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Properties of a classical charged harmonic oscillator accelerated through classical electromagnetic zero-point radiation

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The behavior of a classical charged harmonic oscillator is analyzed when the system is relativistically, uniformly accelerated through classical electromagnetic zero-point radiation. Recently, Boyer considered the same system but with oscillations confined to the plane perpendicular to the direction of acceleration. The statistical behavior of this accelerated transverse oscillator was found to agree with the statistical properties of a similar oscillator situated in an inertial frame and bathed by a thermal electromagnetic spectrum characterized by the Unruh-Davies temperature of $T=\hbar a/2\pi ck$. Here, the restriction to transverse oscillations is removed. The equation of motion describing the system is simplified by using a coordinate system that is Fermi-Walker transported along the trajectory of the oscillator. The results found for the longitudinal oscillator are analogous to those found for the transverse oscillator; in general, though, the frequency of the accelerated oscillator is a function of the proper acceleration of the system.

I. INTRODUCTION

An observer uniformly accelerating through the vacuum of a scalar quantum field was found by Unruh[1] and Davies[2] to observe a Planckian spectrum of the scalar field characterized by the temperature of $T=\hbar a/2\pi ck$. Unfortunately, this beautifully simple relationship was not found for the correlation functions of the quantized electromagnetic field.[3]

Results analogous to those of quantum field theory were shown to exist within the context of classical theory for the situations of an observer uniformly accelerating through classical scalar zero-point radiation and through electromagnetic zero-point radiation.[4] These results were obtained by examining the correlation functions of the zero-point fields along a trajectory in space-time described by uniform acceleration. Hence, even here, the Planckian spectrum seen on acceleration in the scalar case did not carry over to the electromagnetic situation.

Recently, however, Boyer[5] was able to recover Planck's spectrum within classical electromagnetism by considering the behavior of a charged harmonic oscillator uniformly accelerated through classical electromagnetic zero-point radiation. Thus, instead of simply examining the correlation functions of the electromagnetic field along a path described by uniform acceleration, the physical behavior of a uniformly accelerated electromagnetic system was analyzed. A key feature in solving the equation of motion of the oscillator was to retain all terms in the full relativistic radiation-reaction expression except those that were negligible due to the assumed small size of the oscillator. From the solution of this equation and from the assumed statistical properties of the zero-point electromagnetic field, the second-order moments were obtained for the displacement and velocity of the oscillating particle as seen by an observer uniformly accelerating with the system. These properties agreed exactly with those of a similar oscillator in an inertial frame but bathed in a Planckian classical electromagnetic radiation spectrum characterized by the Unruh-Davies temperature of $T=\hbar a/2\pi ck$.

The present article generalizes the classical analysis by removing one of the restrictions previously imposed on the oscillator system. In the model mentioned above,[5] physical constraints were assumed to exist which confined the oscillations of the charged particle to a plane perpendicular to the direction of uniform acceleration. Here, these constraints are removed. Oscillations may then occur along any spatial direction. The equation of motion for longitudinal oscillations, meaning oscillations along the direction of acceleration, is more complicated than the transverse case since there are additional terms due to relativistic effects. In order to ease calculational difficulties, a coordinate system that is Fermi-Walker transported along the trajectory of the equilibrium point of the oscillator is introduced. (This approach should also allow the study of an oscillator moving through electromagnetic zero-point radiation along other space-time trajectories of interest.[6])

The equation obtained for the motion of an oscillator along the direction of acceleration of the system agrees with the equation governing motion perpendicular to the acceleration direction, except for a change in the expression for the oscillator frequency due to "red-shift effects." Indeed, the frequency of the oscillator in the Fermi-Walker coordinate system will in general be a function of the proper acceleration of the system, even for a transverse oscillator; only when the distance between "source and field point" is negligible compared to c^2/a can one expect the dependence of the transverse oscillator frequency upon the proper acceleration to be removed. (Boyer's results apply in this particular limit.) In order to remove the frequency dependence upon acceleration for a longitudinal oscillator, one must in addition impose the restriction that $cT_{os} \ll c^2/a$, where T_{os} is the period of the oscillator. This condition makes negligible the red-shift effects mentioned previously. However, the limiting case of $cT_{os} \ll c^2/a$ is not an interesting situation in which to examine the thermal effects of acceleration, for this condition implies that the thermal energy associated with the acceleration is small compared to the zero-point energy of the oscillator. Thus, $cT_{os} \ll c^2/a$ implies that $kT = \hbar a/2\pi c \ll \hbar/2\pi T_{os} \approx \hbar\omega$. Therefore, in order to observe the thermal effects of acceleration in the longitudinal oscillations, it appears that one cannot impose this restriction.

Consequently, the frequency of the longitudinal oscillator in the accelerating coordinate system will depend upon the value of the proper acceleration of the system. As noted earlier, the same situation exists for the transverse oscillator when the "small source to field point limit" does not apply. This frequency dependence upon acceleration presents an additional complication that did not exist in Boyer's original transverse oscillator model. Nevertheless, the essential conclusion reached by Boyer in regard to the latter model will also hold for the longitudinal and the slightly more general transverse situations considered here, provided the change in frequency with acceleration is taken into account. Thus, let $\omega(a)$ be the frequency of the oscillator accelerating in classical zero-point radiation as seen by an observer moving with the equilibrium point of the oscillator. Let ω' be the frequency of an oscillator situated in an inertial frame with Planckian electromagnetic radiation characterized by $T = \hbar a/2\pi ck$. If the two frequencies are selected so that $\omega(a) = \omega'$, then the statistical behavior of the oscillators, as observed in their respective coordinate systems, will be identical.

II. NEUTRAL OSCILLATOR IN HYPERBOLIC MOTION

The system that will be considered first is a neutral particle of rest mass m oscillating at the end of a massless spring, the equilibrium point of which moves with uniform proper acceleration $\mathbf{a}$. Assume that a constant force $\mathbf{f}_c = m\mathbf{a}$ exists as to provide a uniform acceleration to the particle if the spring was not present. As described in an inertial frame I_*, the equation of motion for the displacement $\mathbf{x}_*(t)$ of the particle from the equilibrium point of the spring is given by

$$\frac{md^2x_*^\mu}{d\tau^2} = F_{*\mathrm{sp}}^\mu + F_{*c}^\mu \ . \tag{1}$$

The quantities $F_{*\mathrm{sp}}^\mu$ and F_{*c}^μ denote the four-vector forces in the I_* frame associated with the spring and the three-vector force $\mathbf{f}_c$ causing the acceleration. The proper time of the particle is given by τ.

If the four-vector forces $F_{*\mathrm{sp}}^\mu$ and F_{*c}^μ were written out in terms of x_*^μ and its derivatives, they would be fairly complicated and highly nonlinear functions of the latter quantities. Hence, it seems appropriate to attempt to transform the coordinates so as to obtain a differential equation that is more manageable. A coordinate system that seems a likely choice to make is one that is Fermi-Walker transported along the path of the equilibrium point of the spring, for in such a coordinate system, the oscillating particle's behavior is naturally described relative to the equilibrium position of the spring. It will be shown that if one chooses this coordinate system, and imposes a small-oscillator restriction as measured in an inertial frame instantaneously at rest with respect to the equilibrium point of the spring, then one obtains a linear differential equation that can easily be solved.

The method for constructing a Fermi-Walker transported coordinate system is described in many standard textbooks on general relativity[7] and will simply be summarized here in order to unify notation. Let the uniform acceleration $\mathbf{a}$ of the spring's equilibrium point be directed along the x axis of the coordinate system. The position of this equilibrium point undergoing relativistic hyperbolic motion is described in the system by[8]

$$X_*^\mu = (ct_*;\mathbf{X}_*(t_*)) = \left[ct_*; \frac{c^2}{a}\left[1+\left[\frac{at_*}{c}\right]^2\right]^{1/2}, 0, 0\right] , \tag{2}$$

where it has been assumed for convenience that $X_* = c^2/a$ at $t_* = 0$. By making use of the relationships

$$\frac{dX_*}{dt_*} = \frac{at_*}{[1+(at_*/c)^2]^{1/2}}$$

and

$$\frac{dt_*}{d\tau_e} = \left[1 - \left|\frac{d\mathbf{X}_*}{dt_*}\right|^2\right]^{-1/2} ,$$

where τ_e is the proper time associated with the equilibrium point of the spring, X_*^μ can be expressed as

$$X_*^\mu(\tau_e) = \left[\frac{c^2}{a}\sinh\left[\frac{a\tau_e}{c}\right]; \frac{c^2}{a}\cosh\left[\frac{a\tau_e}{c}\right], 0, 0\right] . \tag{3}$$

For convenience, the proper time τ_e has been chosen to equal zero when $t_* = 0$.

Using the above description, a coordinate system can be constructed that consists of four unit four-vectors $[e_\nu(\tau_e)]^\mu$ which are Fermi-Walker transported along the path of the equilibrium point of the spring.[9] Coordinates ξ^μ in the accelerated coordinate system can then be defined by $c\tau_e = \xi^0$ and the following conditions:[10]

$$x_*^\mu = \sum_{k=1}^{3} \xi^k [e_k]^\mu + X_*^\mu , \tag{4}$$

or

$$\begin{aligned} ct_* &= \left[\xi^1 + \frac{c^2}{a}\right]\sinh\left[\frac{a\tau_e}{c}\right] , \\ x_* &= \left[\xi^1 + \frac{c^2}{a}\right]\cosh\left[\frac{a\tau_e}{c}\right] , \\ y_* &= \xi^2, \quad z_* = \xi^3 . \end{aligned} \tag{5}$$

Two characteristics of the ξ^μ coordinates make them particularly useful in describing the accelerating oscillator system. First, ξ^0/c equals the proper time associated with the equilibrium point of the spring. Second, differences in the ξ^1, ξ^2, and ξ^3 coordinates are equal to the corresponding differences in the x, y, and z coordinates of an inertial frame instantaneously at rest with respect to the equilibrium point of the spring. Hence, lengths measured in the latter system are equal to lengths expressed in terms of ξ^i, $i=1,2,3$. [This can immediately be deduced from Eqs. (7) below.]

The following set of inertial coordinate systems will now be introduced for future use. Let $I_{\tau_e'}$ be an inertial frame moving with speed $dx_*/dt_* = c\tanh(a\tau_e'/c)$ along the x axis of the I_* inertial frame. The I_* and $I_{\tau_e'}$ coordinate systems can be related by the Lorentz transformation

$$\begin{aligned} ct_{\tau_e'} &= ct_*\cosh\left[\frac{a\tau_e'}{c}\right] - x_*\sinh\left[\frac{a\tau_e'}{c}\right], \\ x_{\tau_e'} &= x_*\cosh\left[\frac{a\tau_e'}{c}\right] - ct_*\sinh\left[\frac{a\tau_e'}{c}\right], \\ y_{\tau_e'} &= y_*, \quad z_{\tau_e'} = z_* . \end{aligned} \tag{6}$$

From Eqs. (6), the I_* inertial frame is equivalent to the $I_{\tau_e'=0}$ system. Due to the above choice in origins of the I_* and $I_{\tau_e'}$ systems, when the equilibrium point of the spring at proper time τ_e' is given by

$$\begin{aligned} x_*^\mu &= X_*^\mu(\tau_e') \\ &= \left[(c^2/a)\sinh\left[\frac{a\tau_e'}{c}\right];(c^2/a)\cosh\left[\frac{a\tau_e'}{c}\right],0,0\right], \end{aligned}$$

then its position in the $I_{\tau_e'}$ inertial frame is described by $x^\mu_{\tau_e'} = (0;c^2/a,0,0)$. Finally, from Eqs. (5) and (6), one immediately obtains

$$\begin{aligned} ct_{\tau_e'} &= \left[\xi^1 + \frac{c^2}{a}\right]\sinh\left[\frac{a}{c}(\tau_e - \tau_e')\right], \\ x_{\tau_e'} &= \left[\xi^1 + \frac{c^2}{a}\right]\cosh\left[\frac{a}{c}(\tau_e - \tau_e')\right], \\ y_{\tau_e'} &= \xi^2, \quad z_{\tau_e'} = \xi^3 . \end{aligned} \tag{7}$$

The relationships given by Eqs. (5) will now be used to express Eq. (1) in terms of the ξ^μ coordinates of the accelerated coordinate system. By substituting Eqs. (5) into the identity $c^2 d\tau^2 = -dx_*^\mu dx_{*\mu}$, it can be shown that

$$\frac{d\tau_e}{d\tau} = \left[\left[1 + \frac{a\xi^1}{c^2}\right]^2 - \frac{1}{c^2}\frac{d\boldsymbol{\xi}}{d\tau_e}\cdot\frac{d\boldsymbol{\xi}}{d\tau_e}\right]^{-1/2} . \tag{8}$$

With the use of Eqs. (5) and (8), all derivatives of x_*^μ with respect to the proper time τ of the particle can be expressed in terms of ξ^i and the differentials of it with respect to τ_e. Hence, Eq. (1) can be rewritten in terms of the ξ^μ coordinates. Again, however, a very nonlinear differential equation will be obtained.

The assumption will now be made that the amplitude of the particle's oscillations about the equilibrium point can be made arbitrarily small.[11] Since one anticipates that the magnitude of $d^n\xi^i/d\tau_e^n$ will be approximately $A\omega^n$, where A is the amplitude of oscillation and ω is the resonant frequency of the oscillator, then the restriction of small amplitude enables $d^n\xi^i/d\tau_e^n$ to be made arbitrarily small. It is under this assumption of small amplitude, where "small" refers to the amplitude measured by an inertial frame instantaneously at rest with respect to the accelerated equilibrium point, that the particle's equation of motion will be linearized. (Clearly the situation is far more complicated than this simple argument since important properties may be masked by linearizing inherently nonlinear differential equations. Fortunately, a harmonic potential is well suited for enforcing mathematical stability at small amplitudes. Consequently, one expects that the above argument holds in some sort of limit whereby the smaller the amplitude of oscillation, the closer the linearized equation approximates the actual one.)

Retaining terms to only first order in the ξ^i coordinates and their derivatives yields

$$\frac{d\tau_e}{d\tau} \approx 1 - \frac{a\xi^1}{c^2} , \tag{8'}$$

$$\begin{aligned} \frac{dx_*^0}{d\tau} &\approx \frac{d\xi^1}{d\tau_e}\sinh\left[\frac{a\tau_e}{c}\right] + c\cosh\left[\frac{a\tau_e}{c}\right], \\ \frac{dx_*^1}{d\tau} &\approx \frac{d\xi^1}{d\tau_e}\cosh\left[\frac{a\tau_e}{c}\right] + c\sinh\left[\frac{a\tau_e}{c}\right], \\ \frac{dx_*^2}{d\tau} &\approx \frac{d\xi^2}{d\tau_e}, \quad \frac{dx_*^3}{d\tau} \approx \frac{d\xi^3}{d\tau_e} , \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{d^2x_*^0}{d\tau^2} &\approx \left[\frac{d^2\xi^1}{d\tau_e{}^2} - \xi^1\left[\frac{a}{c}\right]^2 + a\right]\sinh\left[\frac{a\tau_e}{c}\right] + \left[\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right]\cosh\left[\frac{a\tau_e}{c}\right], \\ \frac{d^2x_*^1}{d\tau^2} &\approx \left[\frac{d^2\xi^1}{d\tau_e{}^2} - \xi^1\left[\frac{a}{c}\right]^2 + a\right]\cosh\left[\frac{a\tau_e}{c}\right] + \left[\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right]\sinh\left[\frac{a\tau_e}{c}\right], \\ \frac{d^2x_*^2}{d\tau^2} &\approx \frac{d^2\xi^2}{d\tau_e{}^2}, \quad \frac{d^2x_*^3}{d\tau^2} \approx \frac{d^2\xi^3}{d\tau_e{}^2} . \end{aligned} \tag{10}$$

Equations (10) reexpress the left-hand side of Eq. (1) in terms of the ξ^μ coordinates. Rewriting the right-hand side of Eq. (1) in terms of the accelerated coordinate system involves transforming the four-vector forces in an appropriate manner. Rather than directly making this transformation from the I_* frame to the Fermi-Walker transported coordinate system, the transformation will be made from I_* to the inertial frame $I_{\tau'_e}$ via the inverse of Eqs. (6). This allows one to utilize the familiar connection between four-vector and three-vector forces in order to make use of the well-known expressions for three-vector forces in inertial frames. As will be shown, these expressions for F_*^μ can then be written in terms of the ξ^μ coordinates by choosing τ'_e to equal τ_e.

Using the expression $F_*^\mu = F_{\tau'_e}^\nu \, \partial x_*^\mu / \partial x_{\tau'_e}^\nu$ and the inverse of Eqs. (6) yields

$$\begin{aligned} F_*^0 &= F_{\tau'_e}^0 \cosh\left[\frac{a\tau'_e}{c}\right] + F_{\tau'_e}^1 \sinh\left[\frac{a\tau'_e}{c}\right], \\ F_*^1 &= F_{\tau'_e}^0 \sinh\left[\frac{a\tau'_e}{c}\right] + F_{\tau'_e}^1 \cosh\left[\frac{a\tau'_e}{c}\right], \\ F_*^2 &= F_{\tau'_e}^2, \quad F_*^3 = F_{\tau'_e}^3 . \end{aligned} \tag{11}$$

The four-vector forces $F_{\tau'_e}^\mu$ can then be written in terms of the three-vector forces $\mathbf{f}_{\tau'_e}$ via

$$F_{\tau'_e}^\nu = \frac{dt_{\tau'_e}}{d\tau} \frac{dp_{\tau'_e}^\mu}{dt_{\tau'_e}} = \frac{dt_{\tau'_e}}{d\tau}\left[\mathbf{f}_{\tau'_e} \cdot \frac{d\mathbf{x}_{\tau'_e}}{dt_{\tau'_e}} \frac{1}{c};\ \mathbf{f}_{\tau'_e}\right], \tag{12}$$

where $p_{\tau'_e}^\mu$ is the particle's four-momentum as measured in the $I_{\tau'_e}$ frame. The quantity

$$\frac{dt_{\tau'_e}}{d\tau} = \left[1 - \frac{1}{c^2} \frac{d\mathbf{x}_{\tau'_e}}{dt_{\tau'_e}} \cdot \frac{d\mathbf{x}_{\tau'_e}}{dt_{\tau'_e}}\right]^{-1/2} \tag{13}$$

can be expressed in terms of the ξ^μ coordinates by using Eqs. (7). One obtains

$$\begin{aligned} \frac{dx_{\tau'_e}^1}{dt_{\tau'_e}} &= \left[\frac{d\xi^1}{d\tau_e} \cosh\left[\frac{a}{c}(\tau_e - \tau'_e)\right] + \left[c + \frac{a\xi^1}{c}\right] \sinh\left[\frac{a}{c}(\tau_e - \tau'_e)\right]\right] \frac{1}{D}, \\ \frac{dx_{\tau'_e}^2}{dt_{\tau'_e}} &= \frac{d\xi^2}{d\tau_e}\frac{1}{D}, \quad \frac{dx_{\tau'_e}^3}{dt_{\tau'_e}} = \frac{d\xi^3}{d\tau_e}\frac{1}{D}, \end{aligned} \tag{14}$$

where

$$D = \frac{1}{c}\frac{d\xi^1}{d\tau_e} \sinh\left[\frac{a}{c}(\tau_e - \tau'_e)\right] + \left[1 + \frac{a\xi^1}{c^2}\right] \cosh\left[\frac{a}{c}(\tau_e - \tau'_e)\right].$$

The inertial reference frame $I_{\tau'_e}$ in which $F_{\tau'_e}^\mu$ is chosen to be evaluated is completely arbitrary since the transformation of Eqs. (11) applies to any value of τ'_e. This arbitrariness can be used to simplify the problem by choosing τ'_e to equal τ_e. This procedure is perfectly well defined, since from the transformations of Eqs. (6), the parameter τ'_e can be treated as a continuous variable to automatically select the inertial frame instantaneously at rest with respect to the spring's equilibrium point for all values of τ_e. Therefore, setting τ'_e equal to τ_e in Eqs. (11) is equivalent to evaluating the four-vector forces in the instantaneous rest frame of the spring's equilibrium point. In this frame, the harmonic restoring force is most simply expressed.

With the above condition, Eqs. (14) reduce to

$$\left.\frac{dx_{\tau'_e}^i}{dt_{\tau'_e}}\right|_{\tau'_e=\tau_e} = \frac{(d\xi^i/d\tau_e)}{(1 + a\xi^1/c^2)}, \quad i = 1,2,3 . \tag{15}$$

From Eqs. (15), (13), and (12), the following results are obtained to first order in the ξ^i coordinates:

$$\left.\frac{dx_{\tau'_e}^i}{dt_{\tau'_e}}\right|_{\tau'_e=\tau_e} \approx \frac{d\xi^i}{d\tau_e}, \quad i = 1,2,3 , \tag{16}$$

$$\left.\frac{dt_{\tau'_e}}{d\tau}\right|_{\tau'_e=\tau_e} \approx 1 , \tag{17}$$

$$F_{\tau_e}^\nu \approx \left[\mathbf{f}_{\tau_e} \cdot \frac{d\xi}{d\tau_e}\frac{1}{c};\ \mathbf{f}_{\tau_e}\right]. \tag{18}$$

The force $\mathbf{f}_{\tau_e c}$ has been assumed to equal $m\mathbf{a}$, as it is the force in the instantaneous rest frame of the spring's equilibrium point that provides the particle with a relativistic hyperbolic motion in the absence of the spring. The force $\mathbf{f}_{\tau_e \mathrm{sp}}$ requires more discussion, however, as it is not so immediately dealt with.

If the equilibrium point of the oscillator is at rest in an inertial frame, one usually defines a harmonic oscillator restoring force as $\mathbf{f}_{\mathrm{sp}} = -k\mathbf{x}$ for an isotropic oscillator and $f_{\mathrm{sp}}^i = -k^i x^i$, $i = 1,2,3$, for an anisotropic oscillator. For a nonrelativistic oscillator obeying the equation of motion

$$m\frac{d^2x^i}{dt^2} = -k_0^i x^i, \quad i = 1,2,3 , \tag{19}$$

this yields a simple harmonic motion of frequency $\omega_0^i = (k_0^i/m)^{1/2}$ along each of the three orthogonal spatial axes. Although one often thinks of a spring obeying Hooke's law as constituting the physical example for Eq. (19), it is well known that the area of applicability of Eq. (19) extends to all stable systems of small amplitude governed by forces that are functions of position alone and are reexpressed in terms of the system's normal coordinates.[12] The force $-k^i x^i$ is then simply the first term in a Taylor's expansion of the forces acting on the particle about the point of equilibrium. [If one instead uses the relativistic expression for the momentum of the particle and writes

$$\frac{d}{dt}\left[m_0\left[1 - \left|\frac{\mathbf{v}}{c}\right|^2\right]^{-1/2} \frac{dx^i}{dt}\right] = -k^i x^i, \quad i = 1,2,3 , \tag{20}$$

then the motion no longer follows a simple harmonic behavior characterized by $A\sin(\omega t-\beta)$. By restricting the motion to small amplitudes, however, so that $v/c\approx\omega_0 A/c \ll 1$, then Eq. (20) is again approximately described by Eq. (19) (Refs. 13 and 14).]

For an oscillator system in which the equilibrium point is undergoing uniform acceleration, the appropriate generalization to the force $-k^i x^i$ must be deduced. To a certain degree, this becomes strictly a matter of definition for what is meant by an "accelerating simple harmonic oscillator." In Appendices A and B of this article, an explicit physical model is examined to help motivate the form for the oscillator's restoring force that will be used in this section of

$$f^i_{\tau_e \mathrm{sp}}=K\delta_{i1}-k^i\xi^i,\quad i=1,2,3\ . \tag{21}$$

Since $\boldsymbol{\xi}=\mathbf{x}_{\tau_e}-\hat{\mathbf{x}}c^2/a$ in I_{τ_e} [$\tau'_e=\tau_e$ in Eqs. (7)], then Eq. (21) may be viewed as applicable to those systems where $f^i_{\tau_e \mathrm{sp}}$ is a function of position and the appropriate conditions of small amplitude, stability, and normal coordinates are satisfied. The simplifying assumption was made that $f^2_{\tau_e \mathrm{sp}}=0$ and $f^3_{\tau_e \mathrm{sp}}=0$ when ξ^2 and ξ^3 equal zero. The constants K and k^i are assumed, in general, to be functions of the proper acceleration of the oscillator's equilibrium point. When $a=0$, it will be assumed that $K=0$ and k^i reduces to the value of the constant k^i_0 in Eq. (19) for an unaccelerated oscillator. The examples in the Appendices bring these points out more clearly. Finally, the presence of the nonzero value of K requires that the uniform acceleration of the equilibrium point be given by $ma=f_c+K$ instead of $ma=f_c$.

Combining the above expressions with Eq. (18) yields

$$F^{\nu}_{\tau_e}\approx\left[(f_c+K)\frac{d\xi^1}{d\tau_e}\frac{1}{c};f_c+K-k^1\xi^1,-k^2\xi^2,-k^3\xi^3\right]. \tag{22}$$

The terms $-\sum_{i=1}^{3}k^i\xi^i(d\xi^i/d\tau_e)(1/c)$ which would occur in $F^0_{\tau_e}$ have been ignored due to the small amplitude assumption. Now combining Eqs. (1), (10), (11), and (22) gives the result that

$$m\left\{\left[\frac{d^2\xi^1}{d\tau_e^2}-\xi^1\left[\frac{a}{c}\right]^2+a\right]\sinh\left[\frac{a\tau_e}{c}\right]+\left[\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right]\cosh\left[\frac{a\tau_e}{c}\right]\right\}=(f_c+K)\frac{d\xi^1}{d\tau_e}\frac{1}{c}\cosh\left[\frac{a\tau_e}{c}\right]+(f_c+K-k^1\xi^1)\sinh\left[\frac{a\tau_e}{c}\right], \tag{23}$$

$$m\left\{\left[\frac{d^2\xi^1}{d\tau_e^2}-\xi^1\left[\frac{a}{c}\right]^2+a\right]\cosh\left[\frac{a\tau_e}{c}\right]+\left[\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right]\sinh\left[\frac{a\tau_e}{c}\right]\right\}=(f_c+K)\frac{d\xi^1}{d\tau_e}\frac{1}{c}\sinh\left[\frac{a\tau_e}{c}\right]+(f_c+K-k^1\xi^1)\cosh\left[\frac{a\tau_e}{c}\right], \tag{24}$$

$$\frac{d^2\xi^2}{d\tau_e^2}=-\frac{k^2}{m}\xi^2\ , \tag{25}$$

$$\frac{d^2\xi^3}{d\tau_e^2}=-\frac{k^3}{m}\xi^3\ . \tag{26}$$

Equations (23) and (24) are equivalent and can be rewritten as

$$\frac{d^2\xi^1}{d\tau_e^2}=-\left[\frac{k^1}{m}-\left[\frac{a}{c}\right]^2\right]\xi^1\ . \tag{27}$$

Provided that $k^1>(a/c)^2m$, then the ξ^i coordinates follow simple harmonic motion of the form $A^i\sin(\omega^i t-\beta^i)$, with angular frequencies given by

$$\omega^1=\left[\frac{k^1}{m}-\left[\frac{a}{c}\right]^2\right]^{1/2},\quad \omega^2=\left[\frac{k^2}{m}\right]^{1/2},\quad \omega^3=\left[\frac{k^3}{m}\right]^{1/2}. \tag{28}$$

The results of this section may cause some puzzlement over the origin of the change in frequency associated with the ξ^1 coordinate due to the $(a/c)^2$ term. The dimensionless quantity $\xi^1 a/c^2$, which gave rise to this change in frequency [see the second term in the first bracket on the left-hand side of Eqs. (23) and (24)], is sometimes referred to as a red-shift effect.[15] In Eq. (8′), this term is the first-order correction to the ratio between the rates of proper time of the oscillating particle and the equilibrium point.

In terms of proper acceleration and three-vector forces, the following explanation helps to understand the frequency change associated with the ξ^1 coordinate. In order for the particle to remain at a constant distance ξ^1 from the equilibrium point, one can show that its proper acceleration must be given by

$$a'=a\bigg/\left[1+\frac{a\xi^1}{c^2}\right]\approx a-\xi^1\left[\frac{a}{c}\right]^2 \quad \text{for } \xi^1\ll\frac{c^2}{a} .$$

Under this condition of constant value ξ^1, no oscillations will take place and both the particle and the equilibrium point will have the same instantaneous rest frame. Since

$$ma'=f_c+K-k^1\xi^1\approx m\left[a-\xi^1\left[\frac{a}{c}\right]^2\right] ,$$

and one requires that $ma=f_c+K$, then k^1 must equal $m(a/c)^2$. This is precisely the limiting condition of oscillatory motion as predicted by Eq. (27), for only if $k^1>m(a/c)^2$ will oscillations occur.[16,17]

III. CHARGED OSCILLATOR IN HYPERBOLIC MOTION

The results of the previous section are easily extended to the case of a particle with rest mass m and charge e. As before, the particle may be pictured as oscillating at the end of a massless spring, the equilibrium point of which undergoes uniform proper acceleration $a\hat{\mathbf{x}}$. The force $f^i_{\tau_e\text{sp}}$ is again taken to be equal to $K\delta_{i1}-k^i\xi^i$. The constant force $\mathbf{f}_c$ may now be obtained from a constant electric field $\mathbf{E}_0$ in the x direction which satisfies $ma=eE_0+K$. The major change in the equation of motion arises from the relativistic radiation reaction terms due to the charged particle's motion. In the I_* frame, the Lorentz-Dirac equation which describes the charged particle's behavior is[18]

$$m\frac{d^2x_*^\mu}{d\tau^2}=F^\mu_{*\text{sp}}+\frac{e}{c}\mathscr{F}_*^{\mu\nu}\frac{dx_{*\nu}}{d\tau}+\Gamma^\mu , \tag{29}$$

where

$$\Gamma^\mu=\frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3x_*^\mu}{d\tau^3}-\frac{1}{c^2}\left[\frac{d^2x_*^\lambda}{d\tau^2}\frac{d^2x_{*\lambda}}{d\tau^2}\right]\frac{dx_*^\mu}{d\tau}\right] . \tag{30}$$

$F^\mu_{*\text{sp}}$ is again the four-force associated with the spring, and $\mathscr{F}_*^{\mu\nu}$ is the electromagnetic field tensor due to the presence of the electric field $\mathbf{E}_0$.

Most of the results from the previous section can be used to evaluate the new terms that have been introduced in the equation of motion. Using Eqs. (8′) and (10) and again invoking the small amplitude approximation yields

$$\begin{aligned}
\frac{d^3x_*^0}{d\tau^3}&\approx\frac{d^3\xi^1}{d\tau_e{}^3}\sinh\left[\frac{a\tau_e}{c}\right]+\left\{2\left[\frac{d^2\xi^1}{d\tau_e{}^2}\frac{a}{c}-\xi^1\left[\frac{a}{c}\right]^3\right]+\frac{a^2}{c}\right\}\cosh\left[\frac{a\tau_e}{c}\right] ,\\
\frac{d^3x_*^1}{d\tau^3}&\approx\frac{d^3\xi^1}{d\tau_e{}^3}\cosh\left[\frac{a\tau_e}{c}\right]+\left\{2\left[\frac{d^2\xi^1}{d\tau_e{}^2}\frac{a}{c}-\xi^1\left[\frac{a}{c}\right]^3\right]+\frac{a^2}{c}\right\}\sinh\left[\frac{a\tau_e}{c}\right] ,\\
\frac{d^3x_*^2}{d\tau^3}&\approx\frac{d^3\xi^2}{d\tau_e{}^3},\quad \frac{d^3x_*^3}{d\tau_e{}^3}\approx\frac{d^3\xi^3}{d\tau_e{}^3} .
\end{aligned} \tag{31}$$

From Eqs. (10),

$$\frac{d^2x_*^\lambda}{d\tau^2}\frac{d^2x_{*\lambda}}{d\tau^2}\approx a^2+2a\left[\frac{d^2\xi^1}{d\tau_e{}^2}-\xi^1\left[\frac{a}{c}\right]^2\right] . \tag{32}$$

Substituting Eqs. (9), (31), and (32) into Eq. (30) results in the expressions

$$\begin{aligned}
\Gamma^0&\approx\frac{2}{3}\frac{e^2}{c^3}\Bigg[\frac{d^3\xi^1}{d\tau_e{}^3}\sinh\left[\frac{a\tau_e}{c}\right]+\left\{2\left[\frac{d^2\xi^1}{d\tau_e{}^2}\frac{a}{c}-\xi^1\left[\frac{a}{c}\right]^3\right]+\frac{a^2}{c}\right\}\cosh\left[\frac{a\tau_e}{c}\right]\\
&\qquad-\frac{1}{c^2}\left\{a^2+2a\left[\frac{d^2\xi^1}{d\tau_e{}^2}-\xi^1\left[\frac{a}{c}\right]^2\right]\right\}\left[\frac{d\xi^1}{d\tau_e}\sinh\left[\frac{a\tau_e}{c}\right]+c\cosh\left[\frac{a\tau_e}{c}\right]\right]\Bigg]\\
&\approx\frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3\xi^1}{d\tau_e{}^3}-\left[\frac{a}{c}\right]^2\frac{d\xi^1}{d\tau_e}\right]\sinh\left[\frac{a\tau_e}{c}\right] ,
\end{aligned} \tag{33}$$

$$\Gamma^1\approx\frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3\xi^1}{d\tau_e{}^3}-\left[\frac{a}{c}\right]^2\frac{d\xi^1}{d\tau_e}\right]\cosh\left[\frac{a\tau_e}{c}\right] , \tag{34}$$

$$\Gamma^i\approx\frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3\xi^i}{d\tau_e{}^3}-\left[\frac{a}{c}\right]^2\frac{d\xi^i}{d\tau_e}\right],\quad i=2,3 . \tag{35}$$

Finally, the term $(e/c)\mathscr{F}_*^{\mu\nu}(dx_{*\nu}/d\tau)$ can be expressed in the ξ^μ coordinates by using the explicit expression for $\mathscr{F}_*^{\mu\nu}$

(Ref. 19) and Eqs. (9). As should be expected, the same results are obtained as in the previous section with f_c replaced by eE_0, since I_* and I_{τ_e} see the same electric field E_0 along the x direction.

After substituting the above quantities into the Lorentz-Dirac equation and observing that the $\mu=0$ and $\mu=1$ equations again yield the same result to first order in ξ^i, one obtains the following equations of motion:

$$\frac{d^2\xi^i}{d\tau_e{}^2}=-(\omega^i)^2\xi^i+\Gamma\left[\frac{d^3\xi^i}{d\tau_e{}^3}-\left[\frac{a}{c}\right]^2\frac{d\xi^i}{d\tau_e}\right],\quad i=1,2,3\,, \tag{36}$$

where $\Gamma=\frac{2}{3}e^2/c^3m$ and ω^i is again given by Eq. (28).

IV. CHARGED OSCILLATOR UNIFORMLY ACCELERATED IN CLASSICAL ZERO-POINT RADIATION

The situation considered next is the oscillating charged particle of Sec. III, again undergoing relativistic hyperbolic motion, but now in the presence of what has been termed classical zero-point electromagnetic radiation.[20] The Lorentz-Dirac equation is now altered simply by including the zero-point radiation fields $\mathbf{E}_*^{\mathrm{zp}}(\mathbf{x}_*,t_*)$ and $\mathbf{B}_*^{\mathrm{zp}}(\mathbf{x}_*,t_*)$ in addition to the $\mathbf{E}_0$ field in the electromagnetic field tensor $\mathcal{F}_*^{\mu\nu}$. Loosely phrased, these additional fields now act as a driving force to the simple harmonic oscillator of Sec. I with damping terms of Sec. II. Hence, an equilibrium behavior of the particle's motion is obtained, since the energy radiated by the particle's oscillations must be supplied by the work done by the zero-point fields in maintaining the particle's oscillations.

Using Eqs. (9) and the appropriate expression for $\mathcal{F}_*^{\mu\nu}$ yields

$$\frac{e}{c}\mathcal{F}_*^{\mu\nu}\frac{dx_{*\nu}}{d\tau}\approx\frac{e}{c}\begin{cases}(E_{*x}^{\mathrm{zp}}+E_0)\left[\dfrac{d\xi^1}{d\tau_e}\cosh\left[\dfrac{a\tau_e}{c}\right]+c\sinh\left[\dfrac{a\tau_e}{c}\right]\right]+E_{*y}^{\mathrm{zp}}\dfrac{d\xi^2}{d\tau_e}+E_{*z}^{\mathrm{zp}}\dfrac{d\xi^3}{d\tau_e}\,,\\ (E_{*x}^{\mathrm{zp}}+E_0)\left[\dfrac{d\xi^1}{d\tau_e}\sinh\left[\dfrac{a\tau_e}{c}\right]+c\cosh\left[\dfrac{a\tau_e}{c}\right]\right]+B_{*z}^{\mathrm{zp}}\dfrac{d\xi^2}{d\tau_e}-B_{*y}^{\mathrm{zp}}\dfrac{d\xi^3}{d\tau_e}\,,\\ E_{*y}^{\mathrm{zp}}\left[\dfrac{d\xi^1}{d\tau_e}\sinh\left[\dfrac{a\tau_e}{c}\right]+c\cosh\left[\dfrac{a\tau_e}{c}\right]\right]-B_{*z}^{\mathrm{zp}}\left[\dfrac{d\xi^1}{d\tau_e}\cosh\left[\dfrac{a\tau_e}{c}\right]+c\sinh\left[\dfrac{a\tau_e}{c}\right]\right]+B_{*x}^{\mathrm{zp}}\dfrac{d\xi^3}{d\tau_e}\,,\\ E_{*z}^{\mathrm{zp}}\left[\dfrac{d\xi^1}{d\tau_e}\sinh\left[\dfrac{a\tau_e}{c}\right]+c\cosh\left[\dfrac{a\tau_e}{c}\right]\right]+B_{*y}^{\mathrm{zp}}\left[\dfrac{d\xi^1}{d\tau_e}\cosh\left[\dfrac{a\tau_e}{c}\right]+c\sinh\left[\dfrac{a\tau_e}{c}\right]\right]-B_{*x}^{\mathrm{zp}}\dfrac{d\xi^2}{d\tau_e}\,.\end{cases} \tag{37}$$

Additional complicated terms due to the zero-point fields now appear in the equation of motion of the charged particle. Solving the resulting differential equations without imposing certain limits would indeed be very difficult. However, the small oscillator limit cannot so arbitrarily be imposed as it was in the previous cases considered. Now the fluctuating zero-point fields will be the determining factor in the size of the amplitude of oscillation. Hence, even the use of Eqs. (9) in obtaining Eqs. (37) must be reexamined.

The following reasoning is intended to provide some rationale for the approximations that will be made subsequently. The amplitude of the frequency component of the zero-point fields near the resonant frequency of the oscillator is anticipated to be the main contributing factor to the amplitude of the oscillator. Let the former quantities be denoted by $\mathbf{E}^{\mathrm{zp}}(\omega^i)$ and $\mathbf{B}^{\mathrm{zp}}(\omega^i)$. If the latter are sufficiently small enough, then one would expect that ξ^i and any derivative $d^n\xi^i/d\tau_e^n$ would roughly be of first order in these quantities. Consequently, any single power of ξ^i or its derivatives and any single power of the fields $\mathbf{E}_*^{\mathrm{zp}}$ and $\mathbf{B}_*^{\mathrm{zp}}$ will be treated as first-order quantities in $\mathbf{E}^{\mathrm{zp}}(\omega^i)$ and $\mathbf{B}^{\mathrm{zp}}(\omega^i)$. Terms of the form $(e/c)B_*^{\mathrm{zp}}d\xi/d\tau_e$, $(e/c)E_*^{\mathrm{zp}}d\xi/d\tau_e$, $-k\xi d\xi/d\tau_e$, and $eE_0(d\xi/d\tau_e)(d\xi/d\tau_e)(1/c^2)$ will be considered of second order in $\mathbf{E}^{\mathrm{zp}}(\omega^i)$ and $\mathbf{B}^{\mathrm{zp}}(\omega^i)$. The quantity $(e/c)\mathcal{F}_*^{\mu\nu}(dx_{*\nu}/d\tau)$ can then be linearized; the earlier linearization steps followed in Secs. I and II will then also hold.[21]

Three equations of motion similar to Eq. (36) are now obtained which include the effects of the zero-point fields,

$$\frac{d^2\xi^i}{d\tau_e{}^2}=-(\omega^i)^2\xi^i+\Gamma\left[\frac{d^3\xi^i}{d\tau_e{}^3}-\left[\frac{a}{c}\right]^2\frac{d\xi^i}{d\tau_e}\right]+\frac{e}{m}E_{\tau_e i}^{\mathrm{zp}}(\xi,\tau_e),\quad i=1,2,3\,, \tag{38}$$

where

$$E^{\mathrm{zp}}_{\tau_e 1}(\xi,\tau_e)=E^{\mathrm{zp}}_{*x}(\xi,\tau_e)\ ,$$

$$E^{\mathrm{zp}}_{\tau_e 2}(\xi,\tau_e)=\cosh\left[\frac{a\tau_e}{c}\right]E^{\mathrm{zp}}_{*y}(\xi,\tau_e)-\sinh\left[\frac{a\tau_e}{c}\right]B^{\mathrm{zp}}_{*z}(\xi,\tau_e)\ ,\tag{39}$$

$$E^{\mathrm{zp}}_{\tau_e 3}(\xi,\tau_e)=\cosh\left[\frac{a\tau_e}{c}\right]E^{\mathrm{zp}}_{*z}(\xi,\tau_e)+\sinh\left[\frac{a\tau_e}{c}\right]B^{\mathrm{zp}}_{*y}(\xi,\tau_e)\ .$$

The quantities $\mathbf{E}^{\mathrm{zp}}_{\tau_e i}$ are the electric fields measured in the inertial frame I_{τ_e} along the three orthogonal coordinate space axes. The quantities (ξ,τ_e) in the arguments of the fields represent the space-time position of the particle at which to evaluate the fields. Using a dipole approximation for the fields, the arguments ξ are then set equal to zero.

Hence, when the small oscillator assumption is made, all three directions of motion are described by the differential equation expressed by Eq. (38). This turns out to be true despite the additional complicating terms that must be taken into account for oscillations occurring in a direction parallel to that of the uniform acceleration **a**.

The above linear stochastic differential equations may now be solved in order to determine the statistical properties imposed upon the oscillating particle by the fluctuating zero-point fields. Fortunately, this work has already been carried out in Ref. 5, where the behavior of the particle was investigated under the restriction that oscillations were confined to the directions perpendicular to **a**. By comparing Eq. (14) of the latter article to Eq. (38) of the present article, it can immediately be seen that they are of the same form when the dipole approximation in the fields is made. The only difference is that in the present paper, ω^i is recognized to be, in general, a function of a, the uniform acceleration of the oscillator's equilibrium point. Of course, this difference in no way effects the method of solution. Hence, the conclusion of Ref. 5 can be immediately applied here, with a slight change in interpretation.

V. CONCLUSION

The results of the previous section lead to the following conclusion. Consider an oscillating charged particle uniformly accelerated through classical electromagnetic zero-point radiation. Let $\omega^i(a)$, for $i=1,2,3$, be the natural frequency of the motion of the particle along each of the spatial axes of the Fermi-Walker transported coordinate system introduced in Sec. II. Now consider a second oscillating charged particle at rest in an inertial frame and bathed in classical electromagnetic zero-point radiation plus Planckian electromagnetic radiation. Let the latter spectrum be characterized by the Unruh-Davies temperature of $T=\hbar a/2\pi ck$. Let this oscillator have a natural frequency $\omega'^i=\omega^i(a)$ along each of the three spatial axes in the inertial frame. One can then conclude that the statistical properties for these two oscillators will be identical, as observed in their respective coordinate systems.

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APPENDIX A: MODEL OF A TRANSVERSE ACCELERATED OSCILLATOR

The following simple model is presented in order to motivate the form assumed for the spring three-vector force given by Eq. (21). A transverse oscillator model is considered here; the following section considers the analogous longitudinal oscillator model. From these examples, it should be apparent how one could construct more general stationary charge distributions in the Fermi-Walker coordinate system, with symmetry axes along the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ directions and that result in the validity of Eq. (21) (21) in the limit of small amplitude A.

In order to create a force which depends linearly upon the oscillating particle's displacement from equilibrium, two charged particles will be placed and held fixed in the positions $\xi_\pm=(0,\pm l,0)$ of the accelerating coordinate system. Consequently, they will possess the same proper acceleration **a** as does the equilibrium point $\xi=(0,0,0)$. Assume that these two particles each have a charge q of the same sign as the charge e of the oscillating particle. Let the latter particle be constrained so that its position is described by $\xi=(0,\xi^2(\tau_e),0)$. By restricting the amplitude of oscillation A to be much smaller than the length l, the force of the outer two charges on the center particle can be expanded in a Taylor series in ξ^2. For $A/l\ll 1$, this force is adequately approximated by retaining only the first-order term in ξ^2, thereby yielding the desired model for a simple harmonic oscillator restoring force. Calculating the value of the proportionality constant k^2 will then determine the dependence of $\omega^2=(k^2/m)^{1/2}$ upon the proper acceleration of the oscillator.

If ones does the above calculation for an unaccelerated oscillator, it is found that $k_0^2=4eq/l^3$. Proceeding to the situation of an accelerated system, one must use the standard expression for the retarded electric field of a point charge given by[22]

$$\mathbf{E}=q\left[\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\mathrm{ret}}+\frac{q}{c}\left[\frac{\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R}\right]_{\mathrm{ret}}\ .\tag{40}$$

In order to conform with the analysis of Sec. II, the force on the oscillating particle should be evaluated along the y axis of an inertial frame I_{τ_e}, instantaneously at rest with respect to the equilibrium point of the oscillator. From Eq. (40), the quantity of interest is

$$E_{y\pm}=\left[\frac{qn_y}{R^2(1-\beta n_x)^3}\left[(1-\beta^2)+\frac{Rn_x\dot{\beta}}{c}\right]\right]_{\text{ret}\pm}, \quad (41)$$

where the $\pm$ signs indicate the field due to the charge at $\xi^2=\pm l$.

In the limit of $A/l \ll 1$, $F_{\tau_e y}=e(E_{y+}+E_{y-})$ can be expanded in terms of ξ^2 to yield

$$F_{\tau_e y}(\xi^2)\approx 2e\left.\frac{dE_{y+}}{d\xi^2}\right|_{\xi=0}\xi^2=-k^2\xi^2\,, \quad (42)$$

where the symmetry of the model has been used to set $(E_{y+}+E_{y-})\,|_{\xi=0}=0$ and

$$\left.\frac{dE_{y+}}{d\xi^2}\right|_{\xi=0}=\left.\frac{dE_{y-}}{d\xi^2}\right|_{\xi=0}.$$

Through rather lengthy calculations, one can now obtain an expression for k^2. This involves calculating the retarded time t_{r+} associated with the charge at $\xi=(0,l,0)$, expressing all quantities in the expression for E_{y+} in terms of t_{r+}, expanding t_{r+} to first order in ξ^2, and finally propagating those first-order terms to Eq. (42).

Clearly, despite the simplicity of this transverse accelerated oscillator model, k^2 will be an extremely complicated function of the proper acceleration a. As one might expect, only in a particular limit, namely, where $l \ll c^2/a$, will k^2 reduce to the value k_0^2 in Eq. (19). Of course, in most cases of interest, this limit is easily satisfied. As discussed in the Introduction, however, the thermal effects of an oscillator associated with acceleration are only expected to be observable when cT_{os} is not small compared with c^2/a. Hence, in order to see thermal effects and yet have $k^2\approx k_0^2$, then $l \ll c^2/a$ and $cT_{\text{os}} \gtrsim c^2/a$ must both be satisfied. (As indicated elsewhere, the conclusion of this article does not depend upon satisfying the condition $k^2=k_0^2$. The present discussion simply examines from a classical point of view some of the subtleties involved with the dependence of k^2 upon a.)

With regard to the present model, one way to satisfy the above conditions is by letting $q\to 0$ as $l\to 0$ in such a way that $4qe/l^3$ remains of constant value k_0^2. Of course, $\xi^2 \ll l$ must remain satisfied and k_0^2 chosen such that $cT_{\text{os}} \gtrsim c^2/a$. Using the following relationships,

$$\begin{aligned}
&t_{r\pm}=-\frac{(l\mp\xi)}{c}\left[1+\left[\frac{a(l\mp\xi)}{2c}\right]^2\right]^{1/2},\\
&R_{r\pm}=-ct_{r\pm},\quad \beta_{r\pm}=\frac{at_{r\pm}/c}{[1+(at_{r\pm}/c)^2]^{1/2}},\\
&n_{xr\pm}=\frac{c^2/a-(c^2/a)[1+(at_{r\pm}/c)^2]^{1/2}}{R_\pm},\quad \dot{\beta}_{r\pm}=\frac{a/c}{[1+(at_{r\pm}/c)^2]^{3/2}},\quad n_{yr\pm}=\left[\frac{\mp l+\xi}{R_\pm}\right],
\end{aligned} \quad (43)$$

and following the operations mentioned earlier, one can then show that

$$k^2=\frac{4qe}{l^3}\left[1+O\left[\frac{al}{c^2}\right]\right]$$

for $al/c^2 \ll 1$.

APPENDIX B: MODEL OF A LONGITUDINAL ACCELERATED OSCILLATOR

A simple model for longitudinal oscillations, analogous to the example in Appendix A, will be briefly examined here. Let two particles of charge q be held fixed at the positions $\xi_\pm=(\pm l,0,0)$ of the accelerating coordinate system. Consequently, they will possess proper accelerations $a_\pm=a/(1\pm la/c^2)$. Assume that the oscillating particle of charge e is constrained so its motion is described by $\xi=(\xi^1(\tau_e),0,0))$. In conformity with the analysis of Sec. II, the force on the oscillating particle should be evaluated in the instantaneous rest frame of the equilibrium point.

For an unaccelerated system, $k_0^1=4qe/l^3$. In the case where $a\neq 0$, one must again use Eq. (40) in order to obtain the force on the oscillating particle. One obtains

$$F_{\tau_e x}=eq\left[-\left[\frac{1-\beta_{r+}}{1+\beta_{r+}}\right]\frac{1}{R_{r+}{}^2}+\left[\frac{1+\beta_{r-}}{1-\beta_{r-}}\right]\frac{1}{R_{r-}{}^2}\right], \quad (44)$$

where the $\pm$ signs again indicate the respective quantities associated with the source particles at $\xi^1=\pm l$. In order to evaluate Eq. (44), the following expressions are needed:

$$\begin{aligned}
&t_{r\pm}=\mp\frac{1}{2c(\xi^1+c^2/a)}\left[\left[\frac{c^2}{a_\pm}\right]^2-\left[\xi^1+\frac{c^2}{a}\right]^2\right],\\
&R_{r\pm}=-ct_{r\pm},\quad \beta_{r\pm}=\frac{a_\pm t_{r\pm}/c}{[1+(a_\pm t_{r\pm}/c)^2]^{1/2}}.
\end{aligned} \quad (45)$$

Substituting Eqs. (45) into Eq. (44) and expanding $F_{\tau_e x}$ to first order in ξ' results in an expression of the form $F_{\tau_e x}=K-k'\xi'$. For $a\neq 0$ $K\neq 0$. If $l \ll c^2/a$ and $4qe/l^3=k_0^1$, then

$$K\approx-\frac{2eq}{l^2}\left[\frac{al}{c^2}\right]\approx 0 \text{ and } k^1=\frac{4qe}{l^3}\left[1+O\left[\frac{al}{c^2}\right]\right].$$

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[1]W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

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[6]Other trajectories that might be of interest are discussed by J. R. Letaw in Phys. Rev. D **23**, 1709 (1981).

[7]In particular, see C. W. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Secs. 6.4—6.6, pp. 169—174.

[8]See, for example, the discussion on hyperbolic motion in Sec. 6.2 of Ref. 7, or the discussion by F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), Secs. 6—11, pp. 169—172. Also, see Refs. 4 and 5.

[9]See Eq. (6.6) in Ref. 7.

[10]See Eqs. (6.16) and (6.17) in Ref. 7.

[11]See the discussion of the small oscillator assumption on p. 1091 of Ref. 5.

[12]See the discussion on small oscillations of stable systems by H. Goldstein, *Classical Mechanics,* 2nd ed. (Addison-Wesley, Reading, Mass., 1980), Chap. 6, p. 243.

[13]See pp. 324 and 325 of Ref. 12. Also, a related simple problem is given by W. Rindler, *Essential Relativity: Special, General, and Cosmological,* 2nd ed. (Springer, New York, 1977), prob. 5.14, p. 263. Using Rindler's relativistic expressions, one can explicitly show that $v/c\rightarrow 0$ as the amplitude$\rightarrow 0$.

[14]Rohrlich chooses to give another definition for a "relativistic oscillator" in Secs. 6—14 of Ref. 8. In the limit of small amplitude, his example also reduces to the nonrelativistic situation of Eq. (19).

[15]See Ref. 7, p. 1008 and the related discussion on p. 393.

[16]See Rindler's discussion on pp. 50 and 51 of Ref. 13. Also, see Ref. 7, problems 37.4, 6.6, and 6.7.

[17]If a force field other than $\mathbf{f}_{\tau_e c}=m\mathbf{a}$ could be constructed that appeared as a stationary force field in the instantaneous rest frame of the spring's equilibrium point, then the frequency shift indicated by Eq. (28) could be altered and even made to vanish. The case of vanishing frequency shift would occur if $\mathbf{f}_{\tau_e c}$ was replaced by $\mathbf{f}_{\tau_e}=mc^2\hat{\mathbf{x}}/(c^2/a+\xi^1)\approx ma\hat{\mathbf{x}}(1-a\xi^1/c^2)$, which corresponds to a force which falls off inversely with distance in the $\hat{\mathbf{x}}$ direction (see Rindler's comments on p. 51 of Ref. 13).

[18]See, for example, Rohrlich's text in Ref. 8, Eq. (6.57).

[19]See Rohrlich's text in Ref. 8, Eqs. (4.47) and (4.48).

[20]See the discussion and references on stochastic electrodynamics given in Refs. 4 and 5.

[21]Obviously, the argument presented here is extremely rough. However, a rigorous justification of the approximations made in obtaining Eqs. (38) would be very difficult to give, if not impossible, without a better understanding of the effects of the high-frequency components of the zero-point fields upon the charged particle's behavior. Unfortunately, such difficulties plague most of the integrals that occur in stochastic electrodynamics. Presumably, some sort of physical mechanism exists so that the high-frequency contributions may be ignored.

[22]See, for example, J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1975), p. 657, Eq. (14.14).