1. Growth and inequality

Consider an economy that is composed of $N$ consumers, indexed by $i = 1, 2, \cdots, N$, and a single representative firm. Each consumer maximizes a lifetime utility function given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

where

$$u(c) = \log(\alpha + c), \quad \forall c > 0 \quad \text{and} \quad \forall(\alpha + c) > 0$$

with $\alpha < 0$ and $\beta \in (0, 1)$.

The representative firm has a technology for producing the homogeneous consumption good using capital. The production function is given by

$$y_t = f(k_t)$$

where $y_t$ is per capita output and $k_t$ is per capita capital at the beginning of period $t$. $f(\cdot)$ is strictly increasing, strictly concave and twice continuously differentiable.

Markets are complete and the price of the consumption good in period $t$ in terms of the consumption good in period 0 is $p_t$.

Define $d_t = f(k_t) + (1 - \delta)k_t - k_{t+1}$ as the per capita dividend of the firm where $\delta \in (0, 1)$ is the depreciation rate of capital. Assume that the firm accumulates the capital stock and household $i$ owns a share $s_t^i$ of the firm with $\sum_{i=1}^{N} s_t^i = 1$.

Let $N\bar{w}_t$ be the total value of the firm. As the firm is the only asset, $\bar{w}_t$ is also the per capita wealth in the economy and the wealth of household $i$ is $w_t^i = s_t^i N\bar{w}_t$.

The initial capital stock and wealth shares are given exogenously.

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1. (5 points) How is $\bar{w}_t$ determined?
2. (20 points) Prove that there are functions \( a(\cdot) \) and \( b(\cdot) \) such that

\[
c_t^1 = a(p_t, p_{t+1}, \cdots) + b(p_t, p_{t+1}, \cdots)w_t^1.
\]

Use the following steps: (a) specify the household’s problem; (b) find the FOC for \( c_t^1 \); (c) derive the functions using the prior results; and (d) comment on the economic interpretation of \( \alpha < 0 \) and its implications for these functions.
3. (10 points) Define a competitive equilibrium for this economy.
4. (10 points) Derive the following three results: (1) The growth of individual wealth:

\[
\frac{w_{t+1}^i}{w_t^i} = \frac{p_t}{p_{t+1}} \left[ 1 - \frac{c_t^i}{w_t^i} \right]
\]

(2) The growth of average wealth:

\[
\frac{\bar{w}_{t+1}}{\bar{w}_t} = \frac{p_t}{p_{t+1}} \left[ 1 - \frac{\bar{c}_t}{\bar{w}_t} \right],
\]

where we define per capita consumption as \( \bar{c}_t \equiv \frac{1}{N} \sum_{i=1}^{N} c_t^i \). (3) The growth of an individuals’ wealth share:

\[
\frac{s_{t+1}^i}{s_t^i} = \frac{w_{t+1}^i}{\bar{w}_{t+1}} \cdot \frac{w_t^i}{\bar{w}_t}.
\]
5. (25 points) For the case $N = 1$, prove that $k_t$ will converge monotonically to a steady state $k^*$. 
6. (10 points) For the case $N > 1$, discuss intuitively why $k_t$ will converge monotonically to a steady state $k^*$. 
7. (20 points) Assume $N = 2$, $s_t^1 < s_t^2$, and $k_t < k^*$. Using your results from part (2) and part (4), prove that $s_{t+1}^1 < s_t^1$, i.e., that the poor will become relatively poorer during the transition process. Link this result to your discussion of $\alpha < 0$ in part (2).
Question 2. Investment in industry equilibrium

Suppose that there is a competitive industry with many firms $j = 1, 2, \ldots, J$. Consider an individual firm $j$ which produces output $y_{jt}$ according to the constant returns to scale production function

$$y_{jt} = Ak_{jt}$$

where $k_{jt}$ is the predetermined quantity of capital and $A$ is the constant level of productivity. The firm's capital stock evolves according to

$$k_{j,t+1} = (1 - \delta)k_{jt} + \phi(z_{jt})k_{jt}$$

where $z_{jt}$ is the ratio of the firm $j$'s investment expenditures to its capital stock ($\frac{i_{jt}}{k_{jt}}$) and $\phi$ is an increasing strictly concave function with $\phi(0) = 0$ that is differentiable as required.

The firm seeks to maximize the present value

$$E_t \sum_{i=0}^{\infty} \beta^i[p_{t+i}Ak_{j,t+i} - z_{j,t+i}k_{j,t+i}]$$

where $p_t$ is the price of the industry's product, $p_t Ak_{jt}$ is its revenue, and $z_{jt}k_{jt} = i_{jt}$ is its investment expenditure. The discount factor $\beta$ is between 0 and 1.

2.1 Firm-level investment (50 points)

Suppose that the price of the product is a function of a stationary Markov process $\zeta_t$,

$$p(\zeta_t)$$
(a) (15 points) Letting $z$ be the firm’s choice variable, what is the Bellman equation for the dynamic optimization problem of firm $j$? (Use $v()$ to denote the value function and clearly identify the state variables).\footnote{This is a model in which the size of the firm can grow without bound. To make sure that the firm’s value is finite, it is necessary to impose a condition on how fast it can grow. Assume that there is some rate of investment $\bar{z}$ which maximizes the growth rate of capital, $\gamma = (1 - \delta) + \phi(\bar{z})$, and that $\beta \gamma < 1$.}
(b) (10 points) What is the first order condition for the optimal rate of investment $z = i/k$?
(c) (5 points) What envelope theorem result governs $v_k(k, \xi)$?
(d) (10 points) For an individual firm, what theory of investment can be derived for this model, using your answer to part (b)?
(e) (10 points) This model has the property that firm value is proportional to $k$ for any policy $z(\gamma)$. What features of the production and investment technologies are important for this result? If firm value is proportional to $k$, show that the FOC implies that the optimal policy is independent of $k$. 
2.2 Industry equilibrium [50 points]

Let \( Y_t = \sum_{j=1}^{J} y_{jt} \), \( I_t = \sum_{j=1}^{J} i_{jt} \) and \( K_t = \sum_{j=1}^{J} k_{jt} \) be industry-wide "aggregates" of output, investment and capital respectively. Let \( Z = I/K \).

When industry supply is equal to industry demand, we now assume that the price of the industry's product obeys

\[
p_t = p(Y_t, \zeta_t)
\]

where \( \zeta_t \) is a demand shifter that is a positive Markov process. Assume further that the price function is such that \( p > 0 \) for all \( Y > 0 \) and \( \zeta > 0 \). Assume that every firm is sufficiently small that it takes price as unrelated to its own actions, an assumption that can be made exact by using a continuum of firms.

A rational expectations competitive equilibrium in this economy consists of a value function \( v(K, \zeta) \) and a policy function \( z(K, \zeta) \), that is the same for each firm \( j \) together with a law of motion for aggregate capital \( K' = H(K, \zeta) \) which satisfies

\[
H(K, \zeta) = (1 - \delta)K + \phi(Z(K, \zeta))K
\]

with

\[
Z(K, \zeta) = z(K, \zeta)
\]

\[
Y = AK
\]

It is thus an equilibrium in which the distribution of capital across firms is irrelevant to the equilibrium outcomes.
(f) (10 points) Explain why the value function and policy function now depend on $K$, but plausibly would continue to have the same properties with respect to $k$ as in part 1 above.
(g) (5 points) In what sense is a fixed point problem an essential feature of this REE? How can it be avoided computationally by using results from above?
(h) (10 points) From the description of the problem and from your work above, extract three equations that comprise a nonlinear rational expectations model in the variables $Z, K$ and a third variable which you need to determine. Indicate how you are evaluating behavioral equations for the individual firm in order to study market equilibrium.
(i) (10 points) Consider a version of the model in which $\varsigma$ is constant for all time. In the stationary state, show that: (i) the market equilibrium price is supply-determined, i.e., independent of the level of $\varsigma$; and (ii) the quantity of capital is demand-determined.
(j) (10 points) If the model is linearized or loglinearized around the stationary point, then how many stable eigenvalues would be required if there is to be a unique, stable rational expectations solution? How would this condition change if an equation for aggregate output,

$$ Y = AK $$

was added to the model.
(k) (5 points) The industry model does not display diminishing returns to capital. Identify the economic mechanism in the model which would lead to stable dynamics of capital around its stationary point.
3. Asset Bubbles in OLG Models with Production

Consider an OLG model. Time is denoted by $t = 1, 2, \ldots$. Population as measured by the number $N_t$ of agents in each generation grows at rate $g$, $N_{t+1} = g N_t$. The initial old agents with $N_0$ members are endowed with one unit of an intrinsically useless asset and $K_1$ units of capital, which are rented to firms. Each of them consumes equally using the reselling value of the asset and the return from his capital. Agents in other generations do not have any endowment and work only when young, supplying inelastically one unit of labor and earning a real wage of $w_t$. They consume part of their wage income $c^t_t$, buy the useless asset $m_t$ at price $P_t$, and save $x_t$. The saving of the young in period $t$ generates the capital stock that is used to produce output in period $t + 1$ in combination with the labor supplied by the young generation of period $t + 1$.

A young agent at date $t$ solves the following problem:

$$\max \ln (c^t_t) + \beta \ln (c^t_{t+1})$$

subject to

$$c^t_t + x_t + P_t m_t = w_t,$$
$$c^t_{t+1} = R_{t+1} x_t + P_{t+1} m_t,$$

where $R_{t+1}$ is the gross interest rate paid on saving held from period $t$ to period $t + 1$.

Firms are identical and hire labor and rent capital from competitive factor markets. Firms combine capital and labor to produce output using a constant-returns-to-scale production function, $F(K, N)$. Capital depreciates at rate $\delta \in [0, 1]$. They solve the following problem:

$$\max_{K_t, N_t} F(K_t, N_t) + (1 - \delta) K_t - w_t N_t - R_t K_t,$$

Define

$$f(k) = F(k, 1) + (1 - \delta) k,$$

where $k = K/N$. Assume that $F(K, N) = K^\alpha N^{1-\alpha}$.

The market clearing conditions are $K_{t+1} = N_t x_t$ and $N_t m_t = 1$.

(a) (10 points) Solve an agent’s optimization problem and derive the optimal consumption $c^t_t$ as a function of $w_t$. Note that $P_t$ can be equal to zero for all $t$ or can be positive for all $t$. Please solve for both cases.
part (a) continued
(b) (5 points) Write down the firm’s first-order conditions and derive the wage rate $w_t$ and the capital return $R_t$ as functions of $k_t$, $w(k_t)$ and $R(k_t)$, where $k_t$ is the capital–labor ratio, $t \geq 1$. 
(c) (10 points) Define the bubble per capita $b_t = P_t m_t$. Show that a competitive equilibrium can be characterized by a system of two nonlinear differences equations in $(k_t, b_t)$. Derive this system.
(d) (5 points) Compute the (positive) steady-state capital stock in the Diamond equilibrium in which $b_t = 0$ or $P_t = 0$. Is this steady state stable?
(e) (5 points) Give conditions such that the Diamond equilibrium is dynamically inefficient.
(f) (10 points) Give conditions such that a bubbly steady state equilibrium in which $k_t = k_b$ and $b_t = b > 0$ for all $t$ can exist. Derive the bubbly steady state $(k_b, b)$. Is it dynamically efficient?
(g) (10 points) On the \((k_t, b_t)\) plane plot the phase diagram of the equilibrium system and study the stability of the bubbly steady state and the Diamond steady state. Analyze multiplicity of equilibria. You can use vector fields to justify your solution.
(h) (20 points) Now suppose that the bubble can crash and agents have rational expectations. All agents believe that the bubble can collapse with probability $1 - \pi$, $\pi \in [0, 1]$. Once the bubble collapses it cannot reemerge. That is, if $P_t > 0$, then $P_{t+1} > 0$ with probability $\pi$ and $P_{t+1} = 0$ with probability $1 - \pi$ for all $t$. But if $P_{t+1} = 0$ then $P_j = 0$ for all $j > t + 1$. Derive first-order conditions for an agent’s optimization problem. Solve for the policy functions of consumption $c^*_t$, saving $x_t$, and asset holdings $m_t$. Explain why a bubble in the same asset cannot reemerge in the future once it collapses.
(i) (10 points) Derive the equilibrium system for \((k_t, b_t)\) with a stochastic bubble.
(j) (15 points) A stationary equilibrium with a stochastic bubble is defined as an equilibrium in which all variables are constant before the bubble collapses. Let $k_s$ and $b_s$ denote the constant equilibrium values of per-capita capital stock and bubble before it collapses. Show that a unique stationary equilibrium with a stochastic bubble exists if and only if

$$\pi > \pi^* \equiv \frac{R(k_D)}{g}.$$ 

When $\pi^* < \pi < 1$, show that $k_b < k_s < k_D$. 
4. Optimal monetary policy with and without commitment

Consider an economy described by the following log-linear dynamic IS and AS curves:

\[ IS : \quad x_t = \sigma r_t + E_t f x_{t+1} g + g_t \]
\[ AS : \quad \pi_t = \kappa x_t + \beta E_t f \pi_{t+1} g + u_t \]

where \( f x_{t+1} g = i_t \).

\( i_t \) denotes the ex ante real interest rate, \( x_t \) denotes the output gap, \( i_t \) is the nominal interest rate, and \( \pi_t \) is the inflation rate. Further assume that \( u_t \) and \( g_t \) satisfy:

\[ u_t = \rho_u u_{t-1} + \varepsilon_t^u \]
\[ g_t = \rho_g g_{t-1} + \varepsilon_t^g \]

with \( \varepsilon_t^u, \varepsilon_t^g \) iid and 0. Assume the objective of the monetary authority is to maximize

\[ \frac{1}{2} E_t \sum_{i=0}^\infty \beta^i (x_{t+i})^2 + \pi_{t+i}^2 \]

Part A) Assume that the monetary authority cannot commit to a monetary policy rule.

1. (15 points) Write down the optimization problem of the monetary policy maker under discretion and derive the optimality conditions of the policy maker. What is the relationship between inflation and real activity implied by the optimality conditions?
2. (15 points) Derive expressions for the equilibrium inflation rate and output as functions of the underlying shock processes. What effect does a shock to $u_t$ and $g_t$ have on each of these variables? Explain.
3. (10 points) Under what circumstances does the monetary authority succeed at completely stabilizing inflation? Provide intuition for these results.
4. (15 points) Show that the interest rate satisfies an equation of the form:

\[ i_t = \gamma_n E_t \pi_{t+1} + \gamma_g g_t \]

and derive algebraic expressions for \( \gamma_n \) and \( \gamma_g \). Provide an economic interpretation for the sign and size of these coefficients in terms of adherence to the Taylor principle and the tradeoffs faced by monetary policy.
Part B) Now suppose that the monetary authority commits to a monetary policy rule such that 

\[ x_t = \omega u_t \]

for all \( t \) where the parameter \( \omega \) is chosen by the monetary authority.

1. (15 point) Derive a condition that determines the optimal choice of \( \omega \) and express this value in terms of the parameters of the model.
2. (15 points) Show that the optimal choice of \( \omega \) implies the following relationship between the output gap and inflation:

\[
x_t = \frac{\kappa}{\alpha(1 - \beta \rho_u)} \pi_t
\]

Compare this relationship to what is obtained in the case of no commitment. How does being able to commit to the policy effectively alter the tradeoff between inflation and output in the model?
3. (15 points) Show that adherence to the monetary policy rule \( x_t = \omega y_t \) with \( \omega \) chosen optimally implies that the nominal interest rate again satisfies an equation of the form:

\[
\hat{\pi}_t = \gamma_i \pi_{t+1} + \gamma_y y_t
\]

How do the coefficients in the Taylor rule differ between the model with versus without commitment? Is the nominal interest rate more or less responsive to expected inflation under commitment relative to the case without commitment? Is the nominal interest rate more or less responsive to shocks to \( y_t \)? Provide economic intuition for your results.