**Instructions.** You have four hours to complete this exam, plus a 15 minute "grace period" to wrap up if needed. Answer all four questions. Questions are equally weighted.

Write on one side of the provided paper only. Start the answer to each question on a new sheet of paper and be sure to write your candidate number, question number, and page number on each sheet.

Be concise in your answers, and think before you write. Good luck!

1. Consider the following variation on the Spence model. There are two types of workers, $\theta_L$ and $\theta_H$, where the prior is that the two types are equally likely. The productivity of either type depends on the level of education as follows:

<table>
<thead>
<tr>
<th></th>
<th>Productivity if $e &lt; 1$</th>
<th>Productivity if $e \geq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_H$</td>
<td>100</td>
<td>110</td>
</tr>
<tr>
<td>$\theta_L$</td>
<td>40</td>
<td>80</td>
</tr>
</tbody>
</table>

The cost of education is $15 per unit for $\theta_H$ and $35$ for $\theta_L$. (I.e., for $\theta_H$, the cost of education level $e$ is $15e$ and analogously for $\theta_L$.)

Find all separating and pooling equilibria. For the equilibrium concept, use the same perfect Bayesian equilibrium notion used by Mas-Colell, Whinston and Green in their treatment of the Spence model.
2. A firm consists of a headquarters H and a manager M. An investment decision $d \geq 0$, which can be interpreted as the scale of a project, is to be undertaken by the firm. The optimal scale depends on a state of the world $\theta$. The payoffs given the state $\theta$ are

\[
1 - 12(\theta - d)^2 \quad \text{for } H \\
1 - 12(\theta + b - d)^2 \quad \text{for } M
\]

where $b > 0$. The state $\theta$ is commonly known to be uniformly distributed on $\Theta = [0, 1]$. Both players are expected utility maximizers.

The decision $d$ is to be taken after the state is realized. Headquarters has no information about the state, other than its prior distribution. The manager M, however observes the state perfectly.

a) Find H's autarkic decision rule $d^A(\theta)$ when she picks $d$ without any input about $\theta$ from M. What is her expected payoff from autarky?

Now consider communication: After observing $\theta$, M sends a costless message $m \in M$ to H using the strategy $\sigma : \Theta \rightarrow M$; upon receiving the message, H chooses $d$ using the strategy $\rho : M \rightarrow \mathbb{R}_+$. We take the message space $M$ to be the set of possible states, $[0, 1]$. The solution concept is perfect Bayesian equilibrium in pure strategies.

b) Show that despite M's perfect knowledge of the state, the full-information outcome, in which H would set $d^F(\theta) = \theta$, cannot be sustained in any equilibrium as long as $b > 0$. Interpret.

c) Consider an equilibrium in which M's strategy is $\sigma(\theta) = \begin{cases} 0 & \text{if } \theta \leq \bar{\theta} \\ 1 & \text{if } \theta > \bar{\theta} \end{cases}$.

(i) What must H's strategy be along the equilibrium path? Specify beliefs following receipt of out-of-equilibrium messages that support this equilibrium.

(ii) What is the value of $\bar{\theta}$, as a function of $b$?

d) What is the maximum value of $b$ consistent with the "two-message" equilibrium in (c)? What happens if $b$ exceeds this maximum value? Interpret.

e) Would H prefer communication or autarky? Explain.
3. Consider the nature of efficient allocations in an exchange economy with uncertainty. There are two consumers, \( i \) and \( j \), two states 1 and 2, and one physical consumption good available in each state. Neither consumer is sure about the likelihoods of the states. Consumer \( i \) believes that state 1 will occur with probability in the interval \([\frac{1}{3}, \frac{1}{2}]\) and consumer \( j \) believes that state 1 will occur with probability in the interval \([p, \frac{2}{3}]\), for some \( 0 < p \leq \frac{2}{3} \). Each is concerned about the worst-case scenario. Accordingly, \( i \) has utility function

\[
U_i(x_{i1}, x_{i2}) = \min\{\frac{1}{3}u(x_{i1}) + \frac{2}{3}u(x_{i2}), \frac{1}{2}u(x_{i1}) + \frac{1}{2}u(x_{i2})\},
\]

and \( j \) has utility function

\[
U_j(x_{j1}, x_{j2}) = \min\{pu(x_{j1}) + (1 - p)u(x_{j2}), \frac{2}{3}u(x_{j1}) + \frac{1}{3}u(x_{j2})\}.
\]

The common utility index \( u \) is concave and strictly increasing. Finally, the aggregate endowment is the same in the two states.

(i) Let \( u \) be linear, \( u(x) = x \). Show that every efficient allocation provides complete insurance to each consumer if \( p \leq \frac{1}{3} \). Is the latter condition also necessary for complete insurance to be Pareto optimal?

(ii) Suppose now that \( u \) is strictly concave and differentiable. How does this affect your answers to both questions in (i)?

(iii) It is often asserted that in the absence of aggregate risk, complete insurance is efficient if and only if consumers have identical beliefs. How do you reconcile this assertion with the finding in (i) that complete insurance is efficient if \( p \leq \frac{1}{3} \) given that the probability intervals of the two consumers do not coincide? Limit your answer to two or three sentences.

In parts (i) and (ii), you must justify all answers fully. Edgeworth box diagrams may be helpful.
4. The Slutsky condition is an important prediction of the preference maximization model. Here you are asked to consider whether it is satisfied in some 'nonstandard' settings. In each of the following cases, determine whether or not the Slutsky condition is satisfied by the indicated demand functions. If so, give or sketch a brief proof. If not, give or sketch a proof or provide a counterexample.

(i) The consumer's utility function depends on prices. Specifically, it is given by $U(x, p) = u(x)g(p)$ where $x \in \mathbb{R}_+^n$ is a vector of goods and $p \in \mathbb{R}_+^m$ is a price vector. The consumer maximizes $U(x, p)$ subject to $p \cdot x \leq w$ where $w \in \mathbb{R}_{++}$ is income. Assume that $u$ is positive, increasing and strictly quasiconcave, while $g$ is positive and decreasing. Consider the demand function $x^*(w, p)$ of this consumer.

(ii) Alice and Bob each consume goods $x$ and $y$. Let $x_A$ denote Alice's consumption of $x$ and $x_B$ denote the amount of good $x$ consumed by Bob; define $y_A$ and $y_B$ similarly. Thus Alice consumes bundles of the form $(x_A, y_A) \in \mathbb{R}_+^2$ and Bob consumes bundles $(x_B, y_B) \in \mathbb{R}_+^2$. Because they are married, they make decisions collectively. Specifically, they determine household consumption by solving

$$\max_{(x_A, y_A, x_B, y_B)} u_A(x_A, y_A) \cdot u_B(x_B, y_B)$$

subject to:

$$p(x_A + x_B) + q(y_A + y_B) \leq w,$$

where $u_A$ and $u_B$ are positive, increasing, continuous and strictly concave utility functions, $p$ and $q$ denote positive prices, and $w > 0$ denotes household wealth. The sum of Alice's and Bob's consumption of $x$ is denoted $x^*(p, q, w)$ and the sum for $y$ is $y^*(p, q, w)$. Consider the demand functions $(x^*(p, q, w), y^*(p, q, w))$.

(iii) This consumer faces uncertainty. There is one physical good and he can trade on a complete set of Arrow-Debreu markets corresponding to a finite state space $S$. He maximizes a state-dependent expected utility function. His demand for consumption in state $s$ is $x^*_s(p, w)$, where $p \in \mathbb{R}_+^2$ is the Arrow-Debreu price vector. Consider the demand function $x^*(p, w) = (x^*_s(p, w))_{s \in S}$. 