Qualifying Exam: CAS MA 582

Boston University, Spring 2008

1. Let X~Pois(θ) with unknown $\theta > 0$ to be estimated.

   a. Find the maximum likelihood estimator (MLE) $\hat{\theta}_{MLE}$ of θ.
   b. Show that $\hat{\theta}_{MLE}$ is unbiased for θ.
   c. Show that $\hat{\theta}_{MLE}$ is consistent for θ.
   d. Show that $\hat{\theta}_{MLE}$ is asymptotically normal.
   e. Compute Fisher's Information for θ.
   f. Is $\hat{\theta}_{MLE}$ an efficient estimator of θ?
   g. Find the MLE of θ^3 and prove it is consistent and asymptotically normal.

2. Recall that X~Exp(λ) if X has pdf f(x) = λ exp(-λx), for x>0, and 0 elsewhere. Recall, too, that E(X) = 1/λ in this parameterization.

   a. Find the mgf of X.

   b. Let X, X_1, X_2, …, X_n be iid Exp(λ) RV's. Show that $S_n = X_1 + X_2 + \ldots + X_n$ has gamma distribution $\Gamma(n, \lambda)$, written $S_n \sim \Gamma(n, \lambda)$.

   c. Suppose R and T are iid with common mgf $M(t) = 1/(1-t)^2$. Identify the distribution of R+T.

   d. If $S_n \sim \Gamma(n, \lambda)$, show that $\sqrt{n}(\frac{S_n}{n} - \frac{1}{\lambda})$ converges in distribution, and identify its asymptotic distribution completely and explicitly. (Hint: Use (b).)

   e. Show that $\sqrt{n}(\sqrt{\frac{S_n}{n}} - \sqrt{\frac{1}{\lambda}})$ converges in distribution, and identify its asymptotic distribution completely and explicitly.

(PLEASE TURN PAGE OVER FOR PROBLEM 3)

3. A random variable (RV) X is said to have a Pareto distribution with parameters $\alpha > 0$ and $\beta > 0$ ($X : Par(\alpha, \beta)$), if the cdf $F(x)$ of X is given by

$$F(x) = F(x; \alpha, \beta) = \begin{cases} 1 - \left(\dfrac{\beta}{x}\right)^{\alpha}, & \text{for } x \geq \beta \\ 0, & \text{for } x < \beta. \end{cases}$$

a. Find the pdf $f(x)$ of X.

b. Find the mean of X.

c. Let $U : Unif(0,1)$ be a RV with uniform distribution on (0,1). Find a transformation $g(u)$ such that the cdf of a RV $g(U)$ coincides with $F(x)$.

d. Let X_1, X_2, …, X_n be a random sample from the Pareto distribution $X : Par(\alpha, \beta)$. Find the distribution (cdf) of the first order statistic $X_{(1)} = \min(X_1, X_2, ..., X_n)$.

e. Let X_1, X_2, …, X_n be a random sample from the Pareto distribution $X : Par(\alpha = 3, \beta)$ with unknown parameter β. Show that the statistic $\hat{\beta} = X_{(1)} = \min(X_1, X_2, ..., X_n)$ is

e-1) a biased estimator for β (compute the bias of $\hat{\beta}$)

e-2) a consistent estimator for β (Hint: use part d.)

e-3) a sufficient statistic for β.

f. Let X_1, X_2, …, X_n be a random sample from the Pareto distribution $X : Par(\alpha, \beta = 3)$ with unknown parameter α. Show that $\hat{\alpha} = \prod_{k=1}^{n} X_k$ is a sufficient statistic for α.