a slightly different interpretation. We can no longer discuss the shocks, $\varepsilon_{ij}$, as simply additions to or deletions from units of effective instruction. Rather, the units of $\varepsilon_{ij}$ are deletions and additions to stocks of student skill where the value of adding one unit to these stocks is $R$.  

13
In this setting, consider the planner’s problem. We assume the planner takes the composition of classes as given and must choose the effort decisions in each classroom given the students in that class. One could imagine a more general problem where the planner chooses the composition of classrooms and the effort vector for each classroom. However, given the optimal composition of classrooms, the planner still needs to choose the optimal levels of effort in each class, and this second step is our focus because we are analyzing the provision of incentives for educators taking as given the sorting of students among schools and classrooms.

Define $\Omega_j$ as the set of baseline achievement types in the class of teacher $j$. We keep class size fixed at $N$ and assume that no two students in one class have the same baseline achievement level. Thus, $\Omega_j$ contains $N$ elements for all $j$. Within each class $j$, the planner chooses effort to solve

$$\max_{e_j, t_j} \sum_{i \in \Omega_j} R[g(a_{ij}) + \gamma(a_{ij})t_j + \alpha(a_{ij})e_{ij} + \varepsilon_{ij}] - C(e_j, t_j) - U_0$$

This problem is strictly concave, and so the first-order conditions are both necessary and sufficient for an optimum. These are given for all $j$ by

$$\frac{\partial C(e_j, t_j)}{\partial e_{ij}} = R\alpha(a_{ij}) \quad \text{for } i = 1, \ldots, N$$
$$\frac{\partial C(e_j, t_j)}{\partial t_j} = R \sum_{i \in \Omega_j} \gamma(a_{ij})$$

For any composition of baseline achievement, $\Omega_j$, there will be a unique effort vector $(e_j, t_j)$ that solves this equation, but this vector will differ for classes with different compositions. In particular, two students with the same benchmark ability but different peers could receive different effort levels because differences in effort allocations to their peers create differences in the cost of effort applied to each of them.

We now show that a slight variation on our pay for percentile scheme can elicit socially optimal effort vectors from all teachers. The bonus scheme is the same as before, but now each student is compared to a set of students who not only have the same initial ability but also peers with the same distribution of baseline achievement. That is, each class $k$ in teacher $j$’s comparison set for a given student must satisfy $a_{ij} = a_{ik}$ for all $i \in \{1, \ldots, N\}$.

Teacher $j$’s problem is given by

$$\max_{e_j, t_j} N X_0 + \sum_{k=1}^{K} \sum_{i \in \Omega_j} H(\alpha(a_{ij})(e_{ij} - e_{ik}) + \gamma(a_{ij})(t_j - t_k))(X_1 - X_0) - C(e_j^*, t_j)$$

Once again, suppose we set $X_1 - X_0 = \frac{R}{K_h(0)}$. If all teachers provide the same effort levels, the first order conditions for teacher $j$ collapse to
\[ \frac{\partial C(e_j, t_j)}{\partial e_{ij}} = R\alpha(a_{ij}) \quad \text{for } i = 1, ..., N \]

\[ \frac{\partial C(e_j, t_j)}{\partial t_j} = R \sum_{i \in \Omega_j} \gamma(a_{ij}) \]

which are the same as the planner’s first order condition. If we assume that other teachers are choosing the socially optimal levels of effort and invoke our usual restriction on \( \sigma \), these first-order conditions are necessary and sufficient for an optimal response. Thus, a Nash equilibrium exists such that all teachers choose the first best effort levels in response to a common prize structure.

Total expected pay will vary among classrooms though. Since the socially efficient effort levels vary with classroom composition, base pay will vary with classroom composition in order to satisfy the teachers participation constraints over all classrooms.

Our result that the optimal prize structure does not vary with baseline achievement even when returns to instruction vary with baseline achievement levels hinges on our assumption that the distribution of \( \varepsilon_{ij} \) and thus \( h(0) \) do not vary with baseline student achievement. Further, this assumption is not testable if we permit returns to effective instruction to vary with baseline achievement in an unrestricted and unspecified manner. It is straightforward to show that, if the populations of students at two different baseline achievement levels differ with respect to both \( h(0) \) and \( \gamma(a_{ij}) \), these differences cannot be separately identified from experiments like those described in section 4 above.

If both \( h(0) \) and \( \gamma(a_{ij}) \) differ with baseline achievement, an efficient pay for percentile scheme still exists that pays different prizes for winning contests involving students of different baseline achievement levels. However, education authorities cannot implement such a scheme without prior information concerning the structure of these differences in \( h(0) \).

The key implication of our analyses is that performance pay for educators should be based on ordinal rankings of student outcomes within properly chosen comparison sets. Whether or not the optimal mapping between rankings and reward money is constant over all comparison sets, the decision to tie rewards to relative performance measures that are scale invariant allows the authority to combat teaching to the test behaviors by using a new form of the assessment in each period while also eliminating incentives to corrupt the system by manipulating the scales used to report assessment results.

7. Lessons for Policy Makers

We analyze the provision of incentives for public educators as a mechanism design problem. We describe a simultaneous contest mechanism that can elicit efficient effort from teachers if the education authority possesses or can acquire the information necessary to choose the correct prize or prizes to associate with these contests. However, the contribution of our analyses extends beyond the specific mechanism we propose. Table 1 summarizes a
number of existing pay for performance schemes that are currently in operation in various
districts in the United States.

<table>
<thead>
<tr>
<th>Name</th>
<th>Location</th>
<th>Performance Indices Linked to Student Achievement</th>
</tr>
</thead>
<tbody>
<tr>
<td>ProComp</td>
<td>Denver</td>
<td>Principals negotiate with teachers to set growth targets for each child. Teachers are judged based on how many targets they meet.</td>
</tr>
<tr>
<td>QComp</td>
<td>Minnesota</td>
<td>Schools submit plans that specify their own methods for measuring gains in student achievement.</td>
</tr>
<tr>
<td>TAP</td>
<td>14 States &amp; DC</td>
<td>Value-added Model that attempts to measure deviations from expected achievement growth.</td>
</tr>
<tr>
<td>MAP</td>
<td>Florida</td>
<td>Districts are free to choose their own methods for measuring and weighting achievement gains and proficiency when forming indices of teacher performance.</td>
</tr>
<tr>
<td>STAR</td>
<td>Florida</td>
<td>Teachers receive points based on the changes in proficiency levels their students experience. A Value Table specifies the point allocations. Teachers receive more points when their students experience positive transitions that are rare.</td>
</tr>
</tbody>
</table>

Each of these systems attempts to evaluate multiple aspects of teacher performance. This table gives a brief description of the methods used to evaluate teacher performance with respect to student achievement. The MAP system has now replaced the STAR system in Florida. STAR created significant controversy when it was implemented in Hillsboro Co, Florida because the Value-Table method used in the first year of the program generated a strikingly high concentration of reward payments in economically advantaged schools.

Our analyses yield several insights that are important but have not been fully recognized in current policy debates concerning the design of these and other performance pay systems for educators.

To begin, our results demonstrate that relative pay for performance systems are advantageous. Systems that base incentive payments on relative student performance within comparison sets avoid the need to know, $g(a_i)$, how achievement growth differs naturally with baseline achievement levels. The Value-Added Model (VAM) approach referenced in Table 1 and advocated by segments of the education research community seeks to create an absolute and universal performance ranking over all teachers by using regression methods to implicitly estimate $g(a_i)$. When considering whether or not this regression approach is likely to be effective, we come to another lesson.

Basing relative performance measures on ordinal comparisons is desirable. Performance measures for teachers should not hinge on how tests are scaled or how they are equated over time. The VAM approach, which implicitly assumes that $s = a$, takes as given that changes in the units of a given psychometric scale $s$ are the correct measure of expected teacher performance even though VAM researchers offer no evidence that the expected hours of effective instruction required to move students up a particular psychometric scale by a fixed amount do not vary with their baseline scores. However, if it is easier to improve scores
over some ranges of a psychometric scale than others, VAM metrics of teacher performance will reflect not only variation in teacher performance but also variation in the baseline achievement distributions among classrooms.

Further, VAM models also require the maintained assumption that scores on both the baseline and follow-up assessment are correctly mapped into a common scale at each point in both achievement distributions. In contrast, our percentile performance indices measure teacher performance using ordinal comparisons between their students and their students peers. These indices are scale invariant and no teacher enjoys a competitive advantage because she teaches any particular type of student since her competitors are always teaching comparable students.

Our results also highlight the value of competition as a means of revealing what efficient achievement targets should be. Several of the systems described in Table 1 involve negotiations between individual teachers and principals concerning achievement targets for individual students. We will not develop an explicit model of this negotiation process, but we note that one possible outcome is that many or all teachers will receive bonus payments for providing less than efficient efforts level because they are able to negotiate less than demanding targets. In contrast, our pay for percentile system involves endogenous thresholds determined by competition among large numbers of contestants. In our scheme, every dollar of bonus pay won by one teacher is a dollar of bonus pay lost by another teacher. This arrangement minimizes the possibility that teachers will find ways to collusively provide low effort levels.

The most important implication of our results is that policy makers should separate two tasks that are at the center of many debates on education policy. The assessments and procedures that make up performance pay systems should be separate and distinct from those used to track student progress over time. Donald Campbell (1976) famously claimed that government performance statistics are always corrupted when high stakes are attached to them, and our analyses indicate that Campbell’s observation may reflect the perils of trying to accomplish two objectives with one set of performance measures. Effective incentive provision does not require the use of a reference assessment scale or procedures for equating the scales of various assessments. Further, the use of incentive systems that require equating may well introduce pressures that corrupt the equating procedures and compromise our understanding of how student achievement is changing over time.

The pay for percentile system we describe elicits effort by inducing competition not by trying to quantify student achievement or the contribution of teachers and schools to achievement. Our scheme makes no attempt to provide information about the evolution of student achievement growth over time or the level of student achievement at a point in time. Our percentile performance indices simply summarize the number of contests won and lost by each teacher. In contrast, systems that try to both provide incentives for teachers and track the evolution of student achievement and educational productivity over time are likely to do neither well. Policy makers cannot make valid judgements about student achievement growth or growth in educational productivity without a series of assessments that involve non-predictable changes in format and item content over time while yet being properly
equated over time. However, if the results from these assessments determine reward pay for teachers, it becomes much less likely that these conditions will be met.

8. Conclusion

In order to implement our system, policy makers must adopt a set of procedures for creating comparison sets, and some may worry that these procedures will be subject to the same type of corruption pressures that compromise the equating of scales on different assessments. However, there is one important difference. Regardless of how our comparison sets are formed, there will be one winner and one loser in each pairwise student comparison. Thus, there is no way to corrupt the construction of comparison sets that benefits all teachers at once. If a group of homogeneous students are placed in the wrong comparison set, either the students’ teachers or the other teachers of students in this set are placed at a disadvantage. Because the total per student reward pay is fixed and each contest has one winner and one loser, there are no manipulations of the rules for constructing comparison sets that benefit all teachers. In contrast, in all existing and proposed systems that tie rewards to the cardinal realizations of assessment results, there are ways to manipulate assessment content or the procedures used to equate scales among various assessments that benefit all teachers.\footnote{In practice, it may prove efficacious to implement our system using a large set of quantile regression models that allow researchers to create, for any set of baseline student and classroom characteristics, a set of predicted scores associated with each percentile in the conditional distribution of scores.}

We have discussed our results in terms of incentives for individual teachers and have not discussed schools as a unit of organization. When implementing the system we describe, it is advisable to make sure that teachers who teach in the same school do not compete against each other. This type of direct competition could undermine useful cooperation among teachers. Further, to the extent that peer monitoring within small teams is effective, education authorities may wish to implement our scheme at the school or school-grade level.\footnote{New York City’s accountability system currently includes a component that ranks school performance within leagues defined by student characteristics.}

Much of the existing research on accountability systems and incentive pay system for educators focuses on ways to solve several difficult statistical inference problems. The task of designing a set of assessments and statistical procedures that will not only allow policy makers to measure secular achievement growth over time but also isolate the contribution of educators and schools to this growth is a daunting one in the best of circumstances. Further, if the results of this endeavor determine rewards and punishments for educators, we can expect political pressure to corrupt the execution of these tasks. Our paper demonstrates that it is neither necessary nor advisable to attempt to measure secular achievement growth or the contribution of schools and teachers to this growth as part of incentive systems for educators. Properly seeded contests where winners are determined by the rank of student outcomes can provide incentives for efficient effort. Policy makers may still want to know how achievement levels are evolving over time or how the contribution of school to
achievement is evolving over time. However, they will do a better job of providing credible answers to these questions if they address them using a measurement system that has no impact on the distribution of rewards and sanctions among teachers and principals.
REFERENCES


Appendix A

Our analysis of two teacher contests involving pairwise comparisons of outcomes for N students yields the following existence result:

**Proposition 1:** Let $\sigma$ denote the variance of $\varepsilon_{ij}$. There exists $\overline{\sigma}$ such that $\forall \sigma > \overline{\sigma}$, both teachers choosing the socially optimal effort levels, $e_1 = \ldots = e_N = e^*$ and $t = t^*$, is a pure strategy Nash equilibrium of the two teacher contest.

**Proof of Proposition 1:** Define $\tilde{v}_i = \tilde{v}_{ij} - \tilde{v}_{ij'}$, and let $\tilde{H}(x) \equiv \Pr(\tilde{v}_i \leq x) = H(x/\sigma)$. Similarly, define $\tilde{h}(x) \equiv \frac{dH(x)}{dx} = \frac{1}{\sigma}h(x/\sigma)$. Note that

$$\tilde{h}(0) = \frac{1}{\sigma}h(0)$$

and

$$\tilde{H}(ae_i - ae^* + t - t^*) = \int_{-\infty}^{ae_i - ae^* + t - t^*} \tilde{h}(x) dx = \int_{-\infty}^{ae_i - ae^* + t - t^*} \frac{1}{\sigma}h(x/\sigma) dx$$

The teacher’s objective function is given by

$$\max_{e_j, t_j} NX_0 + (X_1 - X_0) \sum_{i=1}^{N} \tilde{H}(ae_{ij} - ae^* + t_j - t^*) - C(e_j, t_j) - U_0$$

If we set $X_1 - X_0 = R/\tilde{h}(0)$, this objective function reduces to

(8) $$\max_{e_j, t_j} NX_0 + R \sum_{i=1}^{N} \left[ \int_{-\infty}^{ae_{ij} - ae^* + t_j - t^*} \frac{h(x/\sigma)}{h(0)} dx \right] - C(e_j, t_j) - U_0$$

We first argue that the solution to this problem is bounded in a way that does not depend on $\sigma$. Since $h(\cdot)$ is unimodal with a peak at zero, it follows that $\frac{h(x/\sigma)}{h(0)} \leq 1$, and so

$$\int_{-\infty}^{ae_{ij} - ae^* + t_j - t^*} \frac{h(x/\sigma)}{h(0)} dx = \int_{-\infty}^{-ae^* - t^*} \frac{h(x/\sigma)}{h(0)} dx + \int_{-\infty}^{ae_{ij} - ae^* + t_j - t^*} \frac{h(x/\sigma)}{h(0)} dx \leq \int_{-\infty}^{-ae^* - t^*} \frac{h(x/\sigma)}{h(0)} dx + ae_{ij} + t_j$$

The objective function in (8) is thus bounded above by

$$NX_0 + R \int_{-\infty}^{-ae^* - t^*} \frac{h(x/\sigma)}{h(0)} dx + R \sum_{i=1}^{N} (ae_{ij} + t_j) - C(e_j, t_j) - U_0$$
Next, define the set \( U = \{ \mathbf{u} \in \mathbb{R}_{++}^{N+1} : \sum_{i=1}^{N+1} u_i^2 = 1 \} \). Any vector \((e_j, t_j)\) can be uniquely expressed as \( \lambda \mathbf{u} \) for some \( \lambda \geq 0 \) and some \( \mathbf{u} \in U \). Given our assumptions on \( C(\cdot, \cdot) \), for any vector \( \mathbf{u} \) it must be the case that \( C(\lambda \mathbf{u}) \) is increasing and convex in \( \lambda \) and satisfies the limit \( \lim_{\lambda \to \infty} \frac{\partial C(\lambda \mathbf{u})}{\partial \lambda} = \infty \). Since \( \lambda R \sum_{i=1}^{N} (\alpha e_{ij} + t_j) \) is linear in \( \lambda \), for any \( \mathbf{u} \in U \) there exists a finite cutoff \( \lambda^*(\mathbf{u}) \) such that \( \lambda R \left[ \sum_{i=1}^{N} \alpha u_i + u_{N+1} \right] - C(\lambda \mathbf{u}) < 0 \) for all \( \lambda > \lambda^*(\mathbf{u}) \). Since \( U \) is compact, \( \lambda^* = \sup \{ \lambda^*(\mathbf{u}) : \mathbf{u} \in U \} \) is well defined and finite. Given that \( \lambda R \left[ \sum_{i=1}^{N} \alpha u_i + u_{N+1} \right] - C(\lambda \mathbf{u}) \) for \( \mathbf{u} = 0 \), it follows that the solution to (8) lies in the bounded set \([0, \lambda^*]^{N+1}\).

Next, we argue that there exists a \( \sigma \) such that for \( \sigma > \sigma \), the Hessian matrix of second order partial derivatives for this objective function is negative definite over the bounded set \([0, \lambda^*]^{N+1}\). Define \( \pi(t, e_1, ..., e_N) \equiv R \sum_{i=1}^{N} \int_{-\infty}^{\infty} h(\frac{e_i - \alpha^* + t - t'}{\sigma}) \frac{h(x/\sigma)}{h(0)} \, dx \). Then the Hessian matrix is the sum of two matrices, \( \Pi - C \), where

\[
C \equiv \begin{bmatrix}
\frac{\partial^2 C}{\partial e_1^2} & \frac{\partial^2 C}{\partial e_2 \partial e_1} & \cdots & \frac{\partial^2 C}{\partial e_1 \partial e_N} \\
\frac{\partial^2 C}{\partial e_2 \partial e_1} & \frac{\partial^2 C}{\partial e_2^2} & \cdots & \frac{\partial^2 C}{\partial e_2 \partial e_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 C}{\partial e_1 \partial t} & \frac{\partial^2 C}{\partial e_2 \partial t} & \cdots & \frac{\partial^2 C}{\partial t^2}
\end{bmatrix}
\]

and

\[
\Pi \equiv \begin{bmatrix}
\frac{\partial^2 \pi}{\partial e_1^2} & \frac{\partial^2 \pi}{\partial e_2 \partial e_1} & \cdots & \frac{\partial^2 \pi}{\partial e_1 \partial e_N} \\
\frac{\partial^2 \pi}{\partial e_2 \partial e_1} & \frac{\partial^2 \pi}{\partial e_2^2} & \cdots & \frac{\partial^2 \pi}{\partial e_2 \partial e_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 \pi}{\partial e_1 \partial t} & \frac{\partial^2 \pi}{\partial e_2 \partial t} & \cdots & \frac{\partial^2 \pi}{\partial t^2}
\end{bmatrix}
\]

Since the function \( C(\cdot) \) is strictly convex, \(-C\) must be a negative definite matrix. Turning to \( \Pi \), we have

\[
\Pi = \frac{R}{\sigma h(0)} \times \begin{bmatrix}
\alpha^2 h'(\frac{\alpha e_1 - \alpha^* + t - t'}{\sigma}) & \cdots & 0 \\
0 & \alpha^2 h'(\frac{\alpha e_2 - \alpha^* + t - t'}{\sigma}) & \cdots & \alpha h'(\frac{\alpha e_2 - \alpha^* + t - t'}{\sigma}) \\
\vdots & \ddots & \ddots & \vdots \\
\alpha h'(\frac{\alpha e_1 - \alpha^* + t - t'}{\sigma}) & \cdots & \alpha h'(\frac{\alpha e_2 - \alpha^* + t - t'}{\sigma}) & \sum_{i=1}^{N} h'(\frac{\alpha e_i - \alpha^* + t - t'}{\sigma})
\end{bmatrix}
\]
But $\tilde{h}'(x) = \frac{1}{\sigma^2} h'(x/\sigma)$. For a fixed $x$, all of the elements in $\Pi$ converge to multiples of $h'(0)$ as $\sigma \to \infty$, which is just 0. Hence, within the bounded set $[0, \lambda^*]^{N+1}$, we have $\Pi \to 0$ as $\sigma \to \infty$. Since $C$ is positive definite and $\Pi \to 0$, it follows that there exists a $\bar{\sigma}$ such that for all $\sigma > \bar{\sigma}$, the matrix $\Pi - C$ is negative definite for all values of $(e_j, t_j) \in [0, \lambda^*]^{N+1}$. Hence, the objective function is strictly concave in the region that contains the global optimum, ensuring the first-order conditions are both necessary and sufficient to define a global maximum. ■

**Proposition 2:** In the two teacher contest described here, if a pure strategy Nash equilibrium exists, it involves both teachers choosing the socially optimal effort levels, $e_1 = ... = e_N = e^*$ and $t = t^*$.

**Proof of Proposition 2:**

We begin our proof by establishing the following Lemma:

**Lemma:** Suppose $C(\cdot)$ is a convex differentiable function which satisfies standard boundary conditions concerning the limits of the marginal costs of each dimension of effort as effort on each dimension goes to 0 or $\infty$. Then for any positive real numbers $a_1, ..., a_N$ and $b$, there is a unique solution to the system of equations

$$\frac{\partial C(e_1, ..., e_N, t)}{\partial e_i} = a_i \quad \text{for } i = 1, ..., N$$

$$\frac{\partial C(e_1, ..., e_N, t)}{\partial t} = b$$

**Proof:** Define a function $bt + \sum_{i=1}^{N} a_i e_i - C(e_1, ..., e_N, t)$. Since $C(\cdot)$ is strictly convex, this function is strictly concave, and as such has a unique maximum. The boundary conditions, together with the assumption that $a_1, ..., a_N$ and $b$ are positive, ensure that this maximum must be at an interior point. Because the function is strictly concave, this interior maximum and the solution to the above equations is unique, as claimed. ■

Armed with this lemma, we can demonstrate that any pure strategy Nash equilibrium of the two teacher contest involves both teachers choosing the socially optimal effort levels. Note that, given any pure strategy Nash equilibrium, both teacher’s choices will satisfy the first order conditions for a best response to the other teacher’s actions. Further, since $h(\cdot)$ is symmetric, we know that given the effort choices of $j$ and $j'$,

$$h(\alpha(e_{ij} - e'_{ij}) + t_j - t_{j'}) = h(\alpha(e_{ij'} - e_{ij}) + t_j - t_{j'}) = h(0)$$

In combination, these observations imply that any Nash equilibrium strategies, $(e_{1j}, ... e_{Nj}, t_j)$ and $(e_{1j'}, ... e_{Nj'}, t'_{j'})$, must satisfy:

$$h(0) \frac{\partial C(e_{1j}, ..., e_{Nj}, t_j)}{\partial e_{ij}} = R(h(\alpha(e_{ij} - e_{ij'}) + t_j - t_{j'}) = R(h(\alpha(e_{ij'} - e_{ij}) + t_j - t_{j}) = h(0) \frac{\partial C(e_{1j'}, ..., e_{Nj'}, t_{j'})}{\partial e_{ij'}}$$
\[
h(0) \frac{\partial C(e_1, \ldots, e_N, t_j)}{\partial t_j} = RNh(\alpha(e_{ij} - e_{ij}') + t_j - t_j') = RNh(\alpha(e_{ij} - e_{ij}) + t_j - t_j) = h(0) \frac{\partial C(e_1', \ldots, e_N', t_j')}{\partial t_j'}
\]

Our lemma implies that these equations cannot be satisfied unless \(e_{ij} = e_{ij}' = e^*\) for all \(i = 1, ..., N\) and that \(t_j = t_j' = t^*\). The only pure-strategy equilibrium possible in our two teacher contests is one where teachers invest the classroom instruction effort and common level of tutoring that are socially optimal. ■