Instructions. There are two parts to this exam. Questions in Part A require about 15 minutes. Questions in Part B require about 30 minutes. The weight of each question is proportional to the allocated time that is stated. The exam is designed to take about three hours to be completed.

Budget your time carefully. Try to avoid leaving an answer blank. That may be costly.

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Question A-1
An infinitely lived representative household has preferences of streams of a single consumption good that are ordered by
\[
\sum_{t=0}^{\infty} \beta^t u(c_t), \quad \beta \in (0, 1)
\]
where \( u \) is strictly concave and twice continuously differentiable. The technology is:
\[
c_t + x_t \leq f(k_t)n_t \\
k_{t+1} = (1 - \delta)k_t + \psi_t x_t
\]
where \( \psi_t = 1 \) for \( t < 4 \) and \( \psi_t = 2 \) for \( t \geq 4 \). Here \( f(k_t)n_t \) is output, where \( f' > 0, f'' < 0 \), \( k_t \) is capital and \( n_t \) is labor input. The household supplies one unit of labor inelastically. The initial capital stock \( k_0 \) is given and is owned by the representative household. There is no uncertainty in this economy.

A-1.a Formulate the planning problem for this economy in the space of sequences and form the pertinent Lagrangian. Find a formula for the optimal steady-state level of capital for a constant value of \( \psi \). How does it depend on \( \psi \)?
A-1.b
Formulate the planning problem for this economy recursively (i.e. write a Bellman equation for the planner). Be careful to give the complete description of the state vector and its law of motion.
Question A-2
Consider a one period complete markets economy, in which all agents are risk averse. Prior to the realization of uncertainty, Arrow-Debreu securities are traded and complex claims are priced so that there are no arbitrage profits. The price of a contingent claim to one unit of goods only in state $s$ is $\lambda_s$ for $s = 1, 2, ..., S$. Focus on two complex securities, 1 and 2, which are claims to $\{d_{1s}\}_{s=1}^{S}$ and $\{d_{2s}\}_{s=1}^{S}$ respectively. Suppose that the probability of state $s$ is $\pi_s$, that the expected payout on both securities is $\mu$, and also that the variance of payouts is larger for security 1 than for security 2. What is the value of each security? Do you agree or disagree with the following statement? "Unless contingent claims prices are actuarially fair ($\pi_s = \lambda_s$), the higher variability of returns on security 1 means that it is more risky than security 2 and thus it must have lower value".
Suppose the only assets in the economy are infinitely-lived trees. There are 2 distinct trees per capita indexed $i = 1..2$. Each tree produces an iid dividend stream $D_{it}$. Aggregate output per capita $D_t = \sum_{i=1}^{2} D_{it}$ which is exogenous and cannot be stored. Trees are entirely owned by consumers who have identical preferences, and initial portfolios – each consumer owns the same initial number of shares $S_{it}$ in each tree. Let $P_{it}$ equal the ex-dividend price of one share in the $i^{th}$ tree in period $t$. The representative consumer maximizes

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_{it}^{1-\gamma}}{(1-\gamma)}, \quad 0 < \beta < 1.$$ 

**A-3.a** Write down the optimization problem for the consumer. Derive the optimality conditions.
A-3.b
Provide a closed-form expression for the price of a tree as a function of dividends in the economy
A-3.c
Are risk premia on individual trees positive or negative in this economy? Explain.
**Question A-4**

Time is continuous. There is a continuum of firms of mass one. Each firm can post a vacancy at the cost $c$ per unit of time. If all firms post a vacancy, the mass of new jobs created per unit of time is $h(1 - n_t)$, where $h$ is a fixed parameter ($h > 0$) and where $n_t$ is the mass of employment at time $t$. All vacancies are filled with the same probability, which is therefore $h(1 - n_t)$. A job is terminated according to a Poisson process with an exogenous probability $\delta$ per unit of time. The payoff of a job for a firm is $\alpha n_t$ per unit of time, where $\alpha$ is a parameter. (There is a positive externality between the level of employment and the value of each job). The rate of discount of the firm is $\rho$ (per unit of time).

**A-4.a** Let $V_t$ be the value of a job for a firm. Determine the condition on $V_t$, $c$, $h$ and $n_t$ such that there is hiring.
A-4.b
Assume that $h = \delta = \rho, \alpha = 8, \ c = 3/4$. Show that there is an equilibrium steady state with positive employment. Determine this level. (If you have free time after doing the rest of the exam, you may show that there is another equilibrium. No penalty for not answering this multiplicity).
Question B-1
An economy is populated by a continuum of infinitely lived consumers of types $j \in \{0, 1\}$, with a measure one of each. There is one non-storable consumption good arriving in the form of an endowment stream owned by each consumer. Specifically, the endowments are

$$y^0_t(s_t) = (1 - s_t)\bar{y}^0$$
$$y^1_t(s_t) = s_t\bar{y}^1,$$

where $s_t$ is a two-state time-invariant Markov chain valued in $\{0, 1\}$ and $\bar{y}^0 < \bar{y}^1$. The initial state is $s_0 = 1$. Transition probabilities are denoted $\pi(s'|s)$ for $s$ and $s' \in \{0, 1\}$. The aggregate endowment is $y_t(s_t) \equiv (1 - s_t)\bar{y}^0 + s_t\bar{y}^1$. Thus the economy fluctuates stochastically between recessions $y_t(0) = \bar{y}^0$ and booms $y_t(1) = \bar{y}^1$. In a recession, the aggregate endowment is owned by type 0 consumers, while in a boom it is owned by type 1 consumers. A consumer orders consumption streams according to

$$U(c^j) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t|s_0) c^j_t(s^t)\frac{(s^t)^{1-\gamma}}{1 - \gamma},$$

where $s^t = (s_t, s_{t-1}, \cdots, s_0)$ is a history of the state up to time $t$, $\beta \in (0, 1)$ is the discount factor, and $\gamma > 0$ is the coefficient of relative risk aversion.

B-1.a Define a competitive equilibrium with date 0 trading.
B-1.b
Compute the price system \( \{q^0_t(s^t)\}_{t=0}^{\infty} \) and the equilibrium allocation \( \{c^j_t(s^t)\}_{t=0}^{\infty} \), for \( j \in \{0, 1\} \).
B-1.c
Find a utility function $\bar{U}(c) = E_0 \left( \sum_{t=0}^{\infty} \beta^t u(c_t) \right)$ such that the price system $\{q_t^0(s^t)\}_{t=0}^{\infty}$ and the aggregate endowment is an equilibrium of the single agent economy with the given preferences. How does your answer depend on the initial distribution of endowments among the two agents?
B-1.d
How would your answer to part change if at each node $s^t$, agents can only trade two risk-free assets, namely, a one-period risk-free bond that pays one unit of the good for sure at date $t + 1$ and a two-period risk-free bond that pays one unit of the good for sure at $t + 2$?
Throughout history, governments have conscripted labor to fight wars, to build monuments, etc. Consider the fixed labor neoclassical growth model with constant technology and with a total labor stock of $N$. Assume that saving is endogenously determined, with the utility discount factor $\beta$ being between 0 and 1.

Suppose that there is initially no government use of outputs or inputs.

**B-2.a** How are the long-run levels of the real interest rate, the real wage, consumption, investment and capital determined?
B-2.b
Suppose that, in a surprise event, the government conscripts a fraction $\theta$ of the labor stock permanently and uses this on activities which do not shift the private utility or production functions. How are the long-run levels of the real interest rate, the real wage, consumption, investment, and capital affected?
B-2.c
What are the short-run effects of this conscription on the real interest rate, the real wage, consumption, investment and capital?
Student ID:

**Question B-3**

A representative household chooses consumption goods, labor supply and real balances \((C_t, N_t, M_t/P_t)\) to maximize

\[
E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left( \log(C_t) + \log\left(\frac{M_t}{P_t}\right) - \frac{1}{1+\sigma}N_t^{1+\sigma} \right) \right]
\]

subject to

\[
P_tC_t + M_t + S_tB_t = W_tN_t + M_{t-1} + \Pi_t + B_{t-1}
\]

where \(W_t\) is the nominal wage, \(P_t\) is the aggregate price index, and \(\Pi_t\) represents nominal profits from the production sector, which households take as given. \(S_t = 1/(1+i_t)\) is the discount price of the nominal bond. Final goods output is produced by a competitive sector using the CES production function:

\[
Y_t = \left[ \int_0^1 Y_t(z)^{\frac{\epsilon-1}{\epsilon}} \frac{dz}{z^{\frac{\epsilon}{\epsilon-1}}} \right]^{\frac{\epsilon}{\epsilon-1}}.
\]

Wholesale producer \(z\) is the monopoly supplier of good \(z\). Let \(P_t(z)\) denote the price of wholesale good \(z\). The production function for such a producer satisfies,

\[
Y_t(z) = A_tN_t(z)^\alpha
\]

where \(0 < \alpha \leq 1\) and \(A_t\) indexes the aggregate level of technology. Wholesale good producers choose prices one period in advance to maximize

\[
E_{t-1}\{A_t \left( \frac{P_t(z)}{P_t} Y_t(z) - \frac{W_t}{P_t(z)} N_t(z) \right) \}
\]

subject to the demand curve:

\[
Y_t(z) = \left( \frac{P_t(z)}{P_t} \right)^{-\epsilon} Y_t
\]

where \(MC_t(z)\) denotes real marginal cost and \(A_t = 1/C_t\) is a discount factor common to all firms. Given \(P_t(z), N_t(z)\) is chosen at time \(t\) to meet demand at that price. Assume that technology and money supply are random walks in logs:

\[
\log A_t = \log A_{t-1} + \varepsilon_t
\]

and

\[
\log M_t = \log M_{t-1} + \mu_t
\]

where \(\varepsilon_t\) and \(\mu_t\) are mean-zero iid processes. Finally, consider a symmetric equilibrium where identical firms choose identical prices.
Derive the optimality condition for firms in this economy. Show that if $\alpha = 1$ these conditions imply that firms set a price as a constant markup over expected marginal cost.
B-3.b
Derive the optimality conditions for households and show that these conditions imply a money demand curve of the form:

\[ \frac{M_t}{P_t} = \Phi(i_t)C_t \]

Is the function \( \Phi(i_t) \) an increasing or decreasing function of the nominal interest rate \( i_t \)?
B-3.c
Show that the equilibrium of this economy is characterized by a constant velocity of money.
Consider a 1% shock to technology. Characterize the effect on output, employment and inflation, both the immediate impact as well as the effect on the future path. Be explicit, i.e. what are the short and long-run elasticities of response to the shock? Provide intuition for these results.
Define the natural rate of output as the level of output that would occur if prices are fully flexible. If the goal of monetary policy is to stabilize output around the natural rate of output, how should it conduct monetary policy? Provide an explicit characterization of the monetary policy that achieves this goal.
Question B-4
There is an OG economy where agents live 2 periods and have the utility function

$$U(c, \ell) = (1 - \alpha)\log(c_1) + \alpha \log(c_2).$$

where $c_t$ is the consumption in the period $t$ of the life of an agent. Each went is endowed with a fixed amount of time for work in the first period. The production technology is represented by the linear function $Y = RK + L$ where $K$ and $L$ are the level of capital and labor and $R$ is a constant. There is no depreciation of capital.

There is labor augmenting technological change at the rate $g$ from one period to the next. The economy is on a balanced growth path.

B-4.a Analyze the impact of a decrease of the growth rate $g$ on the capital labor ration in the long run.
Assume that the length of life of agents increases permanently. How would you represent this change in the present model and what would be the impact of a longer life on the capital output ratio.
B-4.c
The growth rate is now assumed to be 0. The population is such that 3/4 of the population is endowed with one effective unit of labor and the remaining quarter is endowed with $\gamma = 2$ unit of labor per individual. What is the fraction of capital that is owned by the top quarter of the people who are in the second period of their life? Denote this ratio by $x$. 
In this question, there is no growth. The production function is such that a government introduces a pay-as-you-go system of benefit in the second period: there is a tax of the fixed amount \( \theta \) on the young (\( \theta \) is not a rate). The tax is independent of the labor income. The revenues of the tax are paid to the old. Determine the level of saving in capital of an individual with one unit of effective labor when \( \theta = 1/4, \alpha = 1/7, R = 1 \). How does the value of \( \theta \) affect the \( x \) ratio? (If you cannot answer the numerical question, at least provide an intuitive answer on the direction of the effect). Discuss the consequence of the pay-as-you-go social security system on the distribution of wealth in the economy.