# Thermal effects of acceleration for a spatially extended electromagnetic system in classical electromagnetic zero-point radiation: Transversely positioned classical oscillators

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Thermal properties of acceleration have been investigated in the past for both quantum-mechanical and classical pointlike systems. Here, for the first time, a spatially extended electromagnetic system is examined. This system consists of two spatially separated classical dipole simple harmonic oscillators that are uniformly accelerated through classical electromagnetic zero-point radiation and that interact with each other through emitted electromagnetic radiation. The two oscillators are assumed to be oriented such that their centers lie in a plane perpendicular to the direction of acceleration; no restrictions are placed upon the direction of oscillations. The behavior of this system is analyzed under the conditions of a small-oscillator assumption, the narrow-linewidth approximation, a small-laboratory approximation, and the unretarded van der Waals condition. The statistical properties investigated for this accelerated spatially extended system are found to agree with the corresponding statistical properties of a pair of similarly constructed, but unaccelerated oscillators that are bathed in a classical electromagnetic Planckian radiation spectrum characterized by the Unruh-Davies temperature of $T=\hbar a/2\pi ck$. The properties examined include the van der Waals force acting between the two oscillators and the correlation functions of the oscillators' positions and their time derivatives. During the course of obtaining these properties, a set of exact relationships are deduced involving the electromagnetic fields of a uniformly accelerated electric dipole and the correlation functions of classical electromagnetic zero-point fields, evaluated along relativistic hyperbolic trajectories.

## I. INTRODUCTION

Recently, the behavior has been analyzed for certain classical electromagnetic dipole systems that are relativistically, uniformly accelerated through classical electromagnetic zero-point radiation.[1–3] Under the assumption of a narrow-linewidth approximation, the statistical properties of these accelerated pointlike electromagnetic systems have been found to agree with the corresponding statistical properties of similarly constructed, but unaccelerated systems that are bathed in a classical electromagnetic Planckian radiation spectrum. This agreement occurs when the temperature $T$ that characterizes the thermal radiation is related to the acceleration $a$ of the accelerated systems by the Unruh-Davies relationship of $T=\hbar a/2\pi ck$ (Refs. 4 and 5).

For the first time in either the classical or quantum literature, the thermal effects of acceleration described above for pointlike electromagnetic systems are shown within this article to also hold for a particular spatially extended situation. Detailed calculations are carried out here for a special case of a spatially extended electromagnetic system that is relativistically, uniformly accelerated through classical electromagnetic zero-point radiation. Under certain specified conditions examined in this article, an equivalency in statistical behavior is shown to occur between this uniformly accelerated spatially extended system and a similar unaccelerated-thermal system. In the process of demonstrating this equivalency in behavior, a set of exact relationships are deduced between the electromagnetic fields of a time-varying electric dipole undergoing uniform acceleration and the two-point correlation functions of classical electromagnetic zero-point fields, as calculated along relativistic hyperbolic trajectories in space-time.[6]

The spatially extended electromagnetic system considered in this article consists of two classical dipole simple harmonic oscillators, oriented such that their equilibrium positions lie in a plane perpendicular to the direction of acceleration. These two spatially separated systems interact with each other via the electromagnetic radiation that each one emits. Hence, the statistical behavior of these two oscillators are correlated because of this interaction and because the two oscillators are being driven at different points in space by correlated values of classical electromagnetic zero-point fields.

The theoretical basis that will be used here for analyzing this electromagnetic system is that of classical electrodynamics with classical electromagnetic zero-point radiation, which has often been termed stochastic electrodynamics. (See Refs. 7–11 for reviews on this field of research.) The van der Waals force between two nonrelativistic classical dipole harmonic oscillators has been previously calculated within the context of stochastic electrodynamics when the temperature equals zero.[12] The result was found to agree with the corresponding result of quantum electrodynamics to all orders in the electronic charge.[12,13] This calculation was generalized, within the domain of stochastic electrodynamics, to include the case where the two oscillators were situated in a thermal plus zero-point classical electromagnetic radiation spectrum.[14]

From the standpoint of stochastic electrodynamics, the van der Waals force is simply the expectation value of the total Lorentz force acting on one of the charged oscillators. Hence, the results of the calculations of Ref. 14 may be compared to the expectation value of the force between the two accelerated oscillators considered in this article, thereby presenting a starting point for a comparison of the statistical properties of the accelerated and unaccelerated-thermal pair of oscillators. These calculations are carried out for a special oscillator system with restricted oscillatory motion in Sec. III B and for the general oscillator system in Sec. IV A.

Additional statistical properties for the accelerated oscillator system are considered in Secs. III C and IV B for the special and general oscillator systems, respectively. These properties consist of the correlation functions of the coordinate positions of each oscillator, as well as correlation functions of higher time derivatives of each oscillator's position.

Certain assumptions and approximations will be made in the analysis presented here. First, the small-oscillator assumption will be imposed (see Refs. 1 and 3), which enables the equations of motion to be linearized in the appropriate Fermi-Walker transported coordinate system.[3] Second, the radiation reaction damping constant of $\Gamma=\frac{2}{3}(e^2/mc^3)$ will be taken to be small compared to the other time constants of the system, thereby enabling the narrow-linewidth approximation to be employed when evaluating integral expressions for the expectation values of certain stochastic quantities. This approximation was also used in Refs. 1–3.

Two additional approximations will be made here that did not enter into the work of Refs. 1–3, which considered only the behavior of single accelerated electromagnetic dipole systems. Both of these assumptions involve the spatial separation $R$ between the equilibrium points of the pair of oscillators discussed in the present article.

First, a "small-laboratory" condition will be imposed of

$$R \ll \frac{c^2}{a} . \tag{1}$$

In a time of $(R/c)$, an oscillator will accelerate an approximate distance of $\frac{1}{2}a(R/c)^2$. Consequently, a light ray travels this distance along the direction of acceleration when propagating from one oscillator to the other. Equation (1) ensures that $\frac{1}{2}a(R/c)^2$ is much less than the distance $c^2/a$ to the event horizon. Moreover, on account of Eq. (1), $\frac{1}{2}a(R/c)^2 \ll R$. Therefore, the light ray described above will make a small angle to the plane perpendicular to the direction of acceleration. Thus, the condition of Eq. (1) reduces the physical distinguishability between an accelerated and unaccelerated pair of charged oscillators interacting via the electromagnetic radiation emitted from each oscillator.

The second approximation that will be imposed here involving the distance $R$ consists of the condition

$$\frac{\omega_0 R}{c} \ll 1 , \tag{2}$$

where $\omega_0$ is the resonant frequency of each oscillator. This condition is traditionally termed the unretarded van der Waals condition. Possibly, the general results of this article hold when this condition is relaxed; this possibility will not be examined here, however.

Experimentally, the unretarded van der Waals condition of Eq. (2) is of interest because it describes the region in which one would be most likely to physically discern the thermal effects of acceleration for the extended system considered here. Roughly speaking, one would expect these thermal-like properties to be discernable from the zero-point motion of each oscillator when

$$\hbar\omega_0 \lesssim kT = \frac{\hbar a}{2\pi c} . \tag{3}$$

Combining Eqs. (1) and (3) results in Eq. (2): namely, the unretarded van der Waals condition.

## II. DESCRIPTION OF ACCELERATING SYSTEM

The accelerated classical dipole harmonic oscillators considered here are assumed to have equilibrium positions lying in a plane undergoing constant proper acceleration **a** along the normal to the plane. As was done in Refs. 1 and 3, a set of inertial reference frames $I_{\tau_e}$ will be introduced here such that the $I_{\tau_e}$ frame constitutes the instantaneous rest frame at proper time $\tau_e$ for the equilibrium positions of the two oscillators. Figure 1 illustrates the configuration assumed for the two oscillators. As discussed in Ref. 3, the equilibrium position of each oscillator is given in the $I_{\tau_e}$ inertial frame at proper time $\tau_e'$ by

$$x^{\mu}_{\binom{A}{B})\tau_e}(\tau_e') = (cT_{\binom{A}{B})\tau_e}(\tau_e');\mathbf{x}_{\binom{A}{B})\tau_e}(\tau_e')) = \left[\frac{c^2}{a}\sinh\left[\frac{a}{c}(\tau_e'-\tau_e)\right];\frac{c^2}{a}\cosh\left[\frac{a}{c}(\tau_e'-\tau_e)\right],\pm\frac{R}{2},0\right] . \tag{4}$$

A Fermi-Walker transported coordinate system will be introduced (see Ref. 3) that is described by the coordinates $\xi^{\mu}=(c\tau_e;\boldsymbol{\xi})$. These coordinates are related to the coordinates $x^{\mu}_{\tau_e'}$ of an inertial frame $I_{\tau_e'}$ by

$$x^0_{\tau_e'} = ct_{\tau_e'} = \left[\xi_1+\frac{c^2}{a}\right]\sinh\left[\frac{a}{c}(\tau_e-\tau_e')\right] , \tag{5a}$$

$$x^1_{\tau_e'} = x_{\tau_e'} = \left[\xi_1+\frac{c^2}{a}\right]\cosh\left[\frac{a}{c}(\tau_e-\tau_e')\right] , \tag{5b}$$

$$x^2_{\tau_e'} = y_{\tau_e'} = \xi_2 , \tag{5c}$$

$$x^3_{\tau_e'} = z_{\tau_e'} = \xi_3 , \tag{5d}$$

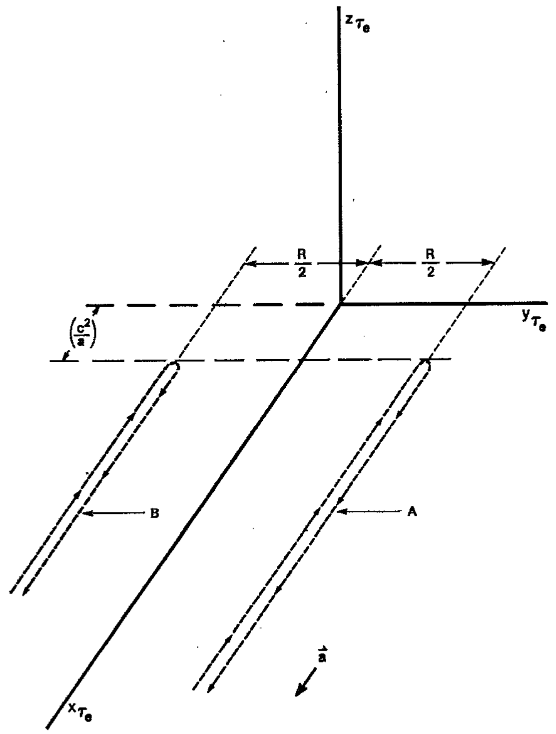


FIG. 1. Trajectories followed by the equilibrium positions of oscillators $A$ and $B$ within an inertial frame $I_{\tau_e}$.

where $x^{i}_{\tau_e'} = x_{\tau_e' i}$ and $\xi^i = \xi_i$ for $i=1,2,3$.

The model assumed for the classical dipole harmonic oscillator consists of a classical charged particle of rest mass $m$ and charge $+e$ that is attracted to a uniformly accelerating equilibrium point by a simple harmonic potential, as measured in the instantaneous rest frame of the equilibrium point of the oscillator (see Ref. 3). As an aid in picturing a specific oscillator model, a continuous negative charge distribution with net charge $-e$ will be assumed to surround the equilibrium point of the oscillator. This charge distribution will be taken to be stationary in the Fermi-Walker transported coordinate system and to possess axes of symmetry along the $i=1,2,3$ coordinate axes. The $+e$ charged particle will be assumed to oscillate inside this cloud distribution of negative charge, which can be constructed to be the source of the simple harmonic oscillator potential. For a sufficiently small volume of negative charge, the total oscillator system approximates an electric dipole in the instantaneous rest frame of the oscillator's equilibrium point.

The $i$th component of this electric dipole in the rest frame of the oscillator's equilibrium point is given by $e\xi_{Li}(\tau_e)$, where $\xi_{Li}(\tau_e)$ is the distance along the rest frame's $i$th axis from the oscillator's equilibrium point to the oscillating particle at proper time $\tau_e$ of the oscillator's equilibrium point. Here, $L$ is a label that takes on the values $A$ or $B$. (It should be noted that a distinction is being made between the general coordinate $\xi_i$ along the $i$th spatial axis in the Fermi-Walker transported coordinate system and the quantity $\xi_{Li}$ described above.)

The equations of motion for the two oscillators may be readily deduced by applying the small-oscillator approximation discussed in Refs. 1 and 3. The dipole fields $\mathbf{E}^{DL}$ and $\mathbf{B}^{DL}$ of each oscillator system will be shown later to depend linearly upon the quantities $\xi_{Li}$, $d\xi_{Li}/d\tau_e$, and $d^2\xi_{Li}/d\tau_e{}^2$. Equation (38) of Ref. 3 may then be generalized to the following set of equations:

$$\frac{d^2\xi_{Ai}}{d\tau_e{}^2} = -(\omega_i)^2\xi_{Ai} + \Gamma\left[\frac{d^3\xi_{Ai}}{d\tau_e{}^3} - \left[\frac{a}{c}\right]^2 \frac{d\xi_{Ai}}{d\tau_e}\right] + \frac{e}{m}\left[E^{\mathrm{ZP}}_{\tau_e i}\left[+\hat{\mathbf{y}}\frac{R}{2},\tau_e\right] + E^{DB}_{\tau_e i}\left[+\hat{\mathbf{y}}\frac{R}{2},\tau_e\right]\right], \tag{6a}$$

$$\frac{d^2\xi_{Bi}}{d\tau_e{}^2} = -(\omega_i)^2\xi_{Bi} + \Gamma\left[\frac{d^3\xi_{Bi}}{d\tau_e{}^3} - \left[\frac{a}{c}\right]^2 \frac{d\xi_{Bi}}{d\tau_e}\right] + \frac{e}{m}\left[E^{\mathrm{ZP}}_{\tau_e i}\left[-\hat{\mathbf{y}}\frac{R}{2},\tau_e\right] + E^{DA}_{\tau_e i}\left[-\hat{\mathbf{y}}\frac{R}{2},\tau_e\right]\right]. \tag{6b}$$

The argument $[\pm\hat{\mathbf{y}}(R/2),\tau_e]$ is used to indicate that the fields are to be evaluated at proper time $\tau_e$ along the trajectory of the equilibrium point of the indicated oscillator. The quantity $E^{\mathrm{ZP}}_{\tau_e i}$ denotes the zero-point electric fields, while $E^{DA}_{\tau_e i}$ and $E^{DB}_{\tau_e i}$ represent the electric dipole fields of the two oscillators; the subscript $\tau_e$ indicates the fields are evaluated in the inertial frame $I_{\tau_e}$. These fields are readily expressed in terms of the fields of another inertial frame $I_{\tau_e'}$ via the Lorentz transformation

$$\mathbf{E}_{\tau_e} = \hat{\mathbf{x}}E_{\tau_e' 1} + \hat{\mathbf{y}}\gamma_{(\tau_e-\tau_e')}(E_{\tau_e' 2} - \beta_{(\tau_e-\tau_e')}B_{\tau_e' 3}) + \hat{\mathbf{z}}\gamma_{(\tau_e-\tau_e')}(E_{\tau_e' 3} + \beta_{(\tau_e-\tau_e')}B_{\tau_e' 2}), \tag{7}$$

$$\mathbf{B}_{\tau_e} = \hat{\mathbf{x}}B_{\tau_e' 1} + \hat{\mathbf{y}}\gamma_{(\tau_e-\tau_e')}(B_{\tau_e' 2} + \beta_{(\tau_e-\tau_e')}E_{\tau_e' 3}) + \hat{\mathbf{z}}\gamma_{(\tau_e-\tau_e')}(B_{\tau_e' 3} - \beta_{(\tau_e-\tau_e')}E_{\tau_e' 2}), \tag{8}$$

where

$$\gamma_{(\tau_e-\tau_e')} = \cosh\left[\frac{a}{c}(\tau_e-\tau_e')\right]$$

and

$$\beta_{(\tau_e-\tau_e')} = \tanh\left[\frac{a}{c}(\tau_e-\tau_e')\right].$$

In order to solve Eqs. (6a) and (6b), expressions for the dipole fields $E^{DB}_{\tau_e i}$ and $E^{DA}_{\tau_e i}$ must be obtained. This work is carried out in Appendix A and will simply be summarized here.

Consider a time-varying electric dipole that is uniformly accelerated and that possesses a spatial position in the associated Fermi-Walker transported coordinate system given by $\xi = \hat{\mathbf{y}}\mathscr{R}_L$. From Appendix A, the electromagnetic fields at the coordinate position $\xi^\mu = (c\tau_e;\hat{\mathbf{y}}\mathscr{R})$, due to this $L$ labeled accelerating electric dipole, are given by

$$E_{\tau_e i}^{DL}(\hat{\mathbf{y}}\mathscr{R},\tau_e)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{E}_i^{DL}(\hat{\mathbf{y}}\mathscr{R},\Omega)\exp(-i\Omega\tau_e)d\Omega\ , \tag{9}$$

$$B_{\tau_e i}^{DL}(\hat{\mathbf{y}}\mathscr{R},\tau_e)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{B}_i^{DL}(\hat{\mathbf{y}}\mathscr{R},\Omega)\exp(-i\Omega\tau_e)d\Omega\ , \tag{10}$$

where

$$\tilde{E}_i^{DL}(\hat{\mathbf{y}}\mathscr{R},\Omega)=\sum_{j=1}^{3}\eta_{ij}^D(\hat{\mathbf{x}}a,\hat{\mathbf{y}}(\mathscr{R}-\mathscr{R}_L),\Omega)[e\tilde{\xi}_{Lj}(\Omega)]\ , \tag{11}$$

$$\tilde{B}_i^{DL}(\hat{\mathbf{y}}\mathscr{R},\Omega)=\sum_{j=1}^{3}\rho_{ij}^D(\hat{\mathbf{x}}a,\hat{\mathbf{y}}(\mathscr{R}-\mathscr{R}_L),\Omega)[e\tilde{\xi}_{Lj}(\Omega)]\ . \tag{12}$$

The electric dipole, as measured in its instantaneous rest frame, is expressed here by

$$[e\xi_{Li}(\tau_e)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[e\xi_{Li}(\Omega)]\exp(-i\Omega\tau_e)d\Omega\ . \tag{13}$$

The quantities $\eta_{ij}^D$ and $\rho_{ij}^D$ that occur in Eqs. (11) and (12) are rather complicated functions of the acceleration $a$, coordinate difference $(\mathscr{R}-\mathscr{R}_L)$, and frequency $\Omega$ [see Eqs. (A37)—(A44)].

In correspondence with Eq. (9), let the zero-point electric fields be written as

$$E_{\tau_e i}^{ZP}(\hat{\mathbf{y}}\mathscr{R},\tau_e)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{E}_i^{ZP}(\hat{\mathbf{y}}\mathscr{R},\Omega)\exp(-i\Omega\tau_e)d\Omega\ . \tag{14}$$

Substituting Eqs. (9), (11), (13), and (14) into Eqs. (6a) and (6b), yields the following set of equations:

$$C_i(a,\Omega)\tilde{\xi}_{Ai}(\Omega)+\sum_{j=1}^{3}\eta_{ij}(\hat{\mathbf{x}}a,+\hat{\mathbf{y}}R,\Omega)\tilde{\xi}_{Bj}(\Omega)=\frac{e}{m}\tilde{E}_i^{ZP}\left[+\hat{\mathbf{y}}\frac{R}{2},\Omega\right]\ , \tag{15a}$$

$$C_i(a,\Omega)\tilde{\xi}_{Bi}(\Omega)+\sum_{j=1}^{3}\eta_{ij}(\hat{\mathbf{x}}a,-\hat{\mathbf{y}}R,\Omega)\tilde{\xi}_{Aj}(\Omega)=\frac{e}{m}\tilde{E}_i^{ZP}\left[-\hat{\mathbf{y}}\frac{R}{2},\Omega\right]\ , \tag{15b}$$

where

$$C_i(a,\Omega)=-\Omega^2+(\omega_i)^2-i\Gamma\left[\Omega^3+\Omega\left[\frac{a}{c}\right]^2\right] \tag{16}$$

and

$$\eta_{ij}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)\equiv-\frac{e^2}{m}\eta_{ij}^D(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)\ . \tag{17}$$

The quantity $\eta_{ij}$ was introduced here simply for notational reasons, since otherwise the factor of $e^2/m$ would occur repeatedly in subsequent calculations.

Equations (15a) and (15b) may be simplified somewhat by noting from Eqs. (A37)—(A41) that

$$\eta_{ii}^D(\hat{\mathbf{x}}a,+\hat{\mathbf{y}}R,\Omega)=\eta_{ii}^D(\hat{\mathbf{x}}a,-\hat{\mathbf{y}}R,\Omega)\ ,$$

$$\eta_{ij}^D(\hat{\mathbf{x}}a,+\hat{\mathbf{y}}R,\Omega)=-\eta_{ij}^D(\hat{\mathbf{x}}a,-\hat{\mathbf{y}}R,\Omega)\quad\text{for } i\neq j\ ,$$

$\eta_{12}^D=-\eta_{21}^D$, and $\eta_{i3}^D=\eta_{3i}^D=0$ for $i\neq3$. Using conventional matrix methods, Eqs. (15a) and (15b) can then be solved for $\tilde{\xi}_{Ai}(\Omega)$ and $\tilde{\xi}_{Bi}(\Omega)$ in terms of $\tilde{E}_i^{ZP}[+\hat{\mathbf{y}}(R/2),\Omega]$ and $\tilde{E}_i^{ZP}[-\hat{\mathbf{y}}(R/2),\Omega]$. These solutions will be used in Secs. III and IV to deduce certain statistical properties of the accelerating oscillator system.

## III. SPECIAL CASE OF TWO ACCELERATING DIPOLE OSCILLATORS

### A. Description of a special case

The full calculations involved in obtaining the statistical properties of the accelerating dipole oscillators are rather long. Consequently, it seems appropriate to examine the simplest case possible that illustrates the essential physics of the system before proceeding with the general situation in Sec. IV. Such a case arises due to the fact that the $i=3$ set of equations obtained from Eqs. (15a) and (15b) are not coupled to the corresponding $i=1,2$ set of equations, since $\eta_{i3}=\eta_{3i}=0$ for $i\neq3$. Hence, the special case of two accelerating electric dipole oscillators, with oscillatory motion confined to only the $\hat{\mathbf{z}}$ direction, may be safely studied for the main underlying physics of the accelerating system.

From Eqs. (15a) and (15b)

$$\tilde{\xi}_{A3}(\Omega)=\frac{e}{2m}\left[\tilde{E}_3^{ZP}\left[+\hat{\mathbf{y}}\frac{R}{2},\Omega\right]\left[\frac{1}{C_3^a-\eta_{33}^a}+\frac{1}{C_3^a+\eta_{33}^a}\right]+\tilde{E}_3^{ZP}\left[-\hat{\mathbf{y}}\frac{R}{2},\Omega\right]\left[\frac{-1}{C_3^a+\eta_{33}^a}+\frac{1}{C_3^a+\eta_{33}^a}\right]\right]\ , \tag{18a}$$

$$\tilde{\xi}_{B3}(\Omega)=\frac{e}{2m}\left[\tilde{E}_3^{ZP}\left[+\hat{\mathbf{y}}\frac{R}{2},\Omega\right]\left[\frac{-1}{C_3^a-\eta_{33}^a}+\frac{1}{C_3^a+\eta_{33}^a}\right]+\tilde{E}_3^{ZP}\left[-\hat{\mathbf{y}}\frac{R}{2},\Omega\right]\left[\frac{1}{C_3^a-\eta_{33}^a}+\frac{1}{C_3^a+\eta_{33}^a}\right]\right]\ , \tag{18b}$$

where $C_3(a,\Omega)$ and $\eta_{33}(\hat{\mathbf{x}}a,+\hat{\mathbf{y}}R,\Omega)$ have been abbreviated by $C_3^a$ and $\eta_{33}^a$.

### B. Expectation value of Lorentz force

Let $\mathbf{F}_{\tau_e}(t_{\tau_e})$ be the Lorentz force on a uniformly accelerating dipole oscillator at time $t_{\tau_e}$ in the $I_{\tau_e}$ inertial frame. According to the construction of the set of inertial reference frames $I_{\tau_e}$ in Ref. 3, $\mathbf{F}_{\tau_e}(t_{\tau_e}=0)$ represents the force in the rest frame of the equilibrium point of the oscillator at proper time $\tau_e$. From Eq. (B3),

$$F_{\tau_e i}(t_{\tau_e}=0)=e\sum_{j=1}^{3}\Delta x_{\tau_e j}(0)\frac{\partial}{\partial x_{\tau_e i}}E_{\tau_e j}(\mathbf{x}_{\tau_e},t_{\tau_e})\Big|_{\mathbf{x}_{\tau_e}(0),0}+\frac{e}{c}\frac{d}{dt_{\tau_e}}[\Delta\mathbf{x}_{\tau_e}(t_{\tau_e})\otimes\mathbf{B}_{\tau_e}(\mathbf{X}(t_{\tau_e}),t_{\tau_e})]_i\Big|_{t_{\tau_e}=0}\,, \tag{19}$$

where $\mathbf{E}_{\tau_e}$ and $\mathbf{B}_{\tau_e}$ are the total electric and magnetic fields due to the zero-point radiation fields and the fields of the opposite accelerating oscillator.

The zero-point electromagnetic fields will be specified in an inertial frame $I_{\tau_{e*}}$. Using the Lorentz transformation of Eqs. (7) and (8), these fields may then be obtained in all $I_{\tau_e}$ inertial frames. The functional form that will be used for the zero-point fields will consist of the following relationships, which have frequently been used in performing calculations in stochastic electrodynamics:[15]

$$\mathbf{E}^{\text{ZP}}_{\tau_{e*}}(\mathbf{x}_{\tau_{e*}},t_{\tau_{e*}})=\sum_{\lambda=1}^{2}\int d^3k\,h(\omega)\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)\cos[\mathbf{k}\cdot\mathbf{x}_{\tau_{e*}}-\omega t_{\tau_{e*}}+\theta(\mathbf{k},\lambda)]\,, \tag{20}$$

$$\mathbf{B}^{\text{ZP}}_{\tau_{e*}}(\mathbf{x}_{\tau_{e*}},t_{\tau_{e*}})=\sum_{\lambda=1}^{2}\int d^3k\,h(\omega)(\hat{\mathbf{k}}\otimes\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda))\cos[\mathbf{k}\cdot\mathbf{x}_{\tau_{e*}}-\omega t_{\tau_{e*}}+\theta(\mathbf{k},\lambda)]\,. \tag{21}$$

The polarization vectors satisfy the identities

$$\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)\cdot\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda')=\delta_{\lambda\lambda'}\,, \tag{22}$$

$$\mathbf{k}\cdot\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)=0\,. \tag{23}$$

The phase angle $\theta(\mathbf{k},\lambda)$ is a random variable, independently distributed for each $\mathbf{k}$ and $\lambda$, that ranges between 0 and $2\pi$ with uniform probability density. The function $h(\omega)$ is given by

$$h^2(\omega)=\frac{\hbar\omega}{2\pi^2}\,. \tag{24}$$

The expectation value of the force component along the direction of separation ($i=2$) between the two oscillators will be calculated in this article. According to the construction of the Fermi-Walker transported coordinate system in Ref. 3, the quantity $\Delta x_{\tau_e j}(0)$ in the first term of Eq. (19) may be replaced by $\xi_{Aj}(\tau_e)$. From Eq. (5c), $\xi_2=x_{\tau_e 2}$; hence, the operator $\partial/\partial x_{\tau_e 2}$ may be replaced by $\partial/\partial\xi_2$. Let $\mathbf{F}_{\tau_e}(t_{\tau_e})$ be relabeled by $\mathbf{F}_{\tau_e}(\tau_e')$, where $t_{\tau_e}$ is related to the proper time $\tau_e'$ by (assuming $\xi_{A1}=0$)

$$t_{\tau_e}(\tau_e')=\frac{c}{a}\sinh\left[\frac{a}{c}(\tau_e'-\tau_e)\right]. \tag{25}$$

Consequently

$$\frac{d\tau_e'}{dt_{\tau_e}}\Bigg|_{\tau_e'=\tau_e}=1\,. \tag{26}$$

For the special case being considered in this section of

$$\Delta\mathbf{x}_{\binom{A}{B}\tau_e}[t_{\tau_e}(\tau_e')]=\hat{\mathbf{z}}\xi_{\binom{A}{B}3}(\tau_e')\,, \tag{27}$$

the expectation value of Eq. (19), when $i=2$, is given by

$$\langle F_{A\tau_e 2}(\tau_e)\rangle=e\left\langle\xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{\text{ZP}}_{\tau_e 3}(\xi,\tau_e)\Bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle+e\left\langle\xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{DB}_{\tau_e 3}(\xi,\tau_e)\Bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle$$
$$+\frac{e}{c}\frac{d}{d\tau_e'}\left\langle\xi_{A3}(\tau_e')\left[B^{\text{ZP}}_{\tau_e 1}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]+B^{DB}_{\tau_e 1}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]\right]\right\rangle\Bigg|_{\tau_e'=\tau_e}. \tag{28}$$

The arguments in the fields have been relabeled here in terms of the $\xi^\mu$ coordinates.

From Eqs. (13), (18), and the inverse of (14), the first term in Eq. (28) becomes

$$e\left\langle\xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{\text{ZP}}_{\tau_e 3}(\xi,\tau_e)\Bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle$$
$$=\frac{e}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\Omega'\exp(-i\Omega'\tau_e)\left\langle\tilde{\xi}_{A3}(\Omega')\frac{\partial}{\partial\xi_2}E^{\text{ZP}}_{\tau_e 3}(\xi,\tau_e)\Bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle$$
$$=\left[\frac{e^2}{2m}\right]\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega'\exp(-i\Omega'\tau_e)\int_{-\infty}^{\infty}d\tau_e'\exp(i\Omega'\tau_e')$$
$$\times\left[\left[\frac{1}{C_3^{a'}-\eta_{33}^{a'}}+\frac{1}{C_3^{a'}+\eta_{33}^{a'}}\right]\left\langle E^{\text{ZP}}_{\tau_e' 3}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]\frac{\partial}{\partial\xi_2}E^{\text{ZP}}_{\tau_e 3}(\xi,\tau_e)\Bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle\right.$$
$$\left.+\left[\frac{-1}{C_3^{a'}-\eta_{33}^{a'}}+\frac{1}{C_3^{a'}+\eta_{33}^{a'}}\right]\left\langle E^{\text{ZP}}_{\tau_e' 3}\left[-\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]\frac{\partial}{\partial\xi_2}E^{\text{ZP}}_{\tau_e 3}(\xi,\tau_e)\Bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle\right]. \tag{29}$$

The two quantities involving the expectation values in Eq. (29) are related to the first of the two-point field correlation functions listed below:

$$\langle E^{\mathrm{ZP}}_{\tau'_e i}(\hat{\mathbf{y}}R',\tau'_e)E^{\mathrm{ZP}}_{\tau''_e j}(\hat{\mathbf{y}}R'',\tau''_e)\rangle \ ,$$

$$\langle B^{\mathrm{ZP}}_{\tau'_e i}(\hat{\mathbf{y}}R',\tau'_e)B^{\mathrm{ZP}}_{\tau''_e j}(\hat{\mathbf{y}}R'',\tau''_e)\rangle \ ,$$

$$\langle B^{\mathrm{ZP}}_{\tau'_e i}(\hat{\mathbf{y}}R',\tau'_e)E^{\mathrm{ZP}}_{\tau''_e j}(\hat{\mathbf{y}}R'',\tau''_e)\rangle \ .$$

These correlation functions are evaluated in Appendix C, where they are shown to be given by the cosine and sine expansions of Eqs. (C1) and (C2). From the work of Appendixes A and C, the quantities $f^{\mathrm{ZP}}_{ij}$ and $g^{\mathrm{ZP}}_{ij}$ appearing in these expansions are found to obey interesting relationships [see Eqs. (C3) and (C4)] to the functions $\eta^{Da}_{ij}$ and $\rho^{Da}_{ij}$ that appear in the dipole fields of Eqs. (11) and (12). Equation (C3) turns out to be the main property that enables the spatially extended accelerated system described in this article to be related to an analogous unaccelerated-thermal system.

Equations (C35) and (C37) relate the expectation values of the two quantities in Eq. (29) to the first of the correlation functions listed above. Hence, one obtains

$$\begin{aligned} e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{\mathrm{ZP}}_{\tau_e 3}(\xi,\tau_e)\Big|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle &= \left[\frac{e^2}{2m}\right]\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega'\exp(-i\Omega'\tau_e)\int_{-\infty}^{\infty}d\tau'_e\exp(i\Omega'\tau'_e) \\ &\quad\times\left[\frac{-1}{C^{a'}_3-\eta^{a'}_{33}}+\frac{1}{C^{a'}_3+\eta^{a'}_{33}}\right]\int_0^{\infty}d\Omega\frac{\partial}{\partial R}f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)\cos[\Omega(\tau'_e-\tau_e)] \\ &= \left[\frac{e^2}{4m}\right]\int_0^{\infty}d\Omega\frac{\partial}{\partial R}f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)\int_{-\infty}^{\infty}d\Omega'\left[\frac{-1}{C^{a'}_3-\eta^{a'}_{33}}+\frac{1}{C^{a'}_3+\eta^{a'}_{33}}\right][\delta(\Omega'-\Omega)+\delta(\Omega'+\Omega)] \ . \end{aligned} \tag{30}$$

From Eq. (16),

$$C_i(a,-\Omega)=C^*_i(a,\Omega) \ . \tag{31}$$

This same property holds for $\eta^{Da}_{ij}$ [see Eq. (A45)]. Hence,

$$e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{\mathrm{ZP}}_{\tau_e 3}(\xi,\tau_e)\Big|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle=\frac{e^2}{2m}\int_0^{\infty}d\Omega\left[-\frac{\mathrm{Re}(C^a_3-\eta^a_{33})}{|C^a_3-\eta^a_{33}|^2}+\frac{\mathrm{Re}(C^a_3+\eta^a_{33})}{|C^a_3+\eta^a_{33}|^2}\right]\frac{\partial}{\partial R}f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega) \ . \tag{32}$$

The second term in Eq. (28) can be calculated in a similar manner. From Eqs. (9), (11), and (13), one obtains

$$\begin{aligned} e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{DB}_{\tau_e 3}(\xi,\tau_e)\Big|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle &= \frac{e}{2\pi}\int_{-\infty}^{\infty}d\Omega'\exp(-i\Omega'\tau_e)\int_{-\infty}^{\infty}d\Omega''\exp(-i\Omega''\tau_e)\left[-\frac{m}{e}\right] \\ &\quad\times\left[\frac{\partial}{\partial R}\eta_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega'')\right]\langle\tilde{\xi}_{A3}(\Omega')\tilde{\xi}_{B3}(\Omega'')\rangle \ . \end{aligned} \tag{33}$$

From Eqs. (18), (C1), (C34), and the inverse of (14), one can prove that

$$\begin{aligned} \langle\tilde{\xi}_{(^A_B)3}(\Omega')\tilde{\xi}_{(^A_B)3}(\Omega'')\rangle &= \langle\tilde{\xi}_{(^B_A)3}(\Omega')\tilde{\xi}_{(^B_A)3}(\Omega'')\rangle \\ &= \left[\frac{e}{2m}\right]^2 2\pi\int_0^{\infty}d\Omega[\delta(\Omega'-\Omega)\delta(\Omega''+\Omega)+\delta(\Omega'+\Omega)\delta(\Omega''-\Omega)] \\ &\quad\times\left[\pm\frac{f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,0,\Omega)-f_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)}{|C^a_3-\eta^a_{33}|^2}+\frac{f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,0,\Omega)+f_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)}{|C^a_3+\eta^a_{33}|^2}\right] \ . \end{aligned} \tag{34}$$

Consequently,

$$\begin{aligned} e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{DB}_{\tau_e 3}(\xi,\tau_e)\Big|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle &= \frac{-e^2}{2m}\int_0^{\infty}d\Omega\left[\frac{-[f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,0,\Omega)-f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)]}{|C^a_3-\eta^a_{33}|^2}\right. \\ &\quad\left.+\frac{f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,0,\Omega)+f^{\mathrm{ZP}}_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)}{|C^a_{33}+\eta^a_{33}|^2}\right]\frac{\partial\,\mathrm{Re}\eta^a_{33}}{\partial R} \ . \end{aligned} \tag{35}$$

The third term in Eq. (28), which came from the second term of Eq. (19), is easily shown to equal zero. From Eq. (8), $B_{\tau_e 1}(\xi,\tau_e')=B_{\tau_e' 1}(\xi,\tau_e')$. This means that all quantities inside the expectation value signs in the third term of Eq. (28) depend only upon $\tau_e'$. Using Eqs. (18), (A36), (C1), and (C2), explicit calculations can then be carried out to demonstrate that upon taking the expectation value, a result is obtained that is totally independent of the value of $\tau_e'$. Alternatively, this demonstration follows more generally from the physical demand that the act of accelerating through the zero-point fields must yield statistical properties that are stationary in the proper time $\tau_e$. Using either argument, however, the net result is that the third term of Eq. (28) is exactly equal to zero.

A few relationships will now be established that will be useful in simplifying the results of both this section and Sec. IV. The series expansion of Eq. (A22) given by

$$\Delta\tau_- = \frac{c}{a}\operatorname{arcsinh}\left[\frac{aR_-}{c^2}\right] = \frac{c}{a}\left[\frac{aR_-}{c^2}-\frac{1}{6}\left[\frac{aR_-}{c^2}\right]^3+\cdots\right], \tag{36}$$

along with Eqs. (A19) and (A20), enable the following identities to be verified:

$$\lim_{R\to 0}\left[\frac{1-\dfrac{c\Delta\tau_-}{R_-}}{R_-^2}\right]=\frac{1}{6}\left[\frac{a}{c^2}\right]^2, \tag{37}$$

$$\lim_{R\to 0}\left[\frac{c\Delta\tau_-}{R_-}\right]=1\,. \tag{38}$$

From Eqs. (37), (38), (A37)–(A39), and (17), one can show that when $\Omega R/c \ll 1$ and $aR/c^2 \ll 1$, then

$$\eta_{ii}^a=\left\{\chi_{ii}-i\Gamma\Omega^3\left[1+\left[\frac{a/c}{\Omega}\right]^2\right]\right\} \times\left[1+O\left[\frac{aR}{c^2}\right]+O\left[\frac{\Omega R}{c^2}\right]\right], \tag{39}$$

where

$$\chi_{11}=\chi_{33}=-\tfrac{1}{2}\chi_{22}=+\frac{3}{2}\Gamma\left[\frac{c}{R}\right]^3. \tag{40}$$

[In order to verify the imaginary part of Eq. (39), a Taylor-series expansion of $\exp(i\Omega\Delta\tau_-)$ should be made, with terms up to the third power in $\Delta\tau_-$ retained.] From Eqs. (A40), (A41), and (16), as well as the above results,

$$\lim_{R\to 0}\operatorname{Im}\eta_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)=+\delta_{ij}\operatorname{Im}C_i(a,\Omega) =-\delta_{ij}\Gamma\left[\Omega^3+\Omega\left[\frac{a}{c}\right]^2\right]. \tag{41}$$

Finally, one can show that

$$\eta_{11}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)\,|_{a=0}=\eta_{33}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)\,|_{a=0}=-\frac{3}{2}\Gamma\Omega^3\left[\frac{1}{kR}+\frac{i}{(kR)^2}-\frac{1}{(kR)^3}\right]\exp(ikR)\,, \tag{42}$$

$$\eta_{22}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)\,|_{a=0}=-3\Gamma\Omega^3\left[\frac{-i}{(kR)^2}+\frac{1}{(kR)^3}\right]\exp(ikR)\,, \tag{43}$$

$$C_i(a,\Omega)\,|_{a=0}=-\Omega^2+(\omega_i)^2-i\Gamma\Omega^3\,. \tag{44}$$

Returning to the evaluation of Eq. (28), Eqs. (32) and (35) can be combined to obtain

$$\langle F_{A\tau_e 2}(\tau_e)\rangle=-\pi\int_0^\infty\frac{d\Omega}{\Omega}h_T{}^2(\Omega)\Bigg|_{T=\hbar a/2\pi ck}\left[\frac{\operatorname{Re}(C_3^a+\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Im}(C_3^a+\eta_{33}^a)-\operatorname{Im}(C_3^a+\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Re}(C_3^a+\eta_{33}^a)}{|C_3^a+\eta_{33}^a|^2} +\frac{\operatorname{Re}(C_3^a-\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Im}(C_3^a-\eta_{33}^a)-\operatorname{Im}(C_3^a-\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Re}(C_3^a-\eta_{33}^a)}{|C_3^a-\eta_{33}^a|^2}\right]. \tag{45}$$

Use was made here of Eqs. (C3) and (41). Using Eq. (C5) and the algebraic relationship of

$$\frac{\operatorname{Re}(C_3^a\pm\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Im}(C_3^a\pm\eta_{33}^a)-\operatorname{Im}(C_3^a\pm\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Re}(C_3^a\pm\eta_{33}^a)}{|C_3^a\pm\eta_{33}^a|^2} =\operatorname{Im}\left[\frac{1}{(C_3^a\pm\eta_{33}^a)}\frac{\partial}{\partial R}(C_3^a\pm\eta_{33}^a)\right]=\frac{\partial}{\partial R}\operatorname{Im}\ln\left[1\pm\frac{\eta_{33}^a}{C_3^a}\right], \tag{46}$$

yields

$$\langle F_{A\tau_e 2}(\tau_e)\rangle=-\frac{\partial}{\partial R}U(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R)\,, \tag{47}$$

where

$$U(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R)=\frac{\hbar}{2\pi}\int_0^\infty d\Omega\coth\left[\frac{\hbar\Omega}{2kT}\right]\Bigg|_{T=\hbar a/2\pi ck}\operatorname{Im}\ln\left[1-\left[\frac{\eta_{33}^a}{C_3^a}\right]^2\right]. \tag{48}$$

An analogous calculation has previously been carried out for the retarded van der Waals force between two unaccelerated, nonrelativistic classical dipole oscillators that are bathed in classical electromagnetic thermal plus zero-point radiation [see Ref. 14, Eqs. (8) and (9)[16]]. This result is valid for all values of the separation distance $R$ and for all orders in the electronic charge.

As one will note, the force expression for the unaccelerated-thermal oscillator system of Ref. 14 differs from the above force expression for the analogous accelerated system. Full agreement between the two expressions may be obtained by replacing the quantities of $\eta_{33}^a$ and $C_3^a$ in Eq. (48) by their $a\to 0$ counterparts of Eqs. (42) and (44). Thus, there exists a direct correspondence, but distinct difference, between the two force expressions. The main reason for this direct correspondence is due to the identities of Eqs. (C1), (C3), and (C5), which relate the $\eta_{ij}^{Da}$ function that appears in the expression for the electric dipole fields to the cosine transform of the correlation function of the zero-point fields. From these equations comes the $\coth(\pi c/a)$ factor, which is the distinguishing factor that relates the accelerating situation to the unaccelerated-thermal case, when $T=\hbar a/2\pi ck$.

Full agreement between the two force expressions can be obtained when the conditions discussed in Sec. I are imposed. The form of Eq. (45) for the force expression will be used for this demonstration. Since $\operatorname{Im}(C_i^a\pm\eta_{ii}^a)$ and $\operatorname{Re}(\eta_{ii}^a)$ are proportional to the damping time $\Gamma=\frac{2}{3}(e^2/mc^3)$, then the quantity

$$\frac{1}{|C_i^a\pm\eta_{ii}^a|^2}=\frac{1}{[-\Omega^2+(\omega_i)^2\pm\operatorname{Re}\eta_{ii}^a]^2+[\operatorname{Im}(C_i^a\pm\eta_{ii}^a)]^2}, \tag{49}$$

becomes a sharply peaked function near $\omega_i$ for small values of $\Gamma$. Consequently, a resonant approximation will be employed in evaluating the integral in Eq. (45). The functions $1/|C_i^a\pm\eta_{ii}^a|^2$, for $i=3$, will be replaced by

$$\frac{1}{|C_i^a\pm\eta_{ii}^a|^2}\approx\frac{1}{(\Omega-\omega_{i\pm})^2(2\omega_{i\pm})^2+[\operatorname{Im}(C_i^a\pm\eta_{ii}^a)\,|_{\omega_{i\pm}}]^2}, \tag{50}$$

where

$$\omega_{i\pm}=[(\omega_i)^2\pm\operatorname{Re}\eta_{ii}^a\,|_{\omega_{i\pm}}]^{1/2}$$
$$\approx\omega_i\pm\frac{1}{2}\frac{\chi_{ii}}{\omega_i}. \tag{51}$$

The second part of Eq. (51) follows from Eqs. (1), (2), (39), and the assumption that

$$\frac{1}{\omega_i^2}|\chi_{ii}|\approx\left[\Gamma\frac{c}{R}\right]\Big/\left[\frac{\omega_i R}{c}\right]^2\ll 1. \tag{52}$$

All quantities in the integrand of Eq. (45) that do not involve $(\Omega-\omega_{3\pm})$ will be replaced by their values at $\omega_{3\pm}$. The lower limit of integration will be replaced by $-\infty$, as the additional contribution to the integral is negligible compared to the resonant part. The two terms in the integrand that contain the quantities

$$\frac{\operatorname{Re}(C_3^a\pm\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Im}(C_3^a\pm\eta_{33}^a)}{|C_3^a\pm\eta_{33}^a|^2}$$

are of negligible contribution to the integral, because $\operatorname{Re}(C_3^a\pm\eta_{33}^a)=0$ when $\Omega=\omega_{3\pm}$. [These two terms come from Eq. (32); hence, under the resonant approximation, the first term of Eq. (28) is negligible compared to the second one.] Using the integral

$$\int_{-\infty}^{\infty}\frac{dx}{A^2x^2+B^2}=\frac{\pi}{|A\cdot B|}, \tag{53}$$

the two main contributing terms in Eq. (45) become

$$\begin{aligned}
&\pi\int_0^\infty\frac{d\Omega}{\Omega}h_T{}^2(\Omega)\Bigg|_{T=\hbar a/2\pi ck}\frac{\operatorname{Im}(C_3^a\pm\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Re}(C_3^a\pm\eta_{33}^a)}{|C_3^a\pm\eta_{33}^a|^2}\\
&\approx(\pm1)\pi\left[\frac{h_T{}^2(\Omega)}{\Omega}\Bigg|_{T=\hbar a/2\pi ck}\operatorname{Im}(C_3^a\pm\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Re}\eta_{33}^a\right]_{\Omega=\omega_{3\pm}}\int_{-\infty}^{\infty}\frac{d\Omega}{(\Omega-\omega_{3\pm})^2(2\omega_{3\pm})^2+[\operatorname{Im}(C_3^a\pm\eta_{33}^a)\,|_{\Omega=\omega_{3\pm}}]^2}\\
&=(\pm1)\pi\left[\frac{h_T{}^2(\Omega)}{\Omega}\Bigg|_{T=\hbar a/2\pi ck}\operatorname{Im}(C_3^a\pm\eta_{33}^a)\frac{\partial}{\partial R}\operatorname{Re}\eta_{33}^a\right]_{\Omega=\omega_{3\pm}}\frac{\pi}{2\omega_{3\pm}\,|\operatorname{Im}(C_3^a\pm\eta_{33}^a)\,|_{\Omega=\omega_{3\pm}}}\\
&\approx(\pm1)\frac{-\hbar}{4\omega_{3\pm}}\coth\left[\frac{\hbar\omega_{3\pm}}{2kT}\right]\Bigg|_{T=\hbar a/2\pi ck}\frac{\partial}{\partial R}\chi_{33}.
\end{aligned} \tag{54}$$

Thus, under the resonant approximation, the factor of $\operatorname{Im}(C_3\pm\eta_{33}^a)$ is removed by cancellation in Eq. (54). Hence, the distinguishing character, namely, $\eta_{33}^a$ and $C_3^a$ versus their $a\to 0$ counterparts, no longer arises between the two force expressions for the accelerated-zero-point (ZP) and unaccelerated-thermal situations. Using Eq. (51) and expanding the result found in Eq. (54) about the point $\omega_3$ yields

$$\langle F_{A\tau_e 2}(\tau_e)\rangle \approx -\frac{\partial}{\partial R}\left\{\frac{\hbar}{8\omega_3}(\chi_{33})^2\frac{\partial}{\partial\Omega}\left[\frac{1}{\Omega}\coth\left[\frac{\hbar\Omega}{2kT}\right]\Bigg|_{T=\hbar a/2\pi ck}\right]\Bigg|_{\Omega=\omega_3}\right\}. \tag{55}$$

Equation (55) agrees exactly with Eq. (38) in Ref. 14. Hence, regarding the expectation value of the force between two dipole linear oscillators, an equivalence has been demonstrated for (1) the special uniformly accelerated dipole oscillator system discussed in this section and (2) a similar unaccelerated oscillator system held fixed in an inertial frame, but bathed with zero-point plus thermal electromagnetic radiation characterized by the temperature of $T=\hbar a/2\pi ck$.

### C. Other properties of special system

Additional properties of the two-oscillator system will be examined in this section in order to determine to what extent the accelerated-ZP and unaccelerated-thermal systems possess identical statistical properties, as observed in their respective coordinate systems. The quantities that will be examined here are the correlation functions of the position and the time derivatives of the position for the two oscillating particles.

From Eq. (34), one can obtain

$$\left\langle \frac{d^m}{d\tau_e^m}\xi_{(A)3}\Bigg|_{\tau_{e0}} \frac{d^n}{d\tau_e^n}\xi_{(B)3}\Bigg|_{(\tau_{e0}+\tau_e)}\right\rangle = \left\langle \frac{d^m}{d\tau_e^m}\xi_{(B)3}\Bigg|_{\tau_{e0}} \frac{d^n}{d\tau_e^n}\xi_{(A)3}\Bigg|_{(\tau_{e0}+\tau_e)}\right\rangle$$
$$= \frac{-\pi}{m}\int_0^\infty d\Omega\,\Omega^{(m+n-1)}h_T^2(\Omega)\Big|_{T=\hbar a/2\pi ck}\left[\pm\frac{\mathrm{Im}(C_3^a-\eta_{33}^a)}{|C_3^a-\eta_{33}^a|^2}+\frac{\mathrm{Im}(C_3^a+\eta_{33}^a)}{|C_3^a+\eta_{33}^a|^2}\right]$$
$$\times\begin{Bmatrix}(-1)^{(n-m)/2}\cos(\Omega\tau_e)\ \ [(m+n)\ \text{even}]\\ (-1)^{(n-m+1)/2}\sin(\Omega\tau_e)\ \ [(m+n)\ \text{odd}]\end{Bmatrix}. \tag{56}$$

A distinct difference exists between Eq. (56) and the analogous unaccelerated-thermal expression. One can show that the latter result is correctly given when $\eta_{33}^a$ and $C_{33}^a$ in Eq. (56) are replaced by their $a\to 0$ limits, and when, of course, the $\xi^\mu$ coordinates are replaced by the coordinates in the rest frame of the oscillator's equilibrium point.

Full agreement between the accelerated-ZP and unaccelerated-thermal systems is again obtained when the resonant approximation is applied to the unretarded van der Waals situation. The width of the resonant peaks in Eq. (56) is approximately given by the left-hand side of the relationship below, while the right-hand side follows from Eqs. (16) and (39):

$$\frac{1}{\omega_3}\,|\,\mathrm{Im}(C_3^a\pm\eta_{33}^a)\,|_{\omega_3}\lesssim 2\Gamma(\omega_3)^2\left[1+\left[\frac{a/c}{\omega_3}\right]^2\right]. \tag{57}$$

Hence, when the condition of

$$2\Gamma(\omega_i)^2\left[1+\left[\frac{a/c}{\omega_i}\right]^2\right]|\tau_e|\ll 1 \tag{58}$$

applies, then the factors of $\cos(\Omega\tau_e)$ and $\sin(\Omega\tau_e)$ in Eq. (56) may be treated as being approximately constant over the width of the resonant peak. The assumption will be made here that a physical mechanism exists that allows the neglect of high-frequency contributions to the integral in Eq. (56), when the oscillator's behavior is fully analyzed.[17] Hence, applying the previously described resonant approximation yields

$$\left\langle \frac{d^m\xi_{A3}}{d\tau_e^m}\Bigg|_{\tau_{e0}} \frac{d^n\xi_{A3}}{d\tau_e^n}\Bigg|_{\tau_{e0}+\tau_e}\right\rangle = \left\langle \frac{d^m\xi_{B3}}{d\tau_e^m}\Bigg|_{\tau_{e0}} \frac{d^n\xi_{B3}}{d\tau_e^n}\Bigg|_{\tau_{e0}+\tau_e}\right\rangle$$
$$\approx\frac{\hbar}{2m}\left\{\omega_3^{(m+n-1)}\coth\left[\frac{\hbar\omega_3}{2kT}\right]\Bigg|_{T=\hbar a/2\pi ck}\begin{Bmatrix}\cos(\omega_3\tau_e)\\ \sin(\omega_3\tau_e)\end{Bmatrix}\right.$$
$$\left.+\frac{1}{8}\left[\frac{\chi_{33}}{\omega_3}\right]^2\left\{\left[-\frac{1}{\omega_3}\frac{\partial}{\partial\Omega}+\frac{\partial^2}{\partial\Omega^2}\right]\left[\Omega^{(m+n-1)}\coth\left[\frac{\hbar\Omega}{2kT}\right]\Bigg|_{T=\hbar a/2\pi ck}\begin{Bmatrix}\cos(\Omega\tau_e)\\ \sin(\Omega_3\tau_e)\end{Bmatrix}\right]\right\}\Bigg|_{\Omega=\omega_3}\right]$$
$$\times\begin{Bmatrix}(-1)^{(n-m)/2}\ \ [(m+n)\ \text{even}]\\ (-1)^{(n-m+1)/2}\ \ [(m+n)\ \text{odd}]\end{Bmatrix}, \tag{59}$$

$$\left\langle \frac{d^m\xi_{A3}}{d\tau_e^m}\bigg|_{\tau_{e0}} \frac{d^n\xi_{B3}}{d\tau_e^n}\bigg|_{\tau_{e0}+\tau_e}\right\rangle \approx \frac{\hbar}{4m}\left[\frac{\chi_{33}}{\omega_3}\right]\frac{\partial}{\partial\Omega}\left[\Omega^{(m+n-1)}\coth\left[\frac{\hbar\Omega}{2kT}\right]\bigg|_{T=\hbar a/2\pi ck}\begin{Bmatrix}\cos(\Omega\tau_e)\\ \sin(\Omega\tau_e)\end{Bmatrix}\right]\bigg|_{\Omega=\omega_3} \times \begin{Bmatrix}(-1)^{(n-m)/2}\ [(m+n)\text{ even}]\\ (-1)^{(n-m+1)/2}\ [(m+n)\text{ odd}]\end{Bmatrix}. \tag{60}$$

[In order to obtain the second term in Eq. (59), the expansion in Eq. (51) must be carried one term further.] The above results are given to lowest order in $\Gamma$.

Equations (59) and (60) hold for both the accelerated-ZP and unaccelerated-thermal systems. Again, the distinguishing factors of $\eta_{33}^a$ and $C_3^a$ in Eq. (56) are removed by the resonant approximation. Moreover, since higher-order moments of the time derivatives of $\xi_{A3}$ and $\xi_{B3}$ can be expressed in terms of the two-point correlation functions of Eq. (56), then a similar analysis also holds for these quantities.[18]

When the limit of $\Gamma\to 0$ is taken, only the first term in Eq. (59) remains, which equals the result one would obtain for a single oscillator. This correlation function for a single oscillator generalizes the work of Ref. 1. Also, when $\Gamma\to 0$, a value of zero is obtained for Eq. (60). This result corresponds to the quantum-mechanical case of two uncharged oscillators separated by a distance $R$ that is large compared to the approximate size of each oscillator, so that their wave functions do not overlap.

## IV. GENERAL CASE OF TWO ACCELERATING DIPOLE OSCILLATORS

### A. Expectation value of force

The situation will now be investigated where all restrictions are removed as to the direction in which oscillations are allowed to occur. The solution to the $i=3$ equation of motion is given by Eqs. (18a) and (18b). Turning to the $i=1,2$ equations of motion, use of the symmetries mentioned at the end of Sec. II results in the following matrix equation:

$$\begin{bmatrix} C_1^a & 0 & \eta_{11}^a & \eta_{12}^a \\ 0 & C_2^a & -\eta_{12}^a & \eta_{22}^a \\ \eta_{11}^a & -\eta_{12}^a & C_1^a & 0 \\ \eta_{12}^a & \eta_{22}^a & 0 & C_2^a \end{bmatrix}\begin{bmatrix}\tilde\xi_{A1}(\Omega)\\ \tilde\xi_{A2}(\Omega)\\ \tilde\xi_{B1}(\Omega)\\ \tilde\xi_{B2}(\Omega)\end{bmatrix} = \frac{e}{m}\begin{bmatrix}\tilde E_1^{ZP}\left[+\hat{y}\frac{R}{2},\Omega\right]\\ \tilde E_2^{ZP}\left[+\hat{y}\frac{R}{2},\Omega\right]\\ \tilde E_1^{ZP}\left[-\hat{y}\frac{R}{2},\Omega\right]\\ \tilde E_2^{ZP}\left[-\hat{y}\frac{R}{2},\Omega\right]\end{bmatrix}, \tag{61}$$

where $\eta_{ij}(\hat{x}a,+\hat{y}R,\Omega)$ has been abbreviated by $\eta_{ij}^a$.

Equation (61) can be immediately solved simply by obtaining the inverse of the matrix on the left-hand side. The Lorentz force acting between the two oscillators can then be calculated in the same manner as in Sec. III. Similar results to those of Sec. III are found, except that additional terms of order $aR/c^2$ now exist. The source of these additional terms arise from the components of the matrix in Eq. (61) given by $\pm\eta_{12}^a$, which couple the $i=1$ and $i=2$ set of equations. As can be seen from Eqs. (A37)—(A40), $\eta_{12}^a$ is of order $aR/c^2$ times the magnitude of $\eta_{11}^a$, $\eta_{22}^a$, and $\eta_{33}^a$.

Assuming the condition of Eq. (1) is satisfied, then the $\eta_{12}^a$ terms in the matrix of Eq. (61) may be ignored. This effectively decouples the $i=1,2$ set of equations and allows the form of the solution for $i=3$ to be applied here also. Hence

$$\tilde\xi_{Ai}(\Omega)=\frac{e}{2m}\left[1+(1-\delta_{i3})O\left[\frac{aR}{c^2}\right]\right] \times\left[\tilde E_i^{ZP}\left[\hat{y}\frac{R}{2},\Omega\right]\left[\frac{1}{C_i^a-\eta_{ii}^a}+\frac{1}{C_i^a+\eta_{ii}^a}\right]+\tilde E_i^{ZP}\left[-\hat{y}\frac{R}{2},\Omega\right]\left[\frac{-1}{C_i^a-\eta_{ii}^a}+\frac{1}{C_i^a+\eta_{ii}^a}\right]\right], \tag{62a}$$

$$\tilde\xi_{Bi}(\Omega)=\frac{e}{2m}\left[1+(1-\delta_{i3})O\left[\frac{aR}{c^2}\right]\right] \times\left[\tilde E_i^{ZP}\left[\hat{y}\frac{R}{2},\Omega\right]\left[\frac{-1}{C_i^a-\eta_{ii}^a}+\frac{1}{C_i^a+\eta_{ii}^a}\right]+\tilde E_i^{ZP}\left[-\hat{y}\frac{R}{2},\Omega\right]\left[\frac{1}{C_i^a-\eta_{ii}^a}+\frac{1}{C_i^a+\eta_{ii}^a}\right]\right]. \tag{62b}$$

Physically, this decoupling of the $i=1,2$ set of equations occurs when the distance $\alpha R_- = (aR/2c^2)R_-$, which an oscillator accelerates in time $(R_-/c)$ (see Appendix A), is small compared to $R_-$; this condition is equivalent to Eq. (1). Under this condition, the angle is small that a light ray would make to the $y$ axis when propagating from one oscillator to the other. For very small angles, the following quantity becomes negligible: namely, the electromagnetic force acting on one of the oscillating particles in the $\hat{\mathbf{x}}$ ($\hat{\mathbf{y}}$) direction due to the oscillations in the $\hat{\mathbf{y}}$ ($\hat{\mathbf{x}}$) direction of the other oscillator. These two sets of oscillations then become independent.

Using Eqs. (19) and (62), one obtains

$$\langle F_{A\tau_e 2}(\tau_e)\rangle = e\sum_{j=1}^{3}\left\langle \xi_{Aj}(\tau_e)\frac{\partial}{\partial \xi_2}E^{ZP}_{\tau_e j}(\xi,\tau_e)\bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle + \sum_{j=1}^{3}\left\langle \xi_{Aj}(\tau_e)\frac{\partial}{\partial \xi_2}E^{DB}_{\tau_e j}(\xi,\tau_e)\bigg|_{\xi=\hat{\mathbf{y}}(R/2)}\right\rangle + \langle A_2\rangle \ . \tag{63}$$

Here, $A_2$ represents the $i=2$ component of the second term of Eq. (19).

Ignoring the $O(aR/c^2)$ terms in Eqs. (62a) and (62b) results in the first term of Eq. (63) being of the same form as Eq. (32), but with a subscript $j$ summed from $j=1$ to $j=3$. In order to obtain the second term of Eq. (63), Eqs. (9) and (11) must be used. The $i\neq j$ terms in Eq. (11) will be ignored here also, as they are only $(aR/c^2)$ times the size of the $i=j$ terms. The second term in Eq. (63) then becomes of the same form as Eq. (35), summed over subscript $j$. The third term in Eq. (63) is evaluated in Appendix D; when the resonant condition is applied, this term becomes negligible.

Following the same steps as led from Eq. (45) to Eqs. (47) and (48) and using Eq. (D11), then yields

$$\langle F_{A\tau_e 2}(\tau_e)\rangle \approx -\frac{\partial}{\partial R}U(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R) + \frac{ea}{c^2}\langle \xi_{A1}(\tau_e)(E^{ZP}_{\tau_e 2}+E^{DB}_{\tau_e 2})\,|_{\hat{\mathbf{y}}R/2,\tau_e}\rangle \ , \tag{64}$$

$$U(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R) = \frac{\hbar}{2\pi}\int_0^\infty d\Omega \coth\left[\frac{\hbar\Omega}{2kT}\right]\bigg|_{T=\hbar a/2\pi ck} \operatorname{Im}\ln\left\{\left[1-\left[\frac{\eta^a_{11}}{C^a_1}\right]^2\right]\left[1-\left[\frac{\eta^a_{22}}{C^a_2}\right]^2\right]\left[1-\left[\frac{\eta^a_{33}}{C^a_3}\right]^2\right]\right\} . \tag{65}$$

Besides the additional terms of order $O(aR/c^2)$ that have been dropped, Eqs. (64) and (65) do not agree with their unaccelerated-thermal counterparts due to the second term in Eq. (64) and the $\eta^a_{ii}$ and $C^a_i$ terms in Eq. (65) [see Eqs. (8) and (9) of Ref. 14].

Agreement is again obtained when the resonant approximation is made in the case of the unretarded van der Waals situation, where

$$\frac{\omega_i R}{c} \ll 1 \quad \text{for } i=1,2,3 \ . \tag{66}$$

From Appendix D, the second term in Eq. (64) is at most of order $(aR/c^2)^2$ times the first term. Writing the first term as a sum of three terms of the form as Eq. (45), and following the steps leading up to Eq. (55), yields

$$\langle F_{A\tau_e 2}(\tau_e)\rangle \approx -\frac{\partial}{\partial R}\sum_{j=1}^{3}\left\{\frac{\hbar}{8\omega_j}(\chi_{jj})^2\frac{\partial}{\partial\Omega}\left[\frac{1}{\Omega}\coth\left[\frac{\hbar\Omega}{2kT}\right]\bigg|_{T=\hbar a/2\pi ck}\right]\bigg|_{\Omega=\omega_j}\right\} . \tag{67}$$

This expression agrees precisely with the corresponding unaccelerated-thermal force expression [see Eq. (39) of Ref. 14].

### B. Other statistical properties of general system

Using Eqs. (62), (C1), and (C3), and recognizing that $\eta^a_{ij} \sim (aR/c^2)\eta^a_{ii}$ for $i\neq j$, enables Eq. (34) and, consequently, Eq. (56) to be readily generalized. The latter result becomes

$$\begin{aligned}\left\langle \frac{d^m}{d\tau_e^m}\xi_{\binom{A}{A}i}\bigg|_{\tau_{e0}} \frac{d^n}{d\tau_e^n}\xi_{\binom{A}{B}j}\bigg|_{\tau_{e0}+\tau_e}\right\rangle &= \left\langle \frac{d^m}{d\tau_e^m}\xi_{\binom{B}{A}i}\bigg|_{\tau_{e0}} \frac{d^n}{d\tau_e^n}\xi_{\binom{B}{B}j}\bigg|_{\tau_{e0}+\tau_e}\right\rangle \\ &= -\left[\delta_{ij}+O\left[\frac{aR}{c^2}\right](1-\delta_{j3})\right]\frac{\pi}{m} \\ &\quad\times \int_0^\infty d\Omega\,\Omega^{(m+n-1)}h_T^2(\Omega)\bigg|_{T=\hbar a/2\pi ck}\left[\pm\frac{\operatorname{Im}(C^a_i-\eta^a_{ii})}{|C^a_i-\eta^a_{ii}|^2}+\frac{\operatorname{Im}(C^a_i+\eta^a_{ii})}{|C^a_i+\eta^a_{ii}|^2}\right] \\ &\quad\times\begin{Bmatrix}(-1)^{(n-m)/2}\cos(\Omega\tau_e) & [(m+n)\text{ even}] \\ (-1)^{(n-m+1)/2}\sin(\Omega\tau_e) & [(m+n)\text{ odd}]\end{Bmatrix} .\end{aligned} \tag{68}$$

The exact expression for the unaccelerated-thermal situation is given by dropping the $O(aR/c^2)$ terms in Eq. (68) and changing the $\eta_{ii}^a$ and $C_i^a$ terms to their $a \to 0$ values.

Agreement is again obtained between the accelerated-ZP and unaccelerated-thermal expressions when the resonant approximation is applied in the case of the unretarded van der Waals situation. Equation (68) then reduces to the same form as that of Eqs. (59) and (60), except that all $i=3$ subscripts on the right-hand side of the latter two equations should be replaced by a general subscript $i$, and a factor of

$$\left[\delta_{ij}+O\left[\frac{aR}{c^2}\right](1-\delta_{i3})(1-\delta_{j3})\right]$$

should be included.

## V. CLOSING REMARKS

During recent years, a close connection in physical behavior has been established between (1) pointlike electromagnetic systems undergoing relativistic hyperbolic motion through electromagnetic zero-point radiation and (2) similar electromagnetic systems, held fixed in an inertial frame, but bathed in zero-point plus thermal electromagnetic radiation, characterized by the temperature $T=\hbar a/2\pi ck$. This connection between these accelerated-zero-point and unaccelerated-thermal electromagnetic systems consist of the agreement in their stochastic properties, when the former accelerated systems are described in a Fermi-Walker transported coordinate reference frame and the latter unaccelerated-thermal systems are described in their inertial rest frame.

The calculations of this article demonstrate that the connection just mentioned for pointlike electromagnetic systems also applies to the spatially extended electromagnetic system considered here: namely, two spatially separated charged simple harmonic oscillators, each taken in the electric dipole limit. Under the four conditions described in Sec. I, a number of stochastic properties were compared between the accelerated-ZP and unaccelerated-thermal situations for such a pair of oscillators. The expectation value of the component of the Lorentz force along the axis separating the two accelerating oscillators was calculated and found to agree with the van der Waals force of a similar unaccelerated, but thermally situated pair of oscillators. Also, agreement was shown to occur for all combinations of the $N$-point correlation functions of the time derivatives of each oscillator's position.

A set of exact relationships, namely, Eqs. (C1)–(C4), were obtained in Appendixes A and C; these identities relate the fields of an accelerating electric dipole to the correlation functions of classical electromagnetic zero-point fields, evaluated along trajectories described by uniform acceleration. These relationships are what enabled the connections to be made between the accelerated-ZP and unaccelerated-thermal systems studied in this article. They should prove to be of aid to researchers in quantum electrodynamics attempting similar work. Appropriate generalizations of these relationships should be helpful to researchers in possible future work involving the thermodynamics of electromagnetic systems suspended in gravitational fields.

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## APPENDIX A: FIELDS OF A UNIFORMLY ACCELERATING FLUCTUATING ELECTRIC DIPOLE

This appendix contains a calculation of the electric and magnetic fields due to a uniformly accelerating electric dipole, which is assumed to be time varying. These fields will be evaluated at the position of the second accelerating oscillator being considered in this article. The electric dipole limit of the oscillator model described in Sec. II will be assumed.

The electric and magnetic fields will first be found in an inertial frame $I_{(\tau_e-\Delta\tau_-)}$, where $\Delta\tau_-$ is defined to be the difference in proper time of the oscillator's equilibrium point for a light signal to travel from the equilibrium point to the space-time point $(c\tau_e;\xi)$. A simple Lorentz transformation of the fields to the rest frame $I_{\tau_e}$ of the other oscillator will then yield the fields that will be used in Eqs. (6a) and (6b) of this article. This indirect method of calculation significantly reduces the algebraic complications that a direct calculation would involve.

According to the above construction, $\tau_e-\Delta\tau_-$ equals the proper time for the start of the light signal just mentioned. Consequently, the inertial coordinate time for the same event is given by $t_{(\tau_e-\Delta\tau_e)}=0$ [see Eq. (5a), with $\xi_1=0$].

For notational purposes, let all quantities in the $I_{(\tau_e-\Delta\tau_-)}$ frame be indicated by a prime. Let the space-time point $x^\mu_{(\tau_e-\Delta\tau_-)}=(ct';\mathbf{x}')$ be given in the Fermi-Walker transported coordinate system by $\xi^\mu=(c\tau_e;\xi)$. At large distances compared to the size of the oscillator, treatment of the oscillator's electric field as being due to a positive and negative charge results in the following expression for the electric field of an accelerating dipole oscillator in the $I_{(\tau_e-\Delta\tau_-)}$ frame, evaluated at point $(c\tau_e;\xi)$:

$$\mathbf{E}_{(\tau_e-\Delta\tau_-)}(\xi,\tau_e)=e\left[\frac{(\mathbf{R}'_+-R'_+\boldsymbol{\beta}'_+)(1-\beta'^2_+)}{(R'_+-\boldsymbol{\beta}'_+\cdot\mathbf{R}'_+)^3}\right]_{t'_{r_+}}+\frac{e}{c}\left[\frac{\mathbf{R}'_+\otimes[(\mathbf{R}'_+-R'_+\boldsymbol{\beta}'_+)\otimes\dot{\boldsymbol{\beta}}'_+]}{(R'_+-\boldsymbol{\beta}'_+\cdot\mathbf{R}'_+)^3}\right]_{t'_{r_+}}$$
$$-e\left[\frac{(\mathbf{R}'_--R'_-\boldsymbol{\beta}'_-)(1-\beta'^2_-)}{(R'_--\boldsymbol{\beta}'_-\cdot\mathbf{R}'_-)^3}\right]_{t'_{r_-}}-\frac{e}{c}\left[\frac{\mathbf{R}'_-\otimes\{(\mathbf{R}'_--R'_-\boldsymbol{\beta}'_-)\otimes\dot{\boldsymbol{\beta}}'_-\}}{(R'_--\boldsymbol{\beta}'_-\cdot\mathbf{R}'_-)^3}\right]_{t'_{r_-}}. \tag{A1}$$

Here, $t'_{r_-}$ and $t'_{r_+}$ are the retarded times associated with the negative charge and the oscillating positive charge, respectively. The other quantities in Eq. (A1) follow the form of conventional usage;[19] their exact functional forms will be specified shortly.

Let $\Delta\mathbf{E}'$ be equal to the top two terms of Eq. (A1) minus the same two terms evaluated at the retarded time $t'_{r_-}$ rather than $t'_{r_+}$. Then, Eq. (A1) can be expressed as the four terms of Eq. (A1), all evaluated at $t'_{r_-}$, plus $\Delta\mathbf{E}'$. The latter term will be evaluated later by using a Taylor-series expansion in $(t'_{r_+}-t'_{r_-})$.

Let $\mathbf{X}'$, $\mathbf{X}'+\Delta\mathbf{x}'$, and $\mathbf{x}'$ be the vector positions of the negative charge, the oscillating positive charge, and the point at which the fields are to be evaluated, respectively, as expressed in the $I_{(\tau_e-\Delta\tau_-)}$ frame. Let the symbol $|_{r_-}$ be a shortened notation for evaluating all quantities at $t'_{r_-}$. The following notation should then be fairly obvious:

$$(\mathbf{R}'_-)_{r_-}=(\mathbf{x}'-\mathbf{X}')_{r_-}\ , \tag{A2}$$

$$(\mathbf{R}'_+)_{r_-}=(\mathbf{x}'-\mathbf{X}'-\Delta\mathbf{x}')_{r_-}=(\mathbf{R}'_--\Delta\mathbf{x}')_{r_-}\ , \tag{A3}$$

$$(\boldsymbol{\beta}'_-)_{r_-}=\left[\frac{1}{c}\frac{d\mathbf{X}'}{dt'}\right]_{r_-}=\left[\frac{(at'/c)\hat{\mathbf{x}}}{\left[1+\left[\frac{at'}{c}\right]^2\right]^{1/2}}\right]_{t'=0}=0\ , \tag{A4}$$

$$(\boldsymbol{\beta}'_+)_{r_-}=\frac{1}{c}\left[\frac{d}{dt'}(\mathbf{X}'+\Delta\mathbf{x}')\right]_{r_-}=\left[\frac{1}{c}\Delta\dot{\mathbf{x}}'\right]_{r_-}\ , \tag{A5}$$

$$(\dot{\boldsymbol{\beta}}'_-)_{r_-}=\left[\frac{1}{c}\frac{d^2\mathbf{X}'}{dt'^2}\right]_{r_-}=\left.\frac{(a/c)\hat{\mathbf{x}}}{\left[1+\left[\frac{at'}{c}\right]^2\right]^{3/2}}\right|_{t'=0}=\frac{a}{c}\hat{\mathbf{x}}\ , \tag{A6}$$

$$(\dot{\boldsymbol{\beta}}'_+)_{r_-}=\frac{1}{c}\left[\frac{d^2}{dt'^2}(\mathbf{X}'+\Delta\mathbf{x}')\right]_{r_-}=\frac{a}{c}\hat{\mathbf{x}}+\left[\frac{1}{c}\Delta\ddot{\mathbf{x}}'\right]_{r_-}\ . \tag{A7}$$

Equations (A4) and (A6) follow from Eq. (2) of Ref. 3. As explained earlier, $t'_{r_-}=0$; hence, $(\boldsymbol{\beta}'_-)_{r_-}=0$, which was the main reason for choosing $I_{(\tau_e-\Delta\tau_-)}$ in which to first evaluate the dipole fields.

All of the terms in Eq. (A1) that are evaluated at $\tau_e-\Delta\tau_-$ may be expressed in terms of the quantities in Eqs. (A2)–(A7). In keeping with the small-oscillator assumption, all quantities in Eq. (A1) will be evaluated only to first order in $\Delta\mathbf{x}'$, $\Delta\dot{\mathbf{x}}'$, and $\Delta\ddot{\mathbf{x}}'$. The following expressions are then obtained:

$$(R'_+)_{r_-}=|\mathbf{x}'-\mathbf{X}'-\Delta\mathbf{x}'|_{r_-}\approx\left[R'_--\frac{\Delta\mathbf{x}'\cdot\mathbf{R}'_-}{R'_-}\right]_{r_-}\ , \tag{A8}$$

$$(R'_+-\boldsymbol{\beta}'_+\cdot\mathbf{R}'_+)_{r_-}\approx\left[R'_--\frac{\Delta\mathbf{x}'\cdot\mathbf{R}'_-}{R'_-}-\frac{\Delta\dot{\mathbf{x}}'}{c}\cdot\mathbf{R}'_-\right]_{r_-}\ , \tag{A9}$$

$$[1-(\boldsymbol{\beta}'_+)^2]_{r_-}=1-\left[\frac{\Delta\dot{\mathbf{x}}'}{c}\right]^2_{r_-}\approx 1\ , \tag{A10}$$

$$(R'_+-\boldsymbol{\beta}'_+\cdot\mathbf{R}'_+)_{r_-}{}^{-3}\approx\left[\frac{1}{R'^{\,3}_-}\left[1+\frac{3\Delta\mathbf{x}'\cdot\mathbf{R}'_-}{R'^{\,2}_-}+\frac{3\Delta\dot{\mathbf{x}}'\cdot\mathbf{R}'_-}{cR'_-}\right]\right]_{r_-}\ , \tag{A11}$$

$$[(\mathbf{R}'_+-R'_+\boldsymbol{\beta}'_+)\otimes\dot{\boldsymbol{\beta}}'_+]_{r_-}\approx\left[\left[\mathbf{R}'_--\Delta\mathbf{x}'-R'_-\frac{\Delta\dot{\mathbf{x}}'}{c}\right]\otimes\dot{\boldsymbol{\beta}}'_-\right]_{r_-}+\left[\mathbf{R}'_-\otimes\frac{\Delta\ddot{\mathbf{x}}'}{c}\right]_{r_-}\ . \tag{A12}$$

Other quantities may be linearized in the same way. Hence, Eq. (A1) becomes

$$\begin{aligned}\mathbf{E}_{(\tau_e-\Delta\tau_-)}(\xi,\tau_e)\approx e&\left[\frac{3(\Delta\mathbf{x}'\cdot\mathbf{R}'_-)\mathbf{R}'_-}{R'^{\,5}_-}+\frac{3(\Delta\dot{\mathbf{x}}'\cdot\mathbf{R}'_-)}{cR'^{\,4}_-}\mathbf{R}'_--\frac{\Delta\mathbf{x}'}{R'^{\,3}_-}-\frac{\Delta\dot{\mathbf{x}}'}{c}\frac{1}{R'^{\,2}_-}\right]_{r_-}\\&+\frac{e}{c}\left\{\left[\mathbf{R}'_-\otimes\left[\mathbf{R}'_-\otimes\frac{\Delta\ddot{\mathbf{x}}'}{c}\right]-\Delta\mathbf{x}'\otimes(\mathbf{R}'_-\otimes\dot{\boldsymbol{\beta}}'_-)-\mathbf{R}'_-\otimes(\Delta\mathbf{x}'\otimes\dot{\boldsymbol{\beta}}'_-)-\mathbf{R}'_-\otimes\left[R'_-\frac{\Delta\dot{\mathbf{x}}'}{c}\otimes\dot{\boldsymbol{\beta}}'_-\right]\right.\right.\\&\left.\left.+[\mathbf{R}'_-\otimes(\mathbf{R}'_-\otimes\dot{\boldsymbol{\beta}}'_-)]\frac{3\Delta\mathbf{x}'\cdot\mathbf{R}'_-}{R'^{\,2}_-}+[\mathbf{R}'_-\otimes(\mathbf{R}'_-\otimes\dot{\boldsymbol{\beta}}'_-)]\frac{3\Delta\dot{\mathbf{x}}'\cdot\mathbf{R}'_-}{R'_-c}\right]\left[\frac{1}{R'^{\,3}_-}\right]\right\}_{r_-}+\Delta\mathbf{E}'\ .\end{aligned} \tag{A13}$$

In order to evaluate $\Delta\mathbf{E}'$, the difference in the retarded times must be found to lowest order in $\Delta x'_i$. Let $\mathbf{X}'=(X',Y',Z')$, $\Delta\mathbf{x}'=(\Delta x',\Delta y',\Delta z')$, and $\mathbf{x}'=(x',y',z')$. Corresponding to the geometrical configuration shown in Fig. 1, let $z'=(Z')_{r_-}$. One can then show from a simple geometrical picture that

$$c(t'_{r_+}-t'_{r_-})\approx+\left[\Delta y'\Big|_{r_-}\left[\frac{y'-Y'}{R'_-}\right]_{r_-}+\Delta x'\Big|_{r_-}\left[\frac{x'-X'}{R'_-}\right]_{r_-}\right]\ . \tag{A14}$$

Using a Taylor-series expansion of the top two terms of Eq. (A1) in terms of the difference in the retarded times of Eq. (A14), yields

$$\Delta\mathbf{E}' \approx \left[ e\left[ \frac{-\dot{\boldsymbol{\beta}}'_-}{R'^2_-} + \frac{3\mathbf{R}'_-(\dot{\boldsymbol{\beta}}'_-\cdot\mathbf{R}'_-)}{R'^4_-} \right]_{r_-} + \frac{e}{c}\left[ \frac{\mathbf{R}'_-\otimes(\mathbf{R}'_-\otimes\ddot{\boldsymbol{\beta}}'_-)}{R'^3_-} \frac{3\dot{\boldsymbol{\beta}}'_-\cdot\mathbf{R}'_-}{R'_-} \right]_{r_-} \right](t'_{r_+} - t'_{r_-}) , \tag{A15}$$

where $\Delta\mathbf{E}'$ is given to first order in the $\Delta x'_i$ coordinates. Here, the coefficient of $(t'_{r_+} - t'_{r_-})$ was obtained by differentiating the top two terms in Eq. (A1) by $t'_r$ and evaluating them at $t'_{r_-}$. The dependence of this coefficient upon the $\Delta x'_i$ coordinates was able to be ignored, because, from Eq. (A14), $(t'_{r_+} - t'_{r_-})$ is first order in $\Delta x'_i$. This fact allowed the Taylor-series expansion to be carried out to only first order in $(t'_{r_+} - t'_{r_-})$. Finally, the following expressions were used in obtaining Eq. (A15):

$$\left[ \frac{d\mathbf{R}'_-}{dt'_r} \right]_{r_-} = -(\boldsymbol{\beta}'_-)_{r_-} = 0 , \tag{A16}$$

$$(\ddot{\boldsymbol{\beta}}'_-)_{r_-} = \left. \frac{-3\hat{\mathbf{x}}\left[\frac{a}{c}\right]^3 t'}{\left[1+\left[\frac{at'}{c}\right]^2\right]^{5/2}} \right|_{t'=0} = 0 . \tag{A17}$$

Equations (A13)–(A15) will now be used to obtain the electric field at the space-time point $\xi^\mu = [c\tau_e; \hat{\mathbf{y}}(\mathscr{R}_L \pm R)]$ due to an accelerated dipole oscillator at $\xi = \hat{\mathbf{y}}\mathscr{R}_L$. Let $(R'_-)_{r_-}$ and $(\mathbf{R}'_-)_{r_-}$ be abbreviated by $R_-$ and $\mathbf{R}_-$, respectively. The following simple relationship then holds:

$$(R_-)^2 = (t' - t'_{r_-})^2 c^2$$
$$= R^2 + \left\{ \frac{c^2}{a}\left[1+\left[\frac{at'}{c}\right]^2\right]^{1/2} - \frac{c^2}{a}\left[1+\left[\frac{at'_{r_-}}{c}\right]^2\right]^{1/2} \right\}^2 . \tag{A18}$$

Substituting in $t'_{r_-} = 0$ and solving for $t'$ results in

$$R_- = ct' = R(1+\alpha^2)^{1/2} , \tag{A19}$$

$$\alpha = \frac{aR}{2c^2} , \tag{A20}$$

$$\mathbf{R}_- = \hat{\mathbf{x}}\alpha R \pm \hat{\mathbf{y}}R , \tag{A21}$$

$$\Delta\tau_- = \frac{c}{a}\,\mathrm{arcsinh}\left[\frac{aR_-}{c^2}\right] . \tag{A22}$$

Combining Eqs. (A13)–(A15) and (A19)–(A21) then yields the following, where the superscript $DL$ indicates dipole fields due to oscillator $L$:

$$\begin{aligned}
\mathbf{E}^{DL}_{(\tau_e-\Delta\tau_-)}(\hat{\mathbf{y}}(\mathscr{R}_L \pm R),\tau_e) = {} & \hat{\mathbf{x}}e\Bigg[ -\Delta x'_L\left[\frac{R^2}{R'^5_-}\right](1+6\alpha^2+8\alpha^4) - \frac{\Delta\dot{x}'_L}{c}\left[\frac{R^2}{R'^4_-}\right](1+4\alpha^2) - \frac{\Delta\ddot{x}'_L}{c^2}\left[\frac{R^2}{R'^3_-}\right] \\
& \mp\Delta y'_L\left[\frac{R^2}{R'^5_-}\right]\alpha(1+4\alpha^2) \mp \frac{\Delta\dot{y}'_L}{c}\left[\frac{R^2}{R'^4_-}\right]\alpha(1-2\alpha^2) \pm \frac{\Delta\ddot{y}'_L}{c^2}\left[\frac{R^2}{R'^3_-}\right]\alpha \Bigg]_{r_-} \\
& + \hat{\mathbf{y}}e\Bigg[ \pm\Delta x'_L\left[\frac{R^2}{R'^5_-}\right]\alpha(1+10\alpha^2+12\alpha^4) \pm \frac{\Delta\dot{x}'_L}{c}\left[\frac{R^2}{R'^4_-}\right]\alpha 3(1+2\alpha^2) \pm \frac{\Delta\ddot{x}'_L}{c^2}\left[\frac{R^2}{R'^3_-}\right]\alpha \\
& + \Delta y'_L\left[\frac{R^2}{R'^5_-}\right](2+9\alpha^2+10\alpha^4) + \frac{\Delta\dot{y}'_L}{c}\left[\frac{R^2}{R'^4_-}\right](2+3\alpha^2-2\alpha^4) - \frac{\Delta\ddot{y}'_L}{c^2}\left[\frac{R^2}{R'^3_-}\right]\alpha^2 \Bigg]_{r_-} \\
& + \hat{\mathbf{z}}e\left[ -\Delta z'_L\frac{1}{(R'_-)^3}(1+2\alpha^2) - \frac{\Delta\dot{z}'_L}{c}\frac{1}{(R'_-)^2}(1+2\alpha^2) - \frac{\Delta\ddot{z}'_L}{c^2}\frac{1}{R'_-} \right]_{r_-} .
\end{aligned} \tag{A23}$$

The magnetic field due to the accelerating charged oscillator can be obtained by using

$$\mathbf{B}^{DL}_{(\tau_e-\Delta\tau_-)}(\hat{\mathbf{y}}(\mathscr{R}_L \pm R),\tau_e) = (\hat{\mathbf{n}}'_+\otimes\mathbf{E}'_+)_{r_+} + (\hat{\mathbf{n}}'_-\otimes\mathbf{E}'_-)_{r_-} , \tag{A24}$$

where $(\mathbf{E}'_+)_{r_+}$ consists of the first two terms of Eq. (A1), $(\mathbf{E}'_-)_{r_-}$ consists of the second two terms, and

$$(\hat{\mathbf{n}}'_-)_{r_-} = (\mathbf{R}_-/R_-) , \tag{A25}$$

$$(\hat{\mathbf{n}}'_+)_{r_+} = \left[ \frac{\mathbf{R}'_- - \Delta\mathbf{x}'_L}{|\mathbf{R}'_- - \Delta\mathbf{x}'_L|} \right]_{r_+} \approx \left[ \frac{\mathbf{R}'_- - \Delta\mathbf{x}'_L}{|\mathbf{R}'_- - \Delta\mathbf{x}'_L|} \right]_{r_-} \approx \left[ \frac{\mathbf{R}'_-}{R'_-} - \frac{\Delta\mathbf{x}'_L}{R'_-} + \frac{\mathbf{R}'_-(\Delta\mathbf{x}'_L \cdot \mathbf{R}'_-)}{(R'_-)^3} \right]_{t_{r_-}} . \tag{A26}$$

The first approximation sign in Eq. (A26) follows from making a Taylor-series expansion in $(t'_{r_+} - t'_{r_-})$, using Eq. (A16), and retaining terms only to first order in $\Delta x'_{Li}$. Because of Eq. (A16), the second term in the Taylor-series expansion does not contain terms linear in $\Delta x'_{Li}$; higher-order terms in the Taylor-series expansion are not linear in $\Delta x'_{Li}$ due to Eq. (A14).

Let $\mathbf{E}^{DL'}$ be an abbreviated form for the electric field in Eq. (A23). From Eqs. (A1) and (A23), $(\mathbf{E}'_+)_{r_+} \approx \mathbf{E}^{DL'} - (\mathbf{E}'_-)_{r_-}$, where $\mathbf{E}^{DL'}$ is given by Eq. (A23) to first order in $\Delta x'_{Li}$. Hence, to this same degree of approximation,

$$\mathbf{B}^{DL}_{(\tau_e - \Delta\tau_-)}(\hat{\mathbf{y}}(\mathscr{R}_L \pm R), \tau_e) \approx \left[ \frac{\mathbf{R}'_- \otimes \mathbf{E}^{DL'}}{R'_-} + \frac{\Delta\mathbf{x}'_L \otimes \mathbf{E}'_-}{R'_-} - \frac{(\mathbf{R}'_- \otimes \mathbf{E}'_-)(\Delta\mathbf{x}'_L \cdot \mathbf{R}'_-)}{(R'_-)^3} \right]_{r_-} , \tag{A27}$$

where $(\mathbf{E}'_-)_{r_-}$ is given by

$$(\mathbf{E}'_-)_{r_-} = \hat{\mathbf{x}} \frac{eR\alpha}{(R_-)^3} \mp \hat{\mathbf{y}} \frac{eR}{(R_-)^3}(1+2\alpha^2) . \tag{A28}$$

From Eqs. (A23), (A27), and (A28), the following expression may be obtained:

$$\begin{aligned}
\mathbf{B}^{DL}_{(\tau_e - \Delta\tau_-)}(\hat{\mathbf{y}}(\mathscr{R}_L \pm R), \tau_e) = {} & \hat{\mathbf{x}} e \left[ \mp \frac{\Delta\dot{z}'_L}{c} \left[ \frac{R}{R'^3_-} \right] (1+2\alpha^2) \mp \frac{\Delta\ddot{z}'_L}{c^2} \left[ \frac{R}{R'^2_-} \right] \right]_{r_-} \\
& + \hat{\mathbf{y}} e \left[ + \Delta z'_L \left[ \frac{R}{R'^4_-} \right] 2\alpha(1+\alpha^2) + \frac{\Delta\dot{z}'_L}{c} \left[ \frac{R}{R'^3_-} \right] \alpha(1+2\alpha^2) + \frac{\Delta\ddot{z}'_L}{c^2} \left[ \frac{R}{R'^2_-} \right] \alpha \right]_{r_-} \\
& + \hat{\mathbf{z}} e \left[ \pm \Delta x'_L \left[ \frac{R}{R'^4_-} \right] (6\alpha^2 + 12\alpha^4) \pm \frac{\Delta\dot{x}'_L}{c} \left[ \frac{R}{R'^3_-} \right] (1+6\alpha^2) \pm \frac{\Delta\ddot{x}'_L}{c^2} \left[ \frac{R}{R'^2_-} \right] \right. \\
& \left. + \Delta y'_L \left[ \frac{R}{R'^4_-} \right] (4\alpha + 10\alpha^3) + \frac{\Delta\dot{y}'_L}{c} \left[ \frac{R}{R'^3_-} \right] \alpha(3 - 2\alpha^2) - \frac{\Delta\ddot{y}'_L}{c^2} \left[ \frac{R}{R'^2_-} \right] \alpha \right]_{r_-} .
\end{aligned} \tag{A29}$$

The fields $\mathbf{E}^{DL}_{\tau_e}$ and $\mathbf{B}^{DL}_{\tau_e}$ in the $I_{\tau_e}$ frame may be obtained by substituting $\mathbf{E}^{DL}_{(\tau_e - \Delta\tau_-)}$ and $\mathbf{B}^{DL}_{(\tau_e - \Delta\tau_-)}$ into the right-hand sides of Eqs. (7) and (8). The quantities $\gamma_{\Delta\tau_-}$ and $\beta_{\Delta\tau_-}$ that will appear in these equations are given by

$$\gamma_{\Delta\tau_-} = \cosh \left[ \frac{a\Delta\tau_-}{c} \right] = (1+2\alpha^2) , \tag{A30}$$

$$\beta_{\Delta\tau_-}\gamma_{\Delta\tau_-} = \sinh \left[ \frac{a\Delta\tau_-}{c} \right] = 2\alpha(1+\alpha^2)^{1/2} . \tag{A31}$$

Because of Eqs. (A23) and (A29), the expressions for $\mathbf{E}^{DL}_{\tau_e}$ and $\mathbf{B}^{DL}_{\tau_e}$ will contain the variables $\Delta\mathbf{x}'_L$, $\Delta\dot{\mathbf{x}}'_L$, and $\Delta\ddot{\mathbf{x}}'_L$, all evaluated at $t_{(\tau_e - \Delta\tau_-)} = 0$. From the discussion in Appendix D [see, in particular, Eqs. (D6) and (D8)], one can show that

$$(\Delta\mathbf{x}_{L(\tau_e - \Delta\tau_-)}) |_{t_{(\tau_e - \Delta\tau_-)} = 0} = \xi_L |_{(\tau_e - \Delta\tau_-)} , \tag{A32}$$

$$\left. \frac{d(\Delta\mathbf{x}_{L(\tau_e - \Delta\tau_-)})}{dt_{(\tau_e - \Delta\tau_-)}} \right|_{t_{(\tau_e - \Delta\tau_-)} = 0} = \left. \frac{d\xi_L}{d\tau_e} \right|_{(\tau_e - \Delta\tau_-)} , \tag{A33}$$

$$\frac{d^2(\Delta \mathbf{x}_{L(\tau_e-\Delta\tau_-)})}{dt_{(\tau_e-\Delta\tau_-)}^2}\bigg|_{t_{(\tau_e-\Delta\tau_-)}=0}=\hat{\mathbf{x}}\left[\frac{d^2\xi_{L1}}{d\tau_e^2}-\left[\frac{a}{c}\right]^2\xi_{L1}\right]_{(\tau_e-\Delta\tau_-)}+\hat{\mathbf{y}}\frac{d^2\xi_{L2}}{d\tau_e^2}\bigg|_{(\tau_e-\Delta\tau_-)}+\hat{\mathbf{z}}\frac{d^2\xi_{L3}}{d\tau_e^2}\bigg|_{(\tau_e-\Delta\tau_-)} . \tag{A34}$$

Combining Eqs. (7), (8), (13), (A23), and (A29)—(A34), yields the following results:

$$E^{DL}_{\tau_e i}(\hat{\mathbf{y}}(\mathscr{R}_L\pm R),\tau_e)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\Omega\exp(-i\Omega\tau_e)\left[\sum_{j=1}^{3}\eta^D_{ij}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)[e\xi_{Lj}(\Omega)]\right] , \tag{A35}$$

$$B^{DL}_{\tau_e i}(\hat{\mathbf{y}}(\mathscr{R}_L\pm R),\tau_e)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\Omega\exp(-i\Omega\tau_e)\left[\sum_{j=1}^{3}\rho^D_{ij}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)[e\xi_{Lj}(\Omega)]\right] , \tag{A36}$$

where

$$\eta^D_{11}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=k^3\left[\frac{R^2}{R_-{}^2}\right]\left[\frac{1}{kR_-}+\frac{i(1+4\alpha^2)}{(kR_-)^2}-\frac{(1+2\alpha^2+4\alpha^4)}{(kR_-)^3}\right]\exp(i\Omega\Delta\tau_-) , \tag{A37}$$

$$\eta^D_{22}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=k^3\left[\frac{R^2}{R_-{}^2}\right]\left[-\frac{\alpha^2}{kR_-}-\frac{i(2+\alpha^2+2\alpha^4)}{(kR_-)^2}+\frac{(2+5\alpha^2)}{(kR_-)^3}\right]\exp(i\Omega\Delta\tau_-) , \tag{A38}$$

$$\eta^D_{33}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=k^3\left[\frac{1}{kR_-}+\frac{i(1+2\alpha^2)}{(kR_-)^2}-\frac{1}{(kR_-)^3}\right]\exp(i\Omega\Delta\tau_-) , \tag{A39}$$

$$\eta^D_{12}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=-\eta^D_{21}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=\pm k^3\left[\frac{R^2}{R_-{}^2}\right]\alpha\left[-\frac{1}{kR_-}+\frac{i(1-2\alpha^2)}{(kR_-)^2}-\frac{(1+4\alpha^2)}{(kR_-)^3}\right]\exp(i\Omega\Delta\tau_-) , \tag{A40}$$

$$\eta^D_{i3}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=\eta^D_{3i}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=0 \quad \text{for } i\neq 3 , \tag{A41}$$

and

$$\rho^D_{13}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=-\rho^D_{31}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=\pm k^3\left[\frac{R}{R_-}\right]\left[\frac{1}{kR_-}+\frac{i(1+2\alpha^2)}{(kR_-)^2}\right]\exp(i\Omega\Delta\tau_-) , \tag{A42}$$

$$\rho^D_{23}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=+\rho^D_{32}(\hat{\mathbf{x}}a,\pm\hat{\mathbf{y}}R,\Omega)=k^3\left[\frac{R}{R_-}\right]\alpha\left[\frac{1}{kR_-}+\frac{i(1+2\alpha^2)}{(kR_-)^2}\right]\exp(i\Omega\Delta\tau_-) , \tag{A43}$$

$$\rho^{Da}_{11}=\rho^{Da}_{22}=\rho^{Da}_{33}=\rho^{Da}_{12}=\rho^{Da}_{21}=0 . \tag{A44}$$

In the above expressions, $k=\Omega/c$, while $R_-$, $\alpha$, and $\Delta\tau_-$ are given by Eqs. (A19), (A20), and (A22).

Two final relationships that will be mentioned here are

$$\eta^D_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,-\Omega)=\eta^{D*}_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega) , \tag{A45}$$

$$\rho^D_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,-\Omega)=\rho^{D*}_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega) . \tag{A46}$$

These relationships, which are easily verified from Eqs. (A37)—(A44), are the appropriate ones that must be satisfied upon the demand that $\mathbf{E}^{DL}_{\tau_e}$, $\mathbf{B}^{DL}_{\tau_e}$, and $\xi_L$ be real quantities.

## APPENDIX B: FORCE ON FLUCTUATING ELECTRIC DIPOLE IN ARBITRARY TRAJECTORY

As was done in Appendix A, the model assumed here for an electric dipole oscillator will be a $+e$ positive point charge, with mass $m$, that oscillates inside a small distribution of negative charge, with net charge $-e$. This charge distribution will be assumed to be constructed in such a way that all points of the negative-charge distribution possess the same instantaneous inertial rest frame throughout the full evolution of their trajectories. For the purposes of this section, the electric dipole oscillator will not be restricted to a trajectory of uniform acceleration.

Let $\mathbf{X}(t)$ be the trajectory of the center of the negative-charge distribution, as expressed in some arbitrary inertial frame. Let $\mathbf{X}(t)+\Delta\mathbf{x}(t)$ be the position of the oscillating positive charge. When the volume of negative charge and $|\Delta\mathbf{x}|$ are made infinitesimally small, then the Lorentz force on this system, due to electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, is approximately given by

$$\mathbf{F}=e\left[\mathbf{E}(\mathbf{X}+\Delta\mathbf{x},t)+\frac{1}{c}\left[\frac{d}{dt}(\mathbf{X}+\Delta\mathbf{x})\right]\otimes\mathbf{B}(\mathbf{X}+\Delta\mathbf{x},t)\right]-e\left[\mathbf{E}(\mathbf{X},t)+\frac{1}{c}\left[\frac{d\mathbf{X}}{dt}\right]\otimes\mathbf{B}(\mathbf{X},t)\right]$$

$$\approx e\left[(\Delta\mathbf{x}\cdot\nabla)\mathbf{E}|_{\mathbf{X},t}+\frac{1}{c}\left[\frac{d\Delta\mathbf{x}}{dt}\right]\otimes\mathbf{B}(\mathbf{X},t)+\frac{1}{c}\left[\frac{d\mathbf{X}}{dt}\right]\otimes[(\Delta\mathbf{x}\cdot\nabla)\mathbf{B}|_{\mathbf{X},t}]\right] . \tag{B1}$$

In the instantaneous inertial rest frame of the charge distribution, $d\mathbf{X}/dt=0$, so the third term in Eq. (B1) drops out. For this case, Eq. (B1) becomes of the same form as the familiar expression

$$\mathbf{F}=(\mathbf{p}\cdot\nabla)\mathbf{E}+\frac{1}{c}\dot{\mathbf{p}}\otimes\mathbf{B}\ , \tag{B2}$$

for the Lorentz force on a stationary electric dipole, provided that the following limit applies: namely, that $|\Delta\mathbf{x}|\to 0$, $e\to\infty$, and the volume of the negative-charge distribution goes to zero such that the first moment of the total charge distribution equals a finite value of $e\Delta\mathbf{x}=\mathbf{p}$.

The first two terms of Eq. (B1) can be reexpressed to yield

$$F_i(t)=e\sum_{j=1}^{3}\Delta x_j\frac{\partial}{\partial x_i}E_j\bigg|_{\mathbf{X}(t),t}+\frac{e}{c}\frac{d}{dt}[\Delta\mathbf{x}\otimes\mathbf{B}(\mathbf{X},t)]_i$$
$$-\frac{e}{c}\left\{\Delta\mathbf{x}\otimes\left[\left[\frac{d\mathbf{X}}{dt}\cdot\nabla\right]\mathbf{B}\right]_{\mathbf{X}(t),t}\right\}_i$$
$$+\frac{e}{c}\left[\frac{d\mathbf{X}}{dt}\otimes[(\Delta\mathbf{x}\cdot\nabla)\mathbf{B}\,|_{\mathbf{X}(t),t}]\right]_i\ . \tag{B3}$$

In the instantaneous inertial rest frame of the negative-charge distribution, the last two terms equal zero.

## APPENDIX C: CORRELATION FUNCTIONS OF FIELDS

In this section, the following relationships will be shown to be true:

$$\langle E^{ZP}_{\tau_{e0}i}(\hat{\mathbf{y}}R_0,\tau_{e0})E^{ZP}_{(\tau_{e0}+\tau_e)j}(\hat{\mathbf{y}}(R_0+R),\tau_{e0}+\tau_e)\rangle=\langle B^{ZP}_{\tau_{e0}i}(\hat{\mathbf{y}}R_0,\tau_{e0})B^{ZP}_{(\tau_{e0}+\tau_e)j}(\hat{\mathbf{y}}(R_0+R),\tau_{e0}+\tau_e)\rangle$$
$$=\int_0^\infty f^{ZP}_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)\cos(\Omega\tau_e)d\Omega\ , \tag{C1}$$

$$\langle B^{ZP}_{\tau_{e0}i}(\hat{\mathbf{y}}R_0,\tau_{e0})E^{ZP}_{(\tau_{e0}+\tau_e)j}(\hat{\mathbf{y}}(R_0+R),\tau_{e0}+\tau_e)\rangle=\int_0^\infty g^{ZP}_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)\sin(\Omega\tau_e)d\Omega\ , \tag{C2}$$

where $f^{ZP}_{ij}$ and $g^{ZP}_{ij}$ are related to the functions $\eta^{Da}_{ij}$ and $\rho^{Da}_{ij}$ [see Eqs. (A35)—(A44)] for an accelerated electric dipole oscillator by[6]

$$f^{ZP}_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)=\frac{2\pi h_T{}^2(\Omega)}{\Omega}\bigg|_{T=\hbar a/2\pi ck}\mathrm{Im}[\eta^D_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)]\ , \tag{C3}$$

$$g^{ZP}_{ij}(\hat{\mathbf{x}}a,\hat{\mathbf{y}}R,\Omega)=\frac{2\pi h_T{}^2(\Omega)}{\Omega}\bigg|_{T=\hbar a/2\pi ck}\mathrm{Re}[\rho^D_{ji}(\hat{\mathbf{x}}_a,\hat{\mathbf{y}}R,\Omega)]\ , \tag{C4}$$

$$h_T{}^2(\Omega)=\frac{\hbar\Omega}{2\pi^2}\coth\left[\frac{\hbar\Omega}{2kT}\right]=\frac{1}{\pi^2}\left[\frac{\hbar\Omega}{2}+\frac{\hbar\Omega}{\exp(\hbar\Omega/kT)-1}\right]\ . \tag{C5}$$

It should be noted that the $i,j$ indices are reversed in order on the left- and right-hand sides of Eq. (C4), but they occur in the same order on both sides of Eq. (C3). The definition given for $h_T{}^2(\Omega)$ in Eq. (C5) generalizes the function $h^2(\Omega)$ in Eq. (24) to the case of a thermal plus zero-point spectrum.

In order to aid in establishing the validity of Eqs. (C1) and (C3), a set of identities will first be proven. Let

$$\beta=\frac{a}{2c}\ , \tag{C6a}$$

$$b=\frac{\sinh(\beta\tau_e)}{\alpha}\ , \tag{C6b}$$

where $\alpha$ is given by Eq. (A20). From Eqs. (A19) and (A22),

$$\Delta\tau_-=\frac{1}{2\beta}\,\mathrm{arcsinh}[2\alpha(1+\alpha^2)^{1/2}]\ . \tag{C7}$$

Let $I_c$ be defined as the integral listed below. From this definition, three useful identities can be established:

$$I_c\equiv\int_0^\infty\coth\left[\frac{\pi\Omega}{2\beta}\right]\sin(\Omega\Delta\tau_-)\cos(\Omega\tau_e)d\Omega=\frac{\beta}{\alpha}(1+\alpha^2)^{1/2}\left[\frac{1}{1-b^2}\right]\ , \tag{C8}$$

$$\left[\frac{dI_c}{d\Delta\tau_-}\right]=\int_0^\infty\coth\left[\frac{\pi\Omega}{2\beta}\right]\cos(\Omega\Delta\tau_-)\cos(\Omega\tau_e)\Omega\,d\Omega=-\left[\frac{\beta}{\alpha}\right]^2\left[\frac{1}{(1-b^2)}+2(1+\alpha^2)\frac{b^2}{(1-b^2)^2}\right]\ , \tag{C9}$$

$$\left[-\frac{d^2 I_c}{d\Delta\tau_-{}^2}\right] = \int_0^\infty \coth\left[\frac{\pi\Omega}{2\beta}\right]\sin(\Omega\Delta\tau_-)\cos(\Omega\tau_e)\Omega^2 d\Omega$$

$$= -2\left[\frac{\beta}{\alpha}\right]^3 (1+\alpha^2)^{1/2}\left[\frac{1}{(1-b^2)} + (2\alpha^2+5)\frac{b^2}{(1-b^2)^2} + 4(1+\alpha^2)\frac{b^4}{(1-b^2)^3}\right] . \quad \text{(C10)}$$

These identities may be verified by utilizing the following relationship, which may be obtained from standard integral tables:[20]

$$\int_0^\infty \coth(b'x)\sin(a'x)dx = \frac{\pi}{2b'}\coth\left[\frac{\pi a'}{2b'}\right] . \quad \text{(C11)}$$

Three more required relationships are

$$\int_0^\infty dw \cos(wb)\sin(w) = \frac{1}{(1-b^2)} , \quad \text{(C12)}$$

$$\int_0^\infty dw\, w\cos(wb)\cos(w) = -\left[\frac{1}{(1-b^2)} + \frac{2b^2}{(1-b^2)^2}\right] , \quad \text{(C13)}$$

$$\int_0^\infty dw\, w^2 \cos(wb)\sin(w) = -2\left[\frac{1}{(1-b^2)} + \frac{5b^2}{(1-b^2)^2}\frac{+4b^4}{(1-b^2)^3}\right] . \quad \text{(C14)}$$

The meaning of these integrals, as well as others that follow in this section, should be interpreted by inserting an effective cutoff into the integral, then evaluating the integral, and finally taking an appropriate limit to remove the cutoff. For example,

$$\int_0^\infty dw \cos(wb')\sin(wc') = \lim_{\epsilon\to 0_+}\int_0^\infty dw\, e^{-\epsilon w}\cos(wb')\sin(wc') = \frac{c'}{(c'^2-b'^2)} . \quad \text{(C15)}$$

Combining Eqs. (C8)–(C10) and Eqs. (C12)–(C14), yields

$$\int_0^\infty dw \cos(wb)\sin(w) = \frac{\alpha}{\beta}\frac{1}{(1+\alpha^2)^{1/2}} I_c , \quad \text{(C16)}$$

$$\int_0^\infty dw\, w\cos(wb)\cos(w) = -\frac{\alpha}{\beta}\frac{\alpha^2}{(1+\alpha^2)^{3/2}} I_c + \frac{\alpha^2}{\beta^2}\frac{1}{(1+\alpha^2)}\frac{dI_c}{d\Delta\tau_-} , \quad \text{(C17)}$$

$$\int_0^\infty dw\, w^2\cos(wb)\sin(w) = \frac{\alpha}{\beta}\frac{(\alpha^2-2\alpha^4)}{(1+\alpha^2)^{5/2}} I_c + \frac{3\left[\frac{\alpha}{\beta}\right]^2\alpha^2}{(1+\alpha^2)^2}\frac{dI_c}{d\Delta\tau_-} + \left[\frac{\alpha}{\beta}\right]^3\frac{1}{(1+\alpha^2)^{3/2}}\left[-\frac{d^2 I_c}{d\Delta\tau_-^2}\right] . \quad \text{(C18)}$$

In order to demonstrate how Eqs. (C1) and (C3) may be verified, consider first the correlation function for the electric fields when $i=j=3$. After a fair amount of algebraic manipulations, the basic procedure of Ref. 21 results in

$$\langle E^{ZP}_{\tau_{e0}3}(\hat{\mathbf{y}}R_0,\tau_{e0})E^{ZP}_{(\tau_{e0}+\tau_e)3}(\hat{\mathbf{y}}(R_0\pm R),\tau_{e0}+\tau_e)\rangle = \mathrm{I}_{33} + \mathrm{II}_{33} , \quad \text{(C19)}$$

where

$$\mathrm{I}_{33} = \frac{\hbar c}{\pi R^4}\int_0^\infty dw \cos(wb)(w^2\sin w + w\cos w - \sin w) , \quad \text{(C20)}$$

$$\mathrm{II}_{33} = \frac{\hbar c}{\pi R^4}(\alpha^2 b^2)\int_0^\infty dw \cos(bw) \times (w^2 \sin w + 3w\cos w - 3\sin w) , \quad \text{(C21)}$$

and $w = kR$. Equation (C21) may be reexpressed by using a temporary cutoff and integrating by parts twice:

$$\mathrm{II}_{33} = \frac{-\hbar c\alpha^2}{\pi R^4}\lim_{\epsilon\to 0_+}\int_0^\infty dw\, e^{-\epsilon w}\left[\frac{d^2}{dw^2}\cos(bw)\right](w^2\sin w + 3w\cos w - 3\sin w)$$

$$= \frac{\hbar c\alpha^2}{\pi R^4}\int_0^\infty dw \cos(bw)(w^2 \sin w - w\cos w + \sin w) . \quad \text{(C22)}$$

Use of Eqs. (C16)–(C18) then results in

$$\langle E^{ZP}_{\tau_{e0}3}(\hat{\mathbf{y}}R_0,\tau_{e0})E^{ZP}_{(\tau_{e0}+\tau_e)3}(\hat{\mathbf{y}}(R_0\pm R),\tau_{e0}+\tau_e)\rangle$$

$$= \frac{\hbar c}{\pi R^4}\left[\left[-\frac{d^2 I_c}{d\Delta\tau_-^2}\right]\left[\frac{\alpha}{\beta}\right]^3\frac{1}{(1+\alpha^2)^{1/2}} + \frac{dI_c}{d\Delta\tau_-}\left[\frac{\alpha}{\beta}\right]^2\frac{(1+2\alpha^2)}{(1+\alpha^2)} - I_c\frac{\alpha}{\beta}\frac{1}{(1+\alpha^2)^{3/2}}\right] , \quad \text{(C23)}$$

which verifies Eqs. (C1) and (C3) when $i=j=3$. [Compare Eqs. (C23), (A39), (C1), (C3), and (C5) by using Eqs. (A19), (A20), (C6a), and (C8)–(C10).] The remaining situations described by Eqs. (C1) and (C3) may be verified in a similar way.

Turning now to the verification of Eqs. (C2) and (C4), the following identities are useful:

$$I_s \equiv \int_0^\infty \coth\left[\frac{\pi\Omega}{2\beta}\right]\cos(\Omega\Delta\tau_-)\sin(\Omega\tau_e)d\Omega = -\left[\frac{\beta}{\alpha}\right]\frac{b\cosh(\beta\tau_e)}{(1-b^2)} , \tag{C24}$$

$$\begin{aligned}\left[-\frac{dI_s}{d\Delta\tau_-}\right] &= \int_0^\infty \coth\left[\frac{\pi\Omega}{2\beta}\right]\sin(\Omega\Delta\tau_-)\sin(\Omega\tau_e)\Omega\,d\Omega \\ &= -2\left[\frac{\beta}{\alpha}\right]^2(1+\alpha^2)^{1/2}\left[\frac{1}{(1-b^2)}+\frac{b^2}{(1-b^2)^2}\right]b\cosh(\beta\tau_e) ,\end{aligned} \tag{C25}$$

$$\begin{aligned}\left[-\frac{d^2I_s}{d\Delta\tau_-^{\,2}}\right] &= \int_0^\infty \coth\left[\frac{\pi\Omega}{2\beta}\right]\cos(\Omega\Delta\tau_-)\sin(\Omega\tau_e)\Omega^2 d\Omega \\ &= 2\left[\frac{\beta}{\alpha}\right]^2\left[\frac{(3+2\alpha^2)}{(1-b^2)}+\frac{(7+6\alpha^2)}{(1-b^2)^2}b^2+\frac{4(1+\alpha^2)b^4}{(1-b^2)^3}\right]b\cosh(\beta\tau_e) .\end{aligned} \tag{C26}$$

In analogy with Eqs. (C12)–(C14), three more required relationships are

$$\int_0^\infty dw\,\sin(wb)\cos(w) = \frac{-b}{(1-b^2)} , \tag{C27}$$

$$\int_0^\infty dw\,w\sin(wb)\sin(w) = -2b\left[\frac{1}{(1-b^2)}+\frac{b^2}{(1-b^2)^2}\right] , \tag{C28}$$

$$\int_0^\infty dw\,w^2\sin(wb)\cos(w) = 2b\left[\frac{3}{(1-b^2)}+\frac{7b^2}{(1-b^2)^2}+\frac{4b^4}{(1-b^2)^3}\right] . \tag{C29}$$

Combining Eqs. (C24)–(C29) yields

$$\cosh(\beta\tau_e)\int_0^\infty dw\,\sin(wb)\cos(w) = \frac{\alpha}{\beta}I_s , \tag{C30}$$

$$\cosh(\beta\tau_e)\int_0^\infty dw\,w\sin(wb)\sin(w) = \left[\frac{\alpha}{\beta}\right]^2\frac{1}{(1+\alpha^2)^{1/2}}\left[-\frac{dI_s}{d\Delta\tau_-}\right] , \tag{C31}$$

$$\cosh(\beta\tau_e)\int_0^\infty dw\,w^2\sin(wb)\cos(w) = \frac{-\alpha^2\left[\dfrac{\alpha}{\beta}\right]^2}{(1+\alpha^2)^{3/2}}\left[-\frac{dI_s}{d\Delta\tau_-}\right]+\frac{\left[\dfrac{\alpha}{\beta}\right]^3}{(1+\alpha^2)}\left[-\frac{d^2I_s}{d\Delta\tau_-^{\,2}}\right] . \tag{C32}$$

As an example of how Eqs. (C2) and (C4) may be verified, consider the case when $i=2$, $j=3$. Earlier steps lead to the first line below, while the second line follows by integrating by parts once and using a temporary cutoff:

$$\begin{aligned}\langle B^{ZP}_{\tau_{e0}2}(\hat{y}R_0,\tau_{e0})E^{ZP}_{(\tau_{e0}+\tau_e)3}(\hat{y}(R_0\pm R),\tau_{e0}+\tau_e)\rangle &= -\frac{\hbar c}{\pi R^4}\cosh\left[\frac{a\tau_e}{2c}\right]\alpha b\int_0^\infty dw\,\cos(wb)(w^2\sin w+3w\cos w-3\sin w) \\ &= +\frac{\hbar c}{\pi R^4}\alpha\cosh\left[\frac{a\tau_e}{2c}\right]\int_0^\infty dw\,\sin(wb)(w^2\cos w - w\sin w) .\end{aligned} \tag{C33}$$

Use of Eqs. (C31) and (C32) then verifies this case. The remaining situations described by Eqs. (C2) and (C4) follow similarly.

Several symmetry relationships may be readily deduced for the correlation functions in Eqs. (C1) and (C2). The spatial symmetries follow from Eqs. (A37)–(A44). In particular, the following ones are of use in this article:

$$f^{ZP}_{ij}(\hat{x}a,+\hat{y}R,\Omega) = \begin{bmatrix}+1\\-1\end{bmatrix}f^{ZP}_{ij}(\hat{x}a,-\hat{y}R,\Omega)\ \text{for}\ \begin{bmatrix}i=j\\i\neq j\end{bmatrix} . \tag{C34}$$

Two final relationships, namely, Eqs. (C35) and (C37), will be discussed here, as these identities are required for the calculations of Sec. III B. From Eq. (C1), the following result can be verified almost immediately:

$$\left\langle E^{ZP}_{\tau'_e i}(\hat{y}R',\tau'_e)\frac{\partial}{\partial\xi_2}E^{ZP}_{\tau''_e j}(\xi,\tau''_e)\Big|_{\xi=\hat{y}R''}\right\rangle = \int_0^\infty d\Omega\,\cos[\Omega(\tau''_e-\tau'_e)]\frac{\partial}{\partial\Delta R}f^{ZP}_{ij}(\hat{x}a,\hat{y}\Delta R,\Omega)\,|_{\Delta R=R''-R'} . \tag{C35}$$

Consider the $i=j=1$ case. One can show that

$$\left\langle E^{\rm ZP}_{\tau'_e 1}(\hat{y}R',\tau'_e)\frac{\partial}{\partial\xi_2}E^{\rm ZP}_{\tau''_e 1}(\xi,\tau''_e)\bigg|_{\xi=\hat{y}R''}\right\rangle=-\frac{1}{2}\int d^3k\,h^2(\Omega)\left[1-\left[\frac{k_1}{k}\right]^2\right]k_2\sin\left[-\frac{2c^2}{a}k\sinh\left[\frac{a\tau_e}{2c}\right]+k_2(R''-R')\right]. \tag{C36}$$

When $R''=R'$, the integrand is an odd function of $k_2$. Provided that $i=j$, this can be shown to be a common feature for the correlation functions of Eq. (C35). Hence,

$$\left\langle E^{\rm ZP}_{\tau'_e i}(\hat{y}R',\tau'_e)\frac{\partial}{\partial\xi_2}E^{\rm ZP}_{\tau''_e i}(\xi,\tau''_e)\bigg|_{\xi=\hat{y}R''}\right\rangle=0 \quad \text{for } R''=R', \tag{C37}$$

since carrying out angular integrations will yield a value of zero.

## APPENDIX D: EVALUATION OF $\langle A_2\rangle$ IN EQ. (63)

The quantity $A_2$ in Eq. (63) arises from the second term in Eq. (19). Of the three vector components of $\Delta\mathbf{x}_{\tau_e}(t_{\tau_e})$ that appear in this term, $\Delta y_{\tau_e}(t_{\tau_e})$ and $\Delta z_{\tau_e}(t_{\tau_e})$ may be immediately replaced by $\xi_{A2}(\tau'_e)$ and $\xi_{A3}(\tau'_e)$. The relationship between $t_{\tau_e}$ and $\tau'_e$ is given by

$$ct_{\tau_e}(\tau'_e)=\left[\xi_{A1}(\tau'_e)+\frac{c^2}{a}\right]\sinh\left[\frac{a}{c}(\tau'_e-\tau_e)\right]. \tag{D1}$$

The $x$ position of the $A$ oscillating particle is given by

$$x_{A\tau_e}(t_{\tau_e})=\left[\xi_{A1}(\tau'_e)+\frac{c^2}{a}\right]\cosh\left[\frac{a}{c}(\tau'_e-\tau_e)\right], \tag{D2}$$

where $\tau'_e$ is again related to $t_{\tau_e}$ by Eq. (D1). The $x$ position of the equilibrium point of the $A$ oscillator at time $t_{\tau_e}=T_{\tau_e}$ is given by

$$X_{A\tau_e}(\tau''_e)=\frac{c^2}{a}\cosh\left[\frac{a}{c}(\tau''_e-\tau_e)\right], \tag{D3}$$

where $\tau''_e$ is related to the time coordinate $T_{\tau_e}$ of the equilibrium point by

$$cT_{\tau_e}(\tau''_e)=\frac{c^2}{a}\sinh\left[\frac{a}{c}(\tau''_e-\tau_e)\right]. \tag{D4}$$

As illustrated in Fig. 2, the value of $T_{\tau_e}(\tau''_e)$ in Eq. (D4) should be set equal to the value of $t_{\tau_e}(\tau'_e)$ in Eq. (D1). Hence, $\tau'_e$ and $\tau''_e$ are related to each other by

$$\frac{c^2}{a}\sinh\left[\frac{a}{c}(\tau''_e-\tau_e)\right]=\left[\xi_{A1}(\tau'_e)+\frac{c^2}{a}\right]\sinh\left[\frac{a}{c}(\tau'_e-\tau_e)\right]. \tag{D5}$$

From Eqs. (D3) and (D5), $X_{A\tau_e}$ can be reexpressed in terms of the value of $\tau'_e$ that occurs in Eq. (D1). Hence,

$$\begin{aligned}\Delta x_{A\tau_e}(t_{\tau_e})&=(x_{A\tau_e}-X_{A\tau_e})\,|_{t_{\tau_e}}\\
&=\left[\xi_{A1}(\tau'_e)+\frac{c^2}{a}\right]\cosh\left[\frac{a}{c}(\tau'_e-\tau_e)\right]-\frac{c^2}{a}\left[1+\left[1+\frac{a\xi_{A1}(\tau'_e)}{c^2}\right]^2\sinh^2\left[\frac{a}{c}(\tau'_e-\tau_e)\right]\right]^{1/2}\\
&\approx\frac{\xi_{A1}(\tau'_e)}{\cosh\left[\frac{a}{c}(\tau'_e-\tau_e)\right]}\left[1+O\left[\frac{a\xi_{A1}}{c^2}\right]\right].\end{aligned} \tag{D6}$$

Thus, the three components of $\Delta\mathbf{x}_{\tau_e}(t_{\tau_e})$ have now been reexpressed in terms of $\xi_{A1}(\tau'_e)$.

The distinction between $\tau'_e$ of Eq. (D1) and $\tau''_e$ of Eq. (D4), where $\tau'_e$ and $\tau''_e$ are related to each other by Eq. (D5), may now be ignored when reexpressing the remaining quantities in the second term of Eq. (19) in terms of the $\xi_{Ai}$ coordinates. The reason for this is that $\tau'_e$ and $\tau''_e$ differ from each other only by terms of order $a\xi_{A1}/c^2$, and $\Delta\mathbf{x}_{A\tau_e}$ has already been shown to be of first order in $\xi_A$.

Consequently, from Eq. (D1), the following time derivative occurring in Eq. (19) may be written as

$$\frac{d}{dt_{\tau_e}}(\;)\bigg|_{t_{\tau_e}=0}=\frac{d\tau_e'}{dt_{\tau_e}}\bigg|_{t_e'=\tau_e}\frac{d}{d\tau_e'}(\;)\bigg|_{\tau_e'=\tau_e}, \tag{D7}$$

where

$$\frac{d\tau_e'}{dt_{\tau_e}}\bigg|_{\tau_e'=\tau_e}=\frac{1}{\left[1+\dfrac{a\xi_{A1}(\tau_e)}{c^2}\right]}=1+O\left[\frac{a\xi_{A1}}{c^2}\right]. \tag{D8}$$

Making the substitutions that have been indicated so far, yields

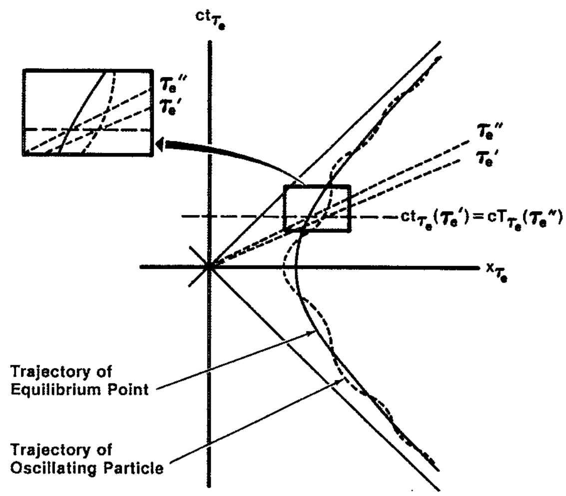


FIG. 2. Distinction between $\tau_e'$ and $\tau_e''$.

$$A_2=\frac{e}{c}\left[1+O\left[\frac{a\xi_{A1}}{c^2}\right]\right]\frac{d}{d\tau_e'}\left[\xi_{A3}(\tau_e')B_{\tau_e1}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]-\frac{\xi_{A1}(\tau_e')}{\cosh\left[\dfrac{a}{c}(\tau_e'-\tau_e)\right]}B_{\tau_e3}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]\right]\Bigg|_{\tau_e'=\tau_e}. \tag{D9}$$

The fields $B_{\tau_e i}$ may be reexpressed in terms of the fields in the $I_{\tau_e'}$ frame via the Lorentz transformations of Eqs. (7) and (8). Making these substitutions, differentiating all the obvious terms of $\cosh[(a/c)(\tau_e'-\tau_e)]$ and $\sinh[(a/c)(\tau_e'-\tau_e)]$, and then evaluating these terms at $\tau_e'=\tau_e$, yields

$$A_2\approx\frac{e}{c}\frac{d}{d\tau_e'}\left[\xi_{A3}(\tau_e')B_{\tau_e'1}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]-\xi_{A1}(\tau_e')B_{\tau_e'3}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e'\right]\right]\Bigg|_{\tau_e'=\tau_e}+\left[\frac{a\xi_{A1}(\tau_e)}{c^2}\right]eE_{\tau_e2}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e\right], \tag{D10}$$

where the $O(a\xi_{A1}/c^2)$ terms in Eq. (D9) have been ignored.

Upon taking the expectation value of $A_2$, the first two terms of Eq. (D10) drop out, leaving

$$\langle A_2\rangle\approx\frac{ea}{c^2}\left[\left\langle\xi_{A1}(\tau_e)E^{\rm ZP}_{\tau_e2}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e\right]\right\rangle+\left\langle\xi_{A1}(\tau_e)E^{DB}_{\tau_e2}\left[\hat{\mathbf{y}}\frac{R}{2},\tau_e\right]\right\rangle\right]. \tag{D11}$$

In order to evaluate this expression, let quantities of order $aR/c^2$ in Eqs. (62a) and (62b) be dropped as an initial approximation. The two terms of Eq. (D11) can then be calculated by following the steps of Sec. III B. When the resonant approximation is employed in the unretarded van der Waals situation, then the first term can be shown to be negligible compared to the second one. This result arises for nearly the same reason that Eq. (32) is negligible compared to Eq. (35) under the same conditions.

Turning attention to the second term of Eq. (D11), one can show that under resonant conditions this quantity is approximately $(aR/c^2)^2$ times the magnitude of the second term of Eq. (63). The factor of $(aR/c^2)^2$ arises essentially from the ratio of

$$\left|\frac{\dfrac{a}{c^2}\,\mathrm{Re}\eta^a_{21}}{\dfrac{\partial}{\partial R}\,\mathrm{Re}\eta^a_{ii}}\right|_{\Omega R/c\ll 1}\sim\frac{\dfrac{a}{c^2}\left[\dfrac{aR}{c^2}\right]\mathrm{Re}\eta_{ii}}{\dfrac{1}{R}\,\mathrm{Re}\eta_{ii}}=\left[\frac{aR}{c^2}\right]^2.$$

If the initial approximation of ignoring terms of order $aR/c^2$ is reexamined and these terms are retained, then one can trace through the calculations indicated above to see that the contribution of these terms will also yield a final result characterized by the factor of $(aR/c^2)^2$.

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[18]Reference 8 describes how to handle higher-order correlation functions for the linear oscillators that involve the random phases.

[19]See, for example, J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1975), p. 657, Eq. (14.14).

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