Working out Muth model 2

R.G. King

The Date

Abstract

Gerzensee 2008 notes.

1 Muth model 2

The second Muthian model is simple enough to be solved easily by hand. In this model, expectations are important in that inventory investment today is based upon expectations about price in the future.

\[
\begin{align*}
\text{Supply} & : q_t = \sigma p_t + k_t \\
\text{Demand} & : q_t = -\alpha p_t + x_t + k_{t+1} \\
\text{Investment} & : k_{t+1} = \phi [\beta E_t p_{t+1} - p_t]
\end{align*}
\]

The three behavioral equations of the model are respectively: a supply curve for the agricultural product that depends positively on the current price \( p_t \) and stock brought into the period \( (k_t) \); a demand that depends negatively on current price but positively on a demand shock \( (x_t) \) and a demand for stock to be carried over \( (k_{t+1}) \); and an investment demand that depends positively on the gap between the discounted (at rate \( \beta < 1 \)) future expected price \( (E_t p_{t+1}) \) and the current price \( p_t \).

1.1 Singular linear difference system

If we write this as a linear difference system, we have:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -\beta \phi & 1 \\
\end{bmatrix}
E_t
\begin{bmatrix}
q_{t+1} \\
p_{t+1} \\
k_{t+1} \\
\end{bmatrix}
= \begin{bmatrix}
-1 & \sigma & 1 \\
1 & \alpha & 0 \\
0 & -\phi & 0 \\
\end{bmatrix}
\begin{bmatrix}
q_t \\
p_t \\
k_t \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-1 \\
0 \\
\end{bmatrix}
x_t
\]

From the presence of a row of zeros, it is clear that the matrix \( A \) is singular and cannot be inverted.
1.2 Reduced System

However, it is possible to reduce the system analytically by hand. Equating supply and demand, we determine a restriction on price, \( \sigma p_t + k_t = -\alpha p_t + x_t + k_{t+1} \). (We also know that we can always determine quantity from the supply curve once we have solve for \( p_t \) and \( k_t \).)

Eliminating \( q_t \), we thus have a simplified system with two equations:

\[
\text{Price : } \sigma p_t + k_t = -\alpha p_t + x_t + k_{t+1} \\
\text{Investment : } k_{t+1} = \phi [\beta E_t p_{t+1} - p_t]
\]

We can write this system as a linear difference system,

\[
\begin{bmatrix}
  -\beta \phi & 1 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  p_{t+1} \\
  k_{t+1}
\end{bmatrix}
= \begin{bmatrix}
  -\phi & 0 \\
  \alpha + \sigma & 1
\end{bmatrix}
\begin{bmatrix}
  p_t \\
  k_t
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  -1
\end{bmatrix} x_t
\]

which has a nonsingular matrix \( A \) so long neither of \( \beta \) and \( \phi \) is equal to zero. Since \( A \) is nonsingular, we can solve for the reduced system by inverting \( A \) and premultiplying,

\[
\begin{bmatrix}
  p_{t+1} \\
  k_{t+1}
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{\beta} + \frac{\alpha + \sigma}{\beta \phi} & \frac{1}{\beta \phi} \\
  \frac{1}{\beta \phi} & \frac{\alpha + \sigma}{1}
\end{bmatrix}
\begin{bmatrix}
  p_t \\
  k_t
\end{bmatrix}
+ \begin{bmatrix}
  -\frac{1}{\beta \phi} \\
  -1
\end{bmatrix} x_t
\]

The eigenvalues of this system are the solutions to

\[
|Az - B| = 0 \text{ or } |Iz - W| = 0
\]

\[
0 = \det\left( \begin{bmatrix}
  -\beta \phi & 1 \\
  0 & 1
\end{bmatrix} z - \begin{bmatrix}
  -\phi & 0 \\
  \alpha + \sigma & 1
\end{bmatrix} \right)
\]

\[
= \det\left( \begin{bmatrix}
  -\phi (\beta z - 1) \\
  -(\alpha + \sigma) & z - 1
\end{bmatrix} \right)
\]

\[
= -\phi (\beta z - 1)(z - 1) + (\alpha + \sigma) z
\]

\[
= -\phi \beta z^2 + Az - \phi
\]

with \( A = (\sigma + \alpha) + \beta \phi + \phi \). Hence, there is one stable and one unstable root (this is the same structure as the growth model). The roots satisfy \(-\phi \beta \mu^2 + A\mu - \phi = 0\) and they also satisfy \( \mu_u + \mu_s = A/(\beta \phi) \) and \( \mu_u \mu_s = 1/\beta \).
1.3 Second order system

Alternatively, this model can be written as a single second order difference equation in capital, as follows

\[
\begin{align*}
\sigma p_t + k_t &= -\alpha p_t + x_t + k_{t+1} \implies (\sigma + \alpha)p_t = k_{t+1} - k_t + x_t \\
\phi[k_{t+1}] &= \phi[\beta E_t p_{t+1} - p_t] \implies (\sigma + \alpha)k_{t+1} = \phi[\beta E_t (k_{t+2} - k_{t+1} + x_{t+1}) - (k_{t+1} - k_t + x_t)]
\end{align*}
\]

Write the last of these expressions as

\[
-\phi \beta E_t k_{t+2} + \phi k_{t+1} = \beta \phi E_t x_{t+1} - \phi x_t
\]

with \( A = (\sigma + \alpha) + \beta \phi + \phi \). Once we have determined the path of the inventory stock, we can use \((\sigma + \alpha)p_t = k_{t+1} - k_t + x_t\) to determine the price of the good. In turn, we can use \(q_t = \sigma p_t + k_t\) to determine the quantity.

1.4 Discussion

There are thus three different systems: (i) an original singular first-order system (1) in \(q_t, p_t, k_t\); (ii) a reduced dimension first-order nonsingular system (2), \(p_t, k_t\) which is supplemented by a non-dynamic identity that can be used to determine \(q_t\); and (iii) a second order system in \(k_t\) (3), which is supplemented by two additional equations that can be used to determine \(q_t, p_t\) after a solution for \(k\) is determined. The last expression is of interest partly because of its similarity to Euler equations in certain dynamic optimization problems.

2 Basic solution along BK lines

Consider the nonsingular system (2)

\[
E_t Y_{t+1} = W Y_t + \Psi x_t
\]

where \(W\) is a 2 by 2 matrix,

\[
W = \begin{bmatrix} w_{pp} & w_{pk} \\
       w_{kp} & w_{kk} \end{bmatrix}
\]

and \(\Psi\) is a two by 1 vector.

2.1 System transformation to separated form

Let \(V\) be a variable transformation matrix (the matrix of left eigenvectors), so that

\[
VW = \mu V
\]
where
\[ \mu = \begin{bmatrix} \mu_u & 0 \\ 0 & \mu_s \end{bmatrix} \]

Then, we can write the system in a separated form
\[ E_t \begin{bmatrix} u_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_u & 0 \\ 0 & \mu_s \end{bmatrix} \begin{bmatrix} u_t \\ s_{tt} \end{bmatrix} + \begin{bmatrix} \theta_u \\ \theta_s \end{bmatrix} x_t \]

with
\[ \begin{bmatrix} u_t \\ s_t \end{bmatrix} = \begin{bmatrix} V_{up} & V_{uk} \\ V_{sp} & V_{sk} \end{bmatrix} \begin{bmatrix} p_t \\ k_t \end{bmatrix} \]

as follows,
\[ E_t Y_{t+1} = W Y_t + \Psi x_t \]
\[ V E_t Y_{t+1} = V W V^{-1} Y_t + V \Psi x_t \]

so that \( \theta = V \Psi \). It is convenient to write the inverse of \( V \) as \( R \), where
\[ R = \begin{bmatrix} R_{pu} & R_{ps} \\ R_{ku} & R_{ks} \end{bmatrix} \]

so that
\[ \begin{bmatrix} p_t \\ k_t \end{bmatrix} = \begin{bmatrix} R_{pu} u_t + R_{ps} s_t \\ R_{ku} u_t + R_{ks} s_t \end{bmatrix} \]

and the definition of the matrix inversion means that we can further write
\[ R = \begin{bmatrix} R_{pu} & R_{ps} \\ R_{ku} & R_{ks} \end{bmatrix} = \begin{bmatrix} V_{up} & V_{uk} \\ V_{sp} & V_{sk} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{V_{ks} V_{pu} - V_{ku} V_{ps}} V_{ks} & -\frac{1}{V_{ks} V_{pu} - V_{ku} V_{ps}} V_{ku} \\ -\frac{1}{V_{ks} V_{pu} - V_{ku} V_{ps}} V_{ps} & \frac{1}{V_{ks} V_{pu} - V_{ku} V_{ps}} V_{pu} \end{bmatrix} \]

if necessary.

### 2.2 Solving the unstable difference equation forward

The separated form
\[ E_t \begin{bmatrix} u_{t+1} \\ s_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_u & 0 \\ 0 & \mu_s \end{bmatrix} \begin{bmatrix} u_t \\ s_{tt} \end{bmatrix} + \begin{bmatrix} \theta_u \\ \theta_s \end{bmatrix} x_t \]

means that
\[ E_t u_{t+1} = \mu_u u_t + \theta_u u_t \]

where \( \mu_u \) is the unstable root.
As with the stock price, suppose that we solve this forward as follows

\[ u_t = \mu_u^{-1}E_t u_{t+1} - \mu_u^{-1}\theta_u x_t \]

\[ = -\theta_u \sum_{j=0}^{\infty} \mu_u^{-j-1}E_t x_{t+j} \]

### 2.3 Solving for the non-predetermined variable \( p \)

From the variable transformation structure above, we can see that

\[ u_t = V_{up}p_t + V_{uk}k_t \]

so that

\[ p_t = -V_{up}^{-1}V_{uk}k_t + V_{up}^{-1}u_t \]

is the solution for the non predetermined variable.

### 2.4 Solving for the predetermined variable \( k \)

There are then a variety of ways of solving for the predetermined variable \( k \). One direct one is

\[ E_t k_{t+1} = w_{kk}k_t + w_{kp}p_t + \Psi_{kx}x_t \]

\[ = [w_{kk} - w_{kp}V_{up}^{-1}V_{uk}]k_t + w_{kp}V_{up}^{-1}u_t + \Psi_{kx}x_t \]

\[ = \mu_s k_t + w_{kp}V_{up}^{-1}u_t + \Psi_{kx}x_t \]

where the last line highlights the stability properties of the inventory stock. It can be derived by a number of routes, but most basically it arises because it must hold in the model without \( x \) variations (as in the neoclassical model of capital formation).

Another way to solve would be via

\[ k_{t+1} = R_{ku}E_t u_{t+1} + R_{ks}E_{s_{t+1}} \]

\[ = R_{ku}E_t u_{t+1} + R_{ks}[\mu_s s_t + \theta_s x_t] \]

\[ = R_{ku}E_t u_{t+1} + R_{ks}[\mu_s (V_{sp}p_t + V_{sk}k_t) + \theta_s x_t] \]

\[ = R_{ku}E_t u_{t+1} + R_{ks}[\mu_s (V_{sp}(-V_{up}^{-1}V_{uk}k_t + V_{up}^{-1}u_t) + V_{sk}k_t) + \theta_s x_t] \]

\[ -V_{up}^{-1}V_{uk}k_t + V_{up}^{-1}u_t \]

### 3 Solution via undetermined coefficients

This is pretty involved, so that one might reasonable ask: "can’t we do something simpler?"
Return to the second order difference equation form,

\[-\phi \beta E_t k_{t+2} + Ak_{t+1} - \phi k_t = \beta \phi E_t x_{t+1} - \phi x_t\]

with \( A = (\sigma + \alpha) + \beta \phi + \phi \).

### 3.1 Undetermined coefficients

Suppose that we specify a conjectured solution

\[ k_{t+1} = \mu k_t + hx_t \]

given the particular driving process

\[ x_t = \rho x_{t-1} + e_t \]

Then, it must be the case that

- updating : \( E_t k_{t+2} = \mu k_{t+1} + hE_t x_{t+1} \)
- forecasting : \( E_t k_{t+2} = \mu k_{t+1} + h\rho x_t \)
- substituting : \( E_t k_{t+2} = \mu^2 k_t + h(\mu + \rho)x_t \)

so that

\[-\phi \beta E_t k_{t+2} + Ak_{t+1} - \phi k_t = \beta \phi E_t x_{t+1} - \phi x_t\]

implies

\[-\phi \beta [\mu^2 k_t + h(\mu + \rho)x_t] + A[\mu k_t + h k_t] - \phi k_t = \beta \phi \rho x_t - \phi x_t\]

For this to hold for all \( k, x \), it must be the case that the following parameter restrictions hold

Restrictions

\[
\begin{align*}
  k & : -\beta \phi \mu^2 + A\mu - \phi = 0 \\
  x & : -\beta \phi h(\mu + \rho) + Ah = \beta \phi \rho - \phi
\end{align*}
\]

and thus that

\[ h = \frac{\beta \phi \rho - \phi}{-\beta \phi (\mu + \rho) + A} \]

for any selected \( \mu \) value. Imposition of "stability" says that we should select \( \mu = \mu_s \).
3.2 Factorization

Another procedure (stressed by Sargent) is to use the lead operator $F$ that shifts variable dating but not conditioning information

\[ F^n E_{t} z_{t+j} = E_t z_{t+j+n} \]

Using this, we can write

\[-\phi \beta E_t k_{t+2} + Ak_{t+1} - \phi k_t = \beta \phi E_t x_{t+1} - \phi x_t \]

as

\[ [-\phi \beta F^2 + AF - \phi] E_t k_t = (\beta \phi F - \phi) E_t x_t \]

and we can factor the polynomial as

\[-\phi \beta (F - \mu_u)(F - \mu_s) E_t k_t = \phi(\beta F - 1) E_t x_t \]

and manipulate it as follows

\[ (\phi \beta \mu_s)(1 - \mu_u^{-1}F)(F - \mu_s) E_t k_t = \phi(\beta F - 1) E_t x_t \]

\[
(F - \mu_s)k_{t+1} = \frac{(\beta F - 1)}{(1 - \mu_u^{-1}F)} \frac{1}{\beta \mu_u} E_t x_t \\
k_{t+1} = \mu_s k_t + \frac{1}{\beta \mu_u} \sum_{j=0}^{\infty} \left( \frac{1}{\mu_u} \right)^j [\beta E_t x_{t+j+1} - E_t x_{t+j}] \\
= \mu_s k_t + \frac{1}{\beta \mu_u} \left[ \beta \rho - 1 \right] x_t \\
= \mu_s k_t + h x_t
\]

where the equivalence of the two $h$ coefficients comes from using $-\phi \beta \mu^2 + A\mu - \phi = 0$ for either root and the related implication that $\mu_u \mu_s = (1/\beta)$.

\[
-\beta \phi (\mu_s + \rho) + A \\
= \frac{1}{\mu_s} [A \mu_s - \beta \phi (\mu_s^2 + \rho \mu_s)] \\
= \frac{1}{\mu_s} [\beta \phi \mu_s^2 + \phi - \beta \phi (\mu_s^2 + \rho \mu_s)] \\
= \frac{\phi}{\mu_s} [\beta \phi \mu_s^2 + \phi - \beta \phi (\mu_s^2 + \rho \mu_s)] \\
= \frac{\phi}{\mu_s} [1 - \beta \mu_s \rho]
\]
implies that the $h$ from the undetermined coefficients solution

$$h = \frac{\beta \phi \rho - \phi}{-\beta \phi (\mu + \rho) + A}$$

$$= \frac{\beta \phi \rho - \phi}{\beta \phi (\mu + \rho) + A}$$

$$= \frac{\phi}{\mu_s [1 - \beta \mu_s \rho]}$$

$$= \frac{(\beta \rho - 1)}{\mu_s [1 - \beta \mu_s \rho]}$$

$$= \frac{1}{\mu_s [1 - \beta \mu_s \rho]}$$

$$= \frac{1}{\beta \mu_u [1 - \beta \mu_u \rho]}$$

which is the same as from the factorization solution.