MA 582 Qualifying Exam Problems

1. The number of breakdowns X per day for a certain machine is a Poisson RV with unknown mean q ($0 < q < \infty$), that is, the probability mass function (pmf) $p(x;q)$ of X is given by

$$p(x;q) = P(X = x) = \begin{cases} \dfrac{q^x \cdot e^{-q}}{x!}, & \text{for } x = 0,1,2,... \\ 0, & \text{elsewhere.} \end{cases}$$

Let X_1, X_2, ..., X_n denote the observed number of breakdowns for n independently selected days.

a) Identify the distribution of the random variable $Y_1 = X_1 + X_2 + \cdots + X_n$.

b) Find the maximum likelihood estimator $\hat{q}_{MLE}$ of q.

c) Show that the MLE $\hat{q}_{MLE}$ is a consistent estimator for q.

d) Show that the MLE $\hat{q}_{MLE}$ is an efficient estimator for q.

e) The daily cost of repairing the breakdowns is given by $Y_2 = 3X^2$. Find a MLE $\hat{h}_{MLE}$ for $h = E[Y_2]$.

f) What is the asymptotic distribution of $\sqrt{n}(\hat{q}_{MLE} - q)$ as $n \to \infty$?

g) Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Obtain the limiting distribution of $Z_n = \dfrac{\bar{X}_n - q}{\sqrt{\bar{X}_n / n}}$ as $n \to \infty$.

h) Let $Y_n = \sqrt{n}[\cos(\bar{X}_n) - \cos(q)]$. Obtain the limiting distribution of Y_n as $n \to \infty$.

i) Find a variance stabilizing transformation, that is, a function $g(\cdot)$ satisfying

$$\sqrt{n}[g(\bar{X}_n) - g(q)] : AN(0,1).$$

2. Let X be a random variable having a power family distribution with parameters $a = 2$ and $b = q > 0$ (written $X : D(q) = P(2,q)$), that is, X has a probability density function (pdf) $f(x;q)$ given by

$$f(x;q) = \begin{cases} \dfrac{2x}{q^2}, & \text{for } 0 < x < q \\ 0, & \text{elsewhere.} \end{cases}$$

Consider the parameterized family $\{D(q):\ q > 0\}$.

i) Let Y_n be the maximum likelihood estimator of the random sample of size n from the distribution of X. Find the cdf and pdf of Y_n.

ii) Show that Y_n is a consistent estimator of q.

iii) Show that Y_n is a biased estimator of q.

iv) Find an unbiased estimator of q, call it T_n.

v) Show that $n(q - Y_n)$) converges in distribution, and find its asymptotic distribution explicitly.

vi) Show that $n(q - T_n)$ converges in distribution, and find its asymptotic distribution explicitly.

vii) Does the family $\{D(q):\ q > 0\}$ obey all the regularity conditions for maximum likelihood estimation? Why or why not?