Qualifying Exam: CAS MA 575

Boston University, Spring 2013

1. Consider the model
$$Y_i = \beta + \epsilon_i \ ,$$
$i = 1, \ldots, n$, where β is a scalar and
$$\epsilon_i = \epsilon_1^* + \cdots + \epsilon_i^* \ ,$$
for $\epsilon_1^*, \ldots, \epsilon_n^*$ a sequence of uncorrelated random variables with mean zero and unit variance[1]

   (a) Provide a simplified expression for the ordinary least squares estimator $\hat{\beta}_{OLS}$ of β.

   (b) Similarly, provide a simplified expression for the generalized least squares estimator $\hat{\beta}_{GLS}$ of β.

   (c) Calculate the variances $\mathrm{Var}(\hat{\beta}_{OLS})$ and $\mathrm{Var}(\hat{\beta}_{GLS})$ of the two estimators in parts (a) and (b) and compare them.

   (d) A colleague of yours, observing the structure of the model assumed here, suggests that it would be natural to instead consider estimating β using the differences $D_i = Y_i - Y_{i-1}$, $i = 1, \ldots, n$, where $Y_0 \equiv 0$. Comment on this idea and compare/contrast what you might get in this case with what you obtained using OLS and GLS.

[1] Hint: For this problem you may wish to use the fact that the inverse of an $m \times m$ matrix of the form
$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & m-1 & m-1 \\ 1 & 2 & 3 & \cdots & m-1 & m \end{pmatrix}$$
can be expressed as
$$M^{-1} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} .$$

2. Consider the multiple regression model

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{e} \ ,$$

where $\mathbf{Y}$ is $n \times 1$, $\mathbf{X}$ is $n \times p$, β is $p \times 1$, and $\mathbf{e}$ is $n \times 1$. The errors $\mathbf{e}$ are assumed to be uncorrelated with mean zero and variance σ^2.

Suppose that it is known that the entries of β should satisfy a set of j linear constraints, which can be encoded in the equations $R\beta = r$, for R a $j \times p$ matrix of linear independent rows, and r a $j \times 1$ vector.

The *restricted ordinary least squares* problem entails estimating β by optimizing the ordinary least squares error, subject to the additional constraints $R\beta = r$. Using the method of Lagrange multipliers, this means solving for that value, say $\hat{\beta}^*$, that minimizes

$$(\mathbf{Y} - \mathbf{X}\beta)^T(\mathbf{Y} - \mathbf{X}\beta) + 2\lambda^T(R\beta - r) \ ,$$

where λ is a $j \times 1$ vector of weights (i.e., the so-called Lagrange multipliers) which form an additional set of parameters to be inferred.

(a) Show that, for fixed λ, the restricted OLS estimator $\hat{\beta}^*$ takes the form

$$\hat{\beta}^* = \hat{\beta} - (\mathbf{X}^T\mathbf{X})^{-1}R^T\lambda \ ,$$

where $\hat{\beta}$ is the standard ordinary least squares estimate. [That is, show that $\hat{\beta}^*$ optimizes the restricted OLS criterion when the latter is treated as a function of β only, keeping λ fixed.]

(b) Using the result in part (a), as well as the fact that we must have $R\hat{\beta}^* = r$, show that

$$\lambda = \left[R(\mathbf{X}^T\mathbf{X})^{-1}R^T\right]^{-1}\left(R\hat{\beta} - r\right) \ .$$

Substituting appropriately, provide a final formula for $\hat{\beta}^*$ in terms of $\mathbf{X}, R, \hat{\beta}$, and r only.

(c) Using the fact that $\hat{\beta}$ is, in general, an unbiased estimate of β, show that under the restricted model assumed here (i.e., that $R\beta = r$), the estimate $\hat{\beta}^*$ is unbiased as well.

(d) Suppose that we only suspect that $R\beta = r$ holds, but are not certain. Define an appropriate F statistic for testing $H_0 : R\beta = r$ versus $H_1 : R\beta \neq r$, specifying the appropriate degrees of freedom of the corresponding F distribution.

(e) Consider the following special case of the general framework analyzed above. Let

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + e_i \ ,$$

and suppose that we restrict $\beta = (\beta_0, \beta_1, \beta_2)^T$ such that $2\beta_1 + 3\beta_2 = 5$. Explain how, <u>without using Lagrange multipliers</u>, you would go about finding the restricted OLS estimate $\hat{\beta}^*$.