Question 1

1. Bidding her own valuation is a weakly dominant strategy for \( i = 1, 2 \). The proof is the same as the proof for a 2nd price auction without collusion.

2. This proof is identical to the standard proof that bidding your own valuation is a weakly dominant strategy in a 2nd price auction.

3. For \( i = 3, 4 \), \( \hat{m}_i(x_i) = m_i(x_i) \): conditional on winning, a non-cartel member always pays the second highest valuation, regardless of whether bidders 1 and 2 collude. For \( i = 1, 2 \), \( \hat{m}_i(x_i) < m_i(x_i) \): under collusion, conditional on winning a cartel member pays either the second highest valuation or the third highest valuation (she pays the third highest valuation when the other cartel member has the second highest valuation; otherwise she pays the second highest valuation).

4. If bidder \( j = 3 \) joins the bidding ring, the seller’s profits go down in expectation. The expected payments of 1, 2 or 3 conditional on winning go down. The cartel member with the highest valuation now pays the valuation of bidder 4, because everyone else is bidding 0. If \( i \in \{1, 2\} \) has the highest valuation and wins the auction, she pays less than before when \( v_4 < v_3 \) and pays the same if \( v_4 \geq v_3 \). If 3 has the highest valuation and wins the auction, she pays less than before if \( v_4 < \max\{v_1, v_2\} \), and pays the same if \( v_4 \geq \max\{v_1, v_2\} \).
Question 2
(i) FOCs for ex ante efficiency are, for all $s$ and $s'$,

$$\frac{\pi_s}{\pi_{s'}} \exp\left(-\alpha_A (x_{As}^* - x_{As'}^*)\right) = \frac{\pi_s}{\pi_{s'}} \exp\left(-\alpha_B (x_{Bs}^* - x_{Bs'}^*)\right) \implies$$

$$-\alpha_A (x_{As}^* - x_{As'}^*) = -\alpha_B (x_{Bs}^* - x_{Bs'}^*).$$

Thus (*) follows from $\alpha_A > \alpha_B > 0$.

(ii) From FOCs, perfect insurance for one implies same for the other consumer and thus also that the aggregate endowment is constant.

(iii) Yes, if security markets are incomplete. For example, if there are no securities, in which case there is no trade and (*) is violated if initial endowments satisfy the opposite inequality. If markets are complete and the other conditions of the "equivalence theorem" are satisfied, then a Radner equilibrium allocation is also ex ante efficient and hence satisfies the above conditions.

(iv) Since there is only one physical good and and vNM utility indices are strictly increasing (by strong monotonicity), every ex post allocation is ex post efficient.
Question 3

(a) Without loss of generality, a landlord with plot quality \( \theta \) will lease out his land if the rent \( r \geq y(\theta) \) and will not if \( r < y(\theta) \). Hence a tenant will demand a plot if \( r < E[t(\theta)|r \geq y(\theta)] \), and not if the inequality is reversed. They will be indifferent if \( r = E[t(\theta)|r \geq y(\theta)] \). Since \( T > 1 \), we cannot have an equilibrium where \( r < E[t(\theta)|r \geq y(\theta)] \), since demand will exceed supply. If the inequality is reversed then demand will be zero, while supply will be positive whenever \( r \geq y(\theta) \). Therefore the definition of a competitive equilibrium is the same as it is in settings with CRS: a rent \( r \geq y(\theta) \) such that \( r = E[t(\theta)|r \geq y(\theta)] \).

(b) Let \( T(r) \equiv E[t(\theta)|r \geq y(\theta)] \), defined over \([y(\theta), \infty)\). The assumptions made ensure that \( T \) is strictly increasing until \( y(\hat{\theta}) \) and constant thereafter. So a sufficient condition for an equilibrium with some rental transactions is that \( T(y(\hat{\theta})) > y(\hat{\theta}) \).

(c) \( T(\hat{\theta}) < \hat{\theta} \)

(d) Plots that will be leased will be those with \( r > y(\hat{\theta}) \), or \( \theta < y^{-1}(r) \).

(e) There is a unique, interior equilibrium if \( T(y(\hat{\theta})) = \alpha \hat{\theta} + \beta > \hat{\theta} \), \( T(y(\hat{\theta})) = \alpha \hat{\theta} + \beta < \hat{\theta} \) and \( T'(r) < 1 \) for all \( r \in (\theta, \bar{\theta}) \). In this case \( T(r) = \alpha r + \beta + \beta \), so the last condition reduces to \( \alpha < 2 \).

Question 4

(a) \( \rightarrow \): Suppose \( x \in B' \subset B \) and \( x \in c(B) \). By nonemptiness of choice, there exist some \( y \in c(B') \). But since \( x, y \in B \cap B' \), WARP implies \( x \in c(B') \), as desired.

\( \leftarrow \): Suppose \( x, y \in B \cap B' \), \( x \in c(B) \) and \( y \in c(B') \). By IIA, \( x, y \in c(B \cap B') \). Since choice is single-valued, it must be that \( x = y \) and thus \( x \in c(B') \) as desired.

(b) (i) Consider the prices \( (p_1, p_2) \) given by \((1, \frac{1}{2})\) and \((1, 2)\), which corresponds to two nested budget sets. In the first case the agent spends all her income on \( x \) and in the second all on \( y \), which violates IIA.

(ii) Consider a change from \((p, w)\) to \((p', w')\) where \( w' = p' \cdot x(p, w) = p'_1(\frac{w}{2p_1}) + p'_2(\frac{w}{2p_2}) \). Then:

\[
\Delta p \cdot \Delta x = (p_1 - p'_1)(\frac{w}{2p_1} - \frac{w'}{2p'_1}) + (p_2 - p'_2)(\frac{w}{2p_2} - \frac{w'}{2p'_2})
\]

\[
= (p_1 - p'_1)(\frac{w}{2p_1} - \frac{p'_1(\frac{w}{2p_1}) + p'_2(\frac{w}{2p_2})}{2p'_1}) + (p_2 - p'_2)(\frac{w}{2p_2} - \frac{p'_1(\frac{w}{2p_1}) + p'_2(\frac{w}{2p_2})}{2p'_2})
\]

\[
= -\frac{1}{4} \frac{w}{p_1 p_2 p_1 p_2} (p'_1 p_2 - p'_2 p_1)^2 < 0
\]