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Derivation of the classical electromagnetic zero-point radiation spectrum via a classical thermodynamic operation involving van der Waals forces

Daniel C. Cole
IBM General Technology Division, Essex Junction, Vermont 05452
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A new fundamental thermodynamic property has been proved for classical electromagnetic zero-point radiation. Aside from a proportionality constant, classical electromagnetic zero-point radiation is shown here to possess the unique spectrum that is suitable for establishing a thermal equilibrium state with a set of fluctuating classical electric dipole harmonic oscillators at the temperature of absolute zero. As a consequence of this analysis, one can see that according to the fundamental thermodynamic definition of absolute zero temperature, the following traditional view in thermodynamics is an unnecessary thermodynamic restriction: namely, that all fluctuating motion and radiation must vanish at $T=0$ for classical physical systems. Indeed, as shown here, this restriction can violate the third law of thermodynamics if the spectrum reduces to zero by being proportional to T. The analysis in this article involves the calculation of (1) the change in internal energy, (2) the work done, and (3) the heat radiated for a quasistatic displacement of electric dipole nonrelativistic harmonic oscillators immersed in random classical electromagnetic radiation. The calculations include full nonperturbative evaluations of retarded van der Waals thermodynamic functions.

I. INTRODUCTION

A reversible isothermal thermodynamic operation is analyzed here that consists of the quasistatic displacement of classical electric dipole oscillators. These fluctuating electric dipoles interact with each other and with random classical electromagnetic radiation at some equilibrium temperature T. The radiation is assumed to have Gaussian stochastic properties that are isotropic and homogeneous in space. A number of thermodynamic quantities are calculated, such as the internal energy for this system and the heat flow during this reversible isothermal process. Aside from a proportionality constant, only one nonzero radiation spectrum is found to result in *no* heat being radiated off into space when the oscillators are slowly displaced from each other: namely, the spectrum of classical electromagnetic zero-point (ZP) radiation (see Refs. 1–5 for reviews).

This result of no heat transfer is precisely what we expect to find for a system in equilibrium at the temperature of absolute zero. More specifically, a system is defined as being at absolute zero when no heat flow Q can occur out of the system during any reversible isothermal process performed on the system.[6] Consequently, aside from a proportionality constant, classical electromagnetic ZP radiation possesses the only nonzero spectrum suitable for establishing an equilibrium state with the electric dipole oscillators at a temperature of absolute zero.

This requirement of $Q=0$ at $T=0$ must also be satisfied in order for the third law of thermodynamics to be true. Indeed, according to the Nernst-Simon form of the third law,[7] not only should $Q=0$ during an isothermal reversible process at $T=0$, but also the ratio of Q/T should approach zero in the limit of $T\rightarrow 0$. Consequently, the third law of thermodynamics is shown here to place a further restriction on the spectrum of incident radiation. This restriction is tested on two radiation spectrums that have been extensively studied in the physics literature.[1–5,8–17] The familiar classical Rayleigh-Jeans (RJ) radiation spectrum is found to violate the third law of thermodynamics, while the classical electromagnetic zero-point plus Planckian (ZPP) radiation spectrum satisfies it.

The above two results fit in nicely with arguments made by a few researchers during the past 27 years or so,[1–5,9–12] that previous analysis[8,14,18] on the thermodynamic behavior of classical charged particle systems have not included an important element in their reasoning:[11,12] namely, classical electromagnetic zero-point radiation. Indeed, as will be argued shortly, if a statistical equilibrium configuration is at all possible for a system of classical charged particles, then at a temperature of absolute zero there must exist a "zero-point" classical electromagnetic radiation, as well as a zero-point oscillating motion for the charges.

Of course, ZP fields and ZP motion are normally associated only with quantum-mechanical systems and are quite foreign to the traditional ideas of classical physics. However, a qualitative way of understanding why ZP fields and ZP motion should be a natural part of the thermodynamic behavior of classical charged particle systems, is to think in terms of Earnshaw's theorem.[19] According to this theorem, a system of classical charged particles cannot exist in static, stable equilibrium. Hence, if an equilibrium situation for charged particles is at all possible, then the charges must be following a fluctuating, oscillatory path in space. The oscillating charges produce fluctuating electromagnetic fields, which in turn act upon the charges. Thus any possible equilibrium situation must involve the presence of electromagnetic radiation, as well as an oscillatory motion for the charges, even at a temperature of absolute zero. All motion of charges would then possess a stochasticlike character. These qualitative ideas correspond to what we observe in nature: at $T=0$, molecular activity does not cease; it has a zero-point motion.

Including ZP radiation in determining the behavior of classical charged particles brings classical theory closer to experimental observation.[1–5,20–23] Indeed, on the basis of the existence of classical electromagnetic ZP radiation, a number of predictions and calculations have been made that agree with quantum theory: for example, (1) classical derivations of blackbody radiation,[11,12,24,25] (2) van der Waals force calculations,[26–32] (3) correct behavior for a charged particle in a simple harmonic-oscillator potential,[2,9] (4) explanations for quantum optic effects,[22,23] as well as qualitative ideas for[1] (5) tunneling, (6) diffraction, (7) interference, (8) the Heisenberg uncertainty principle, and (9) atomic stability.[1,33]

This classical theory is often referred to as stochastic electrodynamics (SED). A major test of SED will be establishing whether or not atomic stability is indeed predicted, and whether the radiation emitted by a classical atomic system yields an appropriate equilibrium state for the net thermal radiation. If true, as in the simplest case of a massive stationary nucleus, then we would have a specific example of statistical equilibrium between a classical charged particle system and classical electromagnetic radiation.

Presently, however, work by other researchers provides arguments for why SED cannot satisfactorily describe the physics of atomic systems in nature (Refs. 16, 17, 20, 21, and 34–36). For example, a charged particle oscillating in certain classes of nonlinear binding potentials has been shown to be in equilibrium with Rayleigh-Jeans radiation, rather than classical electromagnetic ZPP radiation, and to follow the behavior of Maxwell-Boltzmann statistics, rather than the statistical behavior predicted by quantum theory.[16,17] Moreover, some researchers have claimed that a classical hydrogen system must ionize due to the influence of classical electromagnetic ZP radiation.[34,35]

Nevertheless, as I have recently shown,[37] this other work is based on perturbation analysis that is not appropriate for actual atomic systems. Moreover, as Boyer[38] and myself[37] have recently emphasized, past tests of SED on nonlinear mechanical systems (Refs. 13–17, 20, 21, 34–36) should be restricted to systems of charged particles interacting strictly via electromagnetic forces. Only in this way can a proper connection be made with real atomic systems in nature. Testing SED on charged particles oscillating in somewhat arbitrary binding potentials that are not of electromagnetic origin, simply does not adequately approximate the physics of real atomic systems.

Thus the status of SED is still uncertain. This paper does not change that status. Indeed, the analysis described here has a number of limitations. The most significant one is that only a simple model of a classical atomic system is again treated: namely, an electric dipole nonrelativistic simple harmonic oscillator. The advantage of this system is that the mathematical treatment is clear. Consequently, this system has often been treated in both the quantum and classical literature when modeling various properties of atomic systems (e.g., van der Waals forces, optical properties of materials, specific heats of crystals, etc.). A second advantage is that a close connection is known to exist between the classical and quantum theories for this system.[2] Nevertheless, this classical dipole oscillator model is certainly not a sufficiently accurate representation of atomic systems found in nature to claim that the results found here must also hold for a real atomic system described via classical physics. Thus these results would be more significant if, for example, the considerably more difficult case of a classical hydrogen atom was treated.[37]

Moreover, other criticisms can be made about the analysis in this paper. For example, only the change in the total energy is calculated for the thermodynamic operation of quasistatistically displacing the dipole oscillators. This treatment is in keeping with the usual thermodynamic definition of $T=0$, which only requires that we examine the total heat flow out of the system. Still, other interesting and important questions exist about the system, such as what is the spectrum of the radiated energy as the oscillators are slowly displaced, and what is the relationship of the spectrum of the radiated energy to the spectrum of the thermal radiation? These questions are not examined here.[39]

Nevertheless, despite these limitations, this paper does make a contribution to the study of the thermodynamics of classical electrodynamic systems. Specifically, the calculations contained here show that a classical electrodynamic system can possess a nonzero fluctuating energy, yet still satisfy the fundamental *thermodynamic* definition of the temperature at absolute zero. Consequently, the traditional view in classical physics that all motion must stop at $T=0$, is an unnecessary restriction that has been imposed on classical systems. Indeed, as shown here, this restriction can violate the third law of thermodynamics, and may have resulted in a fundamentally flawed view of the equilibrium behavior for classical charged particles and classical electromagnetic radiation.

Instead, by allowing nonzero electromagnetic fields and nonzero particle motion at $T=0$, we are able to fulfill the traditional thermodynamic definition of $T=0$, as well as the third law of thermodynamics, for this electric dipole oscillator system. However, these results could only be obtained by restricting the incident radiation spectrum to be given by $\rho_{\mathrm{in}}(\omega)=\kappa\omega^3/c^3$, where κ is an arbitrary constant greater than or equal to zero, c is the speed of light, and ω is the angular frequency. Since this spectrum is precisely the spectral form for classical electromagnetic ZP radiation, we obtain the derivation that aside from the proportionality constant κ, only the classical electromagnetic ZP radiation spectrum will satisfy the $T=0$ thermodynamic conditions imposed here for this system of dipole oscillators.[40] This result points favorably toward classical electromagnetic ZP and ZPP radiation as possessing the appropriate spectrums to be in equilibrium with classical electrodynamic charged particle systems.

Moreover, in further support of these results, a few suggestive arguments are given here that the proportionality constant κ should be nonzero in nature. These arguments are meant to motivate the speculation that ZP motion and ZP radiation might be an essential part of the thermodynamic behavior of classical charged particles. Clearly, however, these arguments are only suggestive ones, and should not be construed as "proofs." The earlier argument involving Earnshaw's theorem falls under this category, since it was based on the condition of, ". . . *if* an equilibrium situation for charged particles is at all possible, then . . . ."[41]

Thus, despite the proof contained here that $\rho_{\mathrm{in}}(\omega)=\kappa\omega^3/c^3$ yields satisfactory results for the $T=0$ behavior of the electric dipole oscillators, and despite the suggestive arguments that $\kappa\neq 0$ in nature, the final answer on the question of classical electromagnetic ZP radiation has not been shown, and will not be shown, at least until the difficult, but physically important problem of classical atomic systems with Coulombic binding potentials has been carefully treated.

As for the outline of the present paper, Sec. II sets up the calculations needed for finding the change in internal energy of a system of N electric dipole oscillators when they are slowly displaced from each other. Section III finds the steady-state behavior for the oscillators, while Secs. IV and V find the change in internal energy and the work done, respectively, during a reversible isothermal displacement of the oscillators. Section VI calculates the heat radiated off into space and derives the ZP radiation spectrum as a result of the demand that $Q=0$ at $T=0$. In Sec. VII the third law of thermodynamics is shown to hold for ZPP radiation, but not for RJ radiation. Section VIII contains concluding remarks on the work described here.

Finally, before turning to the analysis in this paper, I want to mention that there are a number of quantities calculated here for a classical charged oscillator system that should carry over to the analogous QED case, but which apparently have not been calculated before via the usual methods of QED. A close connection between SED and QED is known to exist for a nonrelativistic charged particle in a simple harmonic-oscillator (SHO) potential,[2] which is precisely the system treated here. In the past, Renne[42] has calculated, via QED, retarded van der Waals forces at $T=0$ for electric dipole nonrelativistic simple harmonic oscillators. His calculation is particularly interesting because it does not require the usual quantum-mechanical perturbation methods, and so is exact, at least within his nonrelativistic treatment of the problem. However, extending this calculation to nonzero temperatures is nontrivial. Boyer accomplished this task quite easily via the different calculational method of SED.[29,30]

In the present paper several additional properties are calculated for this system, such as the change in internal energy, work done, and heat radiated for a slow displacement of these oscillators at a fixed temperature. In a planned, separate paper[43] these quantities will be calculated for even more general reversible thermodynamic processes, so that T does not need to be held fixed. In this way, a general state of entropy can be derived, specific heat can be obtained, and the other usual thermodynamic functions can be calculated.

II. DESCRIPTION OF ENERGY CALCULATION

Here we will set up the calculations needed to analyze the operation of slowly displacing N neutrally charged classical atomic systems from some initial positions in space, to some final configuration, while in equilibrium with incident classical electromagnetic radiation. Let us assume that the atoms are sufficiently far separated at all times that the dipole fields from each atom form the dominant means of interaction between the atoms. In this paper we will calculate the change in energy associated with displacing these atoms.

In order to make this problem tractable, at the present time we must resort to modeling the atoms as electric dipole oscillators. The oscillator model we will consider here is related to the oscillator model treated in Refs. 8, 11, 12, and 29–32, which used a Drude-Lorentz approximation for an atom or molecule. This Drude-Lorentz oscillator-particle model was taken to be electrically neutral, but with a fluctuating electric dipole that obeyed a linearized form of the Lorentz-Dirac equation of motion that would arise from a charged point particle oscillating in a simple harmonic-oscillator potential.

The following further elaboration of this model gives one possible simple means for picturing the physical construction of such an oscillator. We can think of the oscillator as arising from a classical charged point particle with rest mass m and charge $+e$ that oscillates inside a spherical uniform charge distribution with net charge $-e$. When this spherical charge distribution is unaccelerated,[44,45] the oscillating $+e$ particle will be acted upon by an isotropic SHO potential in the rest frame of the $-e$ charge distribution. For points at large distances compared to the radius of the $-e$ spherical charge distribution, the electromagnetic fields of this oscillator-particle model will be that of a time-varying electric dipole.

For times $t\le t_{\mathrm{I}}$, let the centers of these N electric dipoles be held fixed at positions $\mathbf{Z}_{A,\mathrm{I}}$, for $A=1,2,\ldots,N$.

Likewise, for times $t_{\rm II} \le t$, let the centers be held fixed at positions $\mathbf{Z}_{A,\rm II}$. Between times $t_{\rm I}$ and $t_{\rm II}$, let us assume that the electric dipoles are slowly displaced from the initial to the final positions. By making $t_{\rm II}-t_{\rm I}$ sufficiently large, a quasistatic operation can be realized.

Let $\mathcal{V}$ be a volume in space that contains the N particles. Let (1) $\mathcal{U}_{\rm int}(t)$ be the internal energy at time t within this volume $\mathcal{V}$ due to the particles and electromagnetic fields, (2) $\mathcal{Q}$ be the electromagnetic energy that flows between times $t_{\rm I}$ and $t_{\rm II}$ *into* $\mathcal{V}$, and (3) $\mathcal{W}$ be the work done by external forces in moving the N particles from one set of coordinates to another in the time interval between time $t_{\rm I}$ and $t_{\rm II}$. Conservation of energy then demands that

$$\Delta\mathcal{U}_{\rm int} \equiv \mathcal{U}_{\rm int}(t_{\rm II})-\mathcal{U}_{\rm int}(t_{\rm I}) = \mathcal{Q}+\mathcal{W} . \quad (1)$$

where the signs are such that if positive work is done *on* the system of particles, then its internal energy increases.

Upon taking the expectation value of Eq. (1), the first law of thermodynamics is obtained, where $Q \equiv \langle \mathcal{Q} \rangle$ represents the heat flowing into $\mathcal{V}$ in the form of random electromagnetic energy. Here, we will calculate Q by evaluating $W \equiv \langle \mathcal{W} \rangle$ and $\Delta U_{\rm int} \equiv \langle \Delta\mathcal{U}_{\rm int} \rangle$. By finding Q, we can then calculate the change in caloric entropy.

Now let us identify all the parts of $\mathcal{U}_{\rm int}$ that will contribute to $\Delta U_{\rm int}$ between times $t_{\rm I}$ and $t_{\rm II}$. Besides the electromagnetic energy of interaction due to the electric dipoles and any incident radiation, we must include the mechanical internal energy of each fluctuating electric dipole. In keeping with our Drude-Lorentz model, we will treat the mechanical internal energy as arising from a $+e$ point charge oscillating inside a SHO potential, due to a surrounding $-e$ uniform sphere of charge. Hence

$$\mathcal{U}_{\rm int} \equiv \sum_{A=1}^{N} (m^*\gamma_{+A}c^2)+\frac{1}{8\pi}\int_{\mathcal{V}} d^3x(\mathbf{E}_{\rm tot}^2+\mathbf{B}_{\rm tot}^2) , \quad (2)$$

where m^* is the bare mass of the oscillating $+e$ charge of the A-labeled electric dipole,

$$\gamma_{+A} \equiv \left[1-\left|\frac{\dot{\mathbf{z}}_{+A}}{c}\right|^2\right]^{-1/2} ,$$

and $\mathbf{z}_{+A}$ and $\dot{\mathbf{z}}_{+A}$ represent the position and velocity, respectively, of the $(+e)$-A oscillating point charge. Also,

$$\mathbf{E}_{\rm tot}=\mathbf{E}_{\rm in}+\sum_{A=1}^{N}(\mathbf{E}_{+A}+\mathbf{E}_{-A}) , \quad (3)$$

and likewise for $\mathbf{B}_{\rm tot}$, where $\mathbf{E}_{\rm in}$ and $\mathbf{B}_{\rm in}$ are the incident electric and magnetic fields, $\mathbf{E}_{+A}$ and $\mathbf{B}_{+A}$ are the retarded electromagnetic fields due to the oscillating $+e$ charge of the A electric dipole, and $\mathbf{E}_{-A}$ and $\mathbf{B}_{-A}$ are the retarded electromagnetic fields due to the $-e$ sphere of charge in electric dipole A.

The following terms:

$$\mathcal{U}_{{\rm EM},+A} \equiv \frac{1}{8\pi}\int_{\mathcal{V}} d^3x[(\mathbf{E}_{+A})^2+(\mathbf{B}_{+A})^2] , \quad (4)$$

which occur in Eq. (2), are singular; these terms represent the electromagnetic energy in $\mathcal{V}$ due to each of the $(+e)$-A oscillating point charges, while ignoring the presence of the remaining charges. Here, we can use the results of the work of Teitelboim and co-workers[46,47] where a mass renormalization procedure was described to remove these singularities.

Following the procedure in Refs. 46 and 47, the energy term in Eq. (4) can be split into two parts. Let

$$\mathbf{E}_{+A}=\mathbf{E}_{v,+A}+\mathbf{E}_{a,+A} , \quad (5)$$

and likewise for $\mathbf{B}_{+A}$, where $\mathbf{E}_{v,+A}$ and $\mathbf{E}_{a,+A}$ are the velocity and acceleration fields, respectively, associated with the $(+e)$-A particle.[48] Here, $\mathbf{E}_{v,+A}$ and $\mathbf{B}_{v,+A}$ are proportional to R^{-2}, and $\mathbf{E}_{a,+A}$ and $\mathbf{B}_{a,+A}$ are proportional to R^{-1}, where $R=|\mathbf{x}-\mathbf{z}_{+A}(t_r)|$, $\mathbf{x}$ is the point in space at which the fields are to be evaluated, and $\mathbf{z}_{+A}(t_r)$ represents the position of the $(+e)$-A point charge at the retarded time $t_r=t-R/c$.

We can then write that

$$\mathcal{U}_{{\rm EM},+A}=\int_{\mathcal{V}} d^3x[u_{+A(-2)}+u_{+A(-3)}] , \quad (6)$$

where

$$u_{+A(-2)}=\frac{1}{8\pi}[(\mathbf{E}_{a,+A})^2+(\mathbf{B}_{a,+A})^2] , \quad (7)$$

$$u_{+A(-3)}=\frac{1}{8\pi}[2(\mathbf{E}_{a,+A}\cdot\mathbf{E}_{v,+A})+(\mathbf{E}_{v,+A})^2 +2(\mathbf{B}_{a,+A}\cdot\mathbf{B}_{v,+A})+(\mathbf{B}_{v,+A})^2] . \quad (8)$$

Thus $u_{+A(-2)}$ contains only terms proportional to R^{-2}, while $u_{+A(-3)}$ contains the remaining terms that are proportional to R^{-n}, $n \ge 3$.

The first and second terms on the right-hand side (rhs) of Eq. (6) are nonsingular and singular, respectively. If we were to integrate the second term in Eq. (6) over all space, instead of just over $\mathcal{V}$, then the work of Teitelboim and co-workers[46,47] shows that we can identify this energy term as a singular "bound" electromagnetic energy that can be combined with the bare mass energy term $m^*c^2\gamma_{+A}$ to yield a nonsingular result. One then obtains $mc^2\gamma_{+A}$, where m is the experimentally measured mass, plus the term

$$\mathcal{U}_{{\rm Schott},+A}=-\frac{d}{dt}\left[\frac{1}{3}\frac{e^2}{c}(\gamma_{+A})^2\right] , \quad (9)$$

which is the Schott energy term. This result follows[49] from cP^0 in Eq. (8.7) in Ref. 47.

However, since we are considering the case where each of the terms in Eq. (6) is integrated over a finite volume $\mathcal{V}$, rather than over all space, then we must change the above procedure slightly. Let $\tilde{\mathcal{V}}$ represent all of space excluding the volume $\mathcal{V}$. Then,

$$\mathcal{U}_{\text{int}}=\sum_{A=1}^{N}\left\{\left[m\gamma_{+A}c^2-\frac{d}{dt}\left[\frac{1}{3}\frac{e^2}{c}(\gamma_{+A})^2\right]\right]-\int_{\mathcal{V}}d^3x\,u_{+A(-3}\right\}+\int_{\mathcal{V}}d^3x\left[\frac{1}{8\pi}[(\mathbf{E}_{\text{tot}})^2+(\mathbf{B}_{\text{tot}})^2]-\sum_{A=1}^{N}u_{+A(-3}\right]. \tag{10}$$

The sum of the terms in the curly brackets in Eq. (10) equals the sum over all particles of the material energy term plus bound electromagnetic energy term for each particle. In this way we remove the singularities associated with the individual $(+e)$-A particles.

Before proceeding with evaluating $\Delta\mathcal{U}_{\text{int}}$ and its expectation value, however, there remains one other singular quantity that needs to be taken into account. If the incident radiation is one of the two candidates usually considered for providing thermal equilibrium, namely, RJ or ZPP classical electromagnetic radiation, then

$$U_{\text{EM,in}}\equiv\left\langle\frac{1}{8\pi}\int_{\mathcal{V}}d^3x(\mathbf{E}_{\text{in}}^2+\mathbf{B}_{\text{in}}^2)\right\rangle \tag{11}$$

is indeed singular, so that $\langle\mathcal{U}_{\text{int}}\rangle$ is singular as well. However, in our final result we are only interested in the change in $\langle\mathcal{U}_{\text{int}}\rangle$, and not in the actual value of $\langle\mathcal{U}_{\text{int}}\rangle$. Thus, for incident radiation with stochastic properties that are stationary in time, as occurs for RJ or ZPP radiation held at a fixed temperature, then

$$\Delta U_{\text{EM,in}}\equiv U_{\text{EM,in}}(t_{\text{II}})-U_{\text{EM}}(t_{\text{I}})=0\ , \tag{12}$$

thereby removing this singularity. Hence $\Delta U_{\text{EM,in}}$ will only make a contribution in a situation best described via the statistical properties of the radiation changing between times t_{I} and t_{II}, such as occurs when there is a change in temperature of the radiation.[50]

Let us now make a few observations and simplifying assumptions in order to cast $\Delta\mathcal{U}_{\text{int}}$ into a form that can be explicitly evaluated. Consider the terms

$$\mathcal{U}_{\text{EM},-A}\equiv\frac{1}{8\pi}\int_{\mathcal{V}}d^3x[(\mathbf{E}_{-A})^2+(\mathbf{B}_{-A})^2]\ . \tag{13}$$

At time t_{I} the fields of the $(-e)$-A spheres are purely Coulombic. Let us redefine t_{II} slightly to be the time at which the $(-e)$-A spheres have not just stopped moving, but have stopped moving for at least a time $R_{\mathcal{V}}/c$, where $R_{\mathcal{V}}$ is the largest diameter associated with the volume $\mathcal{V}$. Then the fields of the $(-e)$-A spheres will also be Coulombic within $\mathcal{V}$ at time t_{II}. Consequently, at times t_{I} and t_{II} and for points $\mathbf{x}$ within $\mathcal{V}$, $\mathbf{B}_{-A}(\mathbf{x})=0$ and $\mathbf{E}_{-A}(x)=-\nabla\phi_{-A}$, where

$$\phi_{-A}(\mathbf{x})=\begin{cases}\dfrac{e}{2R^3}|\mathbf{x}-\mathbf{Z}_A|^2-\dfrac{3}{2}\dfrac{e}{R}, & |\mathbf{x}-\mathbf{Z}_A|\le R\\[2ex] -\dfrac{e}{|\mathbf{x}-\mathbf{Z}_A|}, & |\mathbf{x}-\mathbf{Z}_A|>R\end{cases}\ . \tag{14}$$

Here, R is the radius of each of the $-e$ spheres of charge, and $\mathbf{Z}_A$ is the position of the center of the $(-e)$-A sphere. Also, in Eq. (14) the zero reference of ϕ_{-A} was conveniently chosen so that $\phi_{-A}=0$ at $|\mathbf{x}|=\infty$. As can be seen, at times t_{I} and t_{II} the force of each $(-e)$-A sphere on the $(+e)$-A particle oscillating inside the sphere is given by the SHO force $e\mathbf{E}_{-A}=-(e^2/R^3)\delta\mathbf{z}_A$, where $\delta\mathbf{z}_A\equiv\mathbf{z}_{+A}-\mathbf{Z}_A$. The oscillating particle's natural angular frequency is then

$$\omega_0^2\equiv\frac{e^2}{mR^3}\ . \tag{15}$$

We can now evaluate the following term that is in Eq. (10):

$$\begin{aligned}\frac{1}{8\pi}\int_{\mathcal{V}}d^3x(2\mathbf{E}_{-A}\cdot\mathbf{E}_{+A}+2\mathbf{B}_{-A}\cdot\mathbf{B}_{+A})&=\frac{1}{4\pi}\int_{\mathcal{V}}d^3x\,\phi_{-A}(\nabla\cdot\mathbf{E}_{+A})-\frac{1}{4\pi}\oint_{\mathcal{S}}d^2x\,\hat{\mathbf{n}}\cdot(\phi_{-A}\mathbf{E}_{+A})\\ &=\frac{m\omega_0^2}{2}(\delta\mathbf{z}_A)^2-\frac{3}{2}\frac{e^2}{R}-\frac{1}{4\pi}\oint_{\mathcal{S}}d^2x\,\hat{\mathbf{n}}\cdot(\phi_{-A}\mathbf{E}_{+A})\ .\end{aligned} \tag{16}$$

Here, the volume integral in the first line on the rhs was evaluated using (1) the fact that $\nabla\cdot\mathbf{E}_{+A}=+e4\pi\delta^3(\mathbf{x}-\mathbf{z}_{+A})$, (2) plus Eq. (14), and (3) the implicit assumption that the $(+e)$-A oscillating particle lies within the $(-e)$-A sphere, which is in keeping with our SHO electric dipole model. As for the constant energy term of $-3e^2/(2R)$ in Eq. (16), this term drops out when computing $\Delta\mathcal{U}_{\text{int}}$. Also, when we compute $\langle\Delta\mathcal{U}_{\text{int}}\rangle$ we will be able to neglect the surface term in Eq. (16), provided that the surface $\mathcal{S}$ of the volume $\mathcal{V}$ is far removed from all particles, and provided that $|\dot{\mathbf{z}}_{+A}|/c\ll1$. (This last nonrelativistic approximation will be used throughout our energy calculations.) Under these conditions, ϕ_{-A} varies as $1/|\mathbf{x}|$ on $\mathcal{S}$, and E_{+A} can safely be approximated on $\mathcal{S}$ by[48]

$$\mathbf{E}_{+A}(\mathbf{x},t)\approx\mathbf{E}_{a,+A}(\mathbf{x},t)=\frac{e}{c}\frac{[\mathbf{x}-\mathbf{z}_{+A}(t_r)]\times\left[\left[[\mathbf{x}-\mathbf{z}_{+A}(t_r)]-\frac{\dot{\mathbf{z}}_{+A}(t_r)}{c}|\mathbf{x}-\mathbf{z}_{+A}(t_r)|\right]\times\frac{\ddot{\mathbf{z}}_{+A}(t_r)}{c}\right]}{\left[|\mathbf{x}-\mathbf{z}_{+A}(t_r)|-\frac{\dot{\mathbf{z}}_{+A}}{c}\cdot[\mathbf{x}-\mathbf{z}_{+A}(t_r)]\right]^3}$$

$$\approx\frac{e}{c}\frac{\mathbf{x}\times\left[\mathbf{x}\times\frac{\ddot{\mathbf{z}}_{+A}(t_r)}{c}\right]}{|\mathbf{x}|^3}, \tag{17}$$

which also varies as $1/|\mathbf{x}|$ on $\mathcal{S}$. (Above, $|\dot{\mathbf{z}}_{+A}|/c$ terms were dropped, and $|\mathbf{z}_{+A}|/|\mathbf{x}|\ll 1$ was assumed.) If the surface $\mathcal{S}$ happens to be a sphere, then $\hat{\mathbf{n}}=\mathbf{x}/|\mathbf{x}|$ and $\hat{\mathbf{n}}\cdot\mathbf{E}_{a,+A}=0$ on $\mathcal{S}$, so then the surface term in Eq. (16) vanishes. In general, though, we will be able to ignore this surface term when computing its expectation value, since

$$\left\langle\oint_{\mathcal{S}}d^2x\,\hat{\mathbf{n}}\cdot(\phi_{-A}\mathbf{E}_{+A})\right\rangle\approx\oint_{\mathcal{S}}d^2x\,\hat{\mathbf{n}}\cdot\left[\frac{-e}{|\mathbf{x}|}\frac{e}{c}\frac{\mathbf{x}\times\left[\mathbf{x}\times\frac{\langle\ddot{\mathbf{z}}_{+A}(t_r)\rangle}{c}\right]}{|\mathbf{x}|^3}\right], \tag{18}$$

and $\langle\ddot{\mathbf{z}}\rangle$ must equal zero for an oscillating bound charge.

Also, the following term in Eq. (10):

$$\frac{1}{8\pi}\int_{\mathcal{V}}d^3x\,\mathbf{E}_{-A}\cdot\mathbf{E}_{-A}=\tfrac{1}{2}\int_{-A\text{ sphere}}d^3x\,\phi_{-A}\rho_{-A}-\frac{1}{8\pi}\oint_{\mathcal{S}}d^2x\,\hat{\mathbf{n}}\cdot(\phi_{-A}\mathbf{E}_{-A}), \tag{19}$$

will be nearly the same value at times $t_{\rm I}$ and $t_{\rm II}$, with the difference being the above surface term that vanishes for $\mathcal{S}$ far removed. Hence we can ignore this term in computing $\Delta\mathcal{U}_{\rm int}$.

Similarly, let us assume that the surface $\mathcal{S}$ of $\mathcal{V}$ is far enough from all the electric dipole oscillators that

$$\int_{\mathcal{V}}d^3x\,u_{+A(-3}$$

makes a negligible contribution when computing $\Delta\mathcal{U}_{\rm int}$.

Combining the above, we then obtain that the left-hand side (lhs) of Eq. (1) is given by

$$\Delta\mathcal{U}_{\rm int}=\left[\sum_{A=1}^{N}\left[m\gamma_{+A}c^2-\frac{1}{3}\frac{e^2}{c}\frac{d}{dt}(\gamma_{+A})^2\right]+\sum_{A=1}^{N}\left[\frac{m\omega_0^2}{2}(\delta\mathbf{z}_A)^2-\frac{1}{4\pi}\oint_{\mathcal{S}}d^2x\,\hat{\mathbf{n}}\cdot(\phi_{-A}\mathbf{E}_{+A})\right]+\int_{\mathcal{V}}d^3x\,u'_{\rm EM}\right]\Bigg|_{t_{\rm I}}^{t_{\rm II}}, \tag{20}$$

where

$$u'_{\rm EM}=\frac{1}{8\pi}\left[\sum_{A=1}^{N}[(\mathbf{E}_{\text{osc-a},A})^2+(\mathbf{B}_{\text{osc-a},A})^2]+\sum_{\substack{A,B=1\\A\neq B}}^{N}[\mathbf{E}_{\text{osc},A}\cdot\mathbf{E}_{\text{osc},B}+\mathbf{B}_{\text{osc},A}\cdot\mathbf{B}_{\text{osc},B}]\right.$$
$$\left.+2\sum_{A=1}^{N}[\mathbf{E}_{\text{osc},A}\cdot\mathbf{E}_{\rm in}+\mathbf{B}_{\text{osc},A}\cdot\mathbf{B}_{\rm in}]+[(\mathbf{E}_{\rm in})^2+(\mathbf{B}_{\rm in})^2]\right]. \tag{21}$$

Here, $\mathbf{E}_{\text{osc},A}\equiv\mathbf{E}_{+A}+\mathbf{E}_{-A}$, so $\mathbf{E}_{\text{osc-}a,A}\equiv\mathbf{E}_{a,+A}$, and similarly for the **B** fields. When we compute the expectation value of Eq. (20) in Sec. IV, we will obtain that the Schott energy term and the surface term in Eq. (20) will drop out. Also, for the isothermal process considered in the present paper, the $[(\mathbf{E}_{\rm in})^2+(\mathbf{B}_{\rm in})^2]$ term in Eq. (21) will not contribute to $\langle\Delta\mathcal{U}_{\rm int}\rangle$, due to Eq. (12). (However, these terms will be important in the planned work of Ref. 43, which will consider reversible processes where the temperature does change.)

Now $\Delta\mathcal{U}_{\rm int}$ is in a convenient form to begin calculating its expectation value.

III. STEADY-STATE SOLUTION FOR OSCILLATORS

At time $t_{\rm I}$, the electric dipole oscillators are in steady-state equilibrium with the incident radiation. Let us again constrain our definition of $t_{\rm II}$ slightly and require that $t_{\rm II}$ be sufficiently greater than the time at which all the electric dipoles have stopped being moved, so that we may also treat the electric dipoles as being in steady-state equilibrium with the incident radiation at time $t_{\rm II}$.

Let us confine our attention to incident radiation fields that are described by a Gaussian process and that satisfy conditions of homogeneity and isotropy in space. As discussed in Ref. 1 (see also Ref. 3), the free incident fields in a large cubic region of space with side length L can be expressed by

$$\mathbf{E}_{\rm in}(\mathbf{x},t)=\sum_{\mathbf{n}}\sum_{\lambda=1,2}\left[\frac{2\pi}{L}\right]^{3/2} h_{\rm in}(\omega)\hat{\epsilon}_{\mathbf{n},\lambda} \times\cos(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_{\mathbf{n},\lambda}) , \quad (22)$$

$$\mathbf{B}_{\rm in}(\mathbf{x},t)=\sum_{\mathbf{n}}\sum_{\lambda=1,2}\left[\frac{2\pi}{L}\right]^{3/2} h_{\rm in}(\omega)(\hat{\mathbf{k}}\times\hat{\epsilon}_{\mathbf{n},\lambda}) \times\cos(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_{\mathbf{n},\lambda}) . \quad (23)$$

Here, periodic conditions have been applied,

$$\mathbf{k}=\frac{2\pi}{L}\mathbf{n},\quad n_x,n_y,n_z=\{0,\pm1,\pm2,\ldots\} , \quad (24)$$

and

$$\hat{\epsilon}_{\mathbf{n},\lambda}\cdot\hat{\epsilon}_{\mathbf{n}\lambda'}=\delta_{\lambda,\lambda'} , \quad (25)$$

$$\mathbf{k}\cdot\hat{\epsilon}_{\mathbf{n},\lambda}=0 . \quad (26)$$

Here, $\omega=c|\mathbf{k}|$. The above representation of the fields conveniently attributes the stochastic properties of the incident fields to the phase angle $\theta_{\mathbf{n},\lambda}$, which is taken to be a random variable with a uniform probability density distribution between 0 and 2π. For each $\mathbf{n}$ and λ, $\theta_{\mathbf{n},\lambda}$ is assumed to be independently distributed.

For the RJ and ZPP situations, respectively, $h_{\rm in}$ is given by

$$(h_{\rm RJ})^2\equiv\frac{k_BT}{\pi^2} , \quad (27)$$

$$(h_{\rm ZPP})^2\equiv\left[\frac{\hbar\omega}{2\pi^2}\right]\coth\left[\frac{\hbar\omega}{2k_BT}\right] . \quad (28)$$

When $T\to0$, $h_{\rm ZPP}$ reduces to the zero-point case of

$$(h_{\rm ZP})^2\equiv\frac{\hbar\omega}{2\pi^2} . \quad (29)$$

As shown in Ref. 37, when $L\to\infty$, so as to enclose all of space, then we can let

$$a_\lambda(\mathbf{k})\equiv\left[\frac{L}{2\pi}\right]^{3/2} e^{i\theta_{\mathbf{k},\lambda}} \quad (30)$$

and obtain

$$\mathbf{E}_{\rm in}(\mathbf{x},t)=\sum_{\lambda=1,2}\int d^3k\, h_{\rm in}(\omega)\hat{\epsilon}_\lambda(\mathbf{k})\tfrac{1}{2}[a_\lambda(\mathbf{k})e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}+a_\lambda^*(\mathbf{k})e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}] , \quad (31)$$

$$\mathbf{B}_{\rm in}(\mathbf{x},t)=\sum_{\lambda=1,2}\int d^3k\, h_{\rm in}(\omega)[\hat{\mathbf{k}}\times\hat{\epsilon}_\lambda(\mathbf{k})]\tfrac{1}{2}[a_\lambda(\mathbf{k})e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}+a_\lambda^*(\mathbf{k})e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}] . \quad (32)$$

Here, $\hat{\epsilon}_\lambda(\mathbf{k})$ is simply a relabeling of $\hat{\epsilon}_{\mathbf{n},\lambda}$ via the transformation of $\mathbf{n}\to\mathbf{k}$ in Eq. (24). Also, one can show that[37]

$$\langle a_{\lambda_1}(\mathbf{k}_1)a_{\lambda_2}^*(\mathbf{k}_2)\rangle=\delta^3(\mathbf{k}_1-\mathbf{k}_2)\delta_{\lambda_1,\lambda_2} , \quad (33)$$

$$\langle a_{\lambda_1}(\mathbf{k}_1)a_{\lambda_2}(\mathbf{k}_2)\rangle=\langle a_{\lambda_1}^*(\mathbf{k}_1)a_{\lambda_2}^*(\mathbf{k}_2)\rangle=0 . \quad (34)$$

In this paper we will find restrictions imposed on the function $h_{\rm in}^2$ in order for the incident radiation to qualify as thermal radiation. Since $[h_{\rm in}(\omega)]^2$ can be shown to satisfy

$$\rho_{\rm in}(\omega)=\frac{\omega^2}{c^3}[h_{\rm in}(\omega)]^2 , \quad (35)$$

where $\rho_{\rm in}(\omega)$ is the incident spectral electromagnetic energy density

$$\frac{1}{8\pi}\langle(\mathbf{E}_{\rm in})^2+(\mathbf{B}_{\rm in})^2\rangle=\int_0^\infty d\omega\,\rho_{\rm in}(\omega) , \quad (36)$$

then our restrictions on $h_{\rm in}^2$ immediately apply to $\rho_{\rm in}$. [From Eqs. (27)–(29) and (36), we see that $U_{\rm EM,in}$ in Eq. (11) is indeed singular for the above spectra; however, this singularity is eliminated in our calculations via Eq. (12).]

We now need to determine the effect of the incident fields on the behavior of the fluctuating electric dipoles, which we are treating here as though they each consist of a small $-e$ uniform sphere of charge with a $+e$ charged point particle making small oscillations inside. As usual in SED,[1–3] let us approximate the equation of motion of the $(+e)$-A oscillating particle by

$$m\delta\ddot{\mathbf{z}}_A=-m\omega_0^2\delta\mathbf{z}_A+m\Gamma\delta\dddot{\mathbf{z}} +e\left[\mathbf{E}_{\rm in}(\mathbf{Z}_A,t)+\sum_{\substack{B=1\\(B\neq A)}}^{N}\mathbf{E}_{{\rm osc},B}(\mathbf{Z}_A,t)\right] , \quad (37)$$

where $\Gamma\equiv\frac{2}{3}(e^2/mc^3)$. Here, (1) the particle was treated nonrelativistically, (2) the Abraham-Lorentz approximation was used for the radiation reaction force, (3) the $\dot{\mathbf{z}}/c\times\mathbf{B}$ part of the Lorentz force was ignored, and (4) the dipole approximation was used in replacing the $\mathbf{Z}_A+\delta\mathbf{z}_A$ spatial argument in the $\mathbf{E}$ fields by $\mathbf{Z}_A$.

We will treat $\mathbf{E}_{{\rm osc},B}=\mathbf{E}_{+B}+\mathbf{E}_{-B}$ and $\mathbf{B}_{{\rm osc},B}=\mathbf{B}_{+B}+\mathbf{B}_{-B}$ as being due to a time-varying electric dipole, which for our model equals $\mathbf{p}_B(t)\equiv+e\delta\mathbf{z}_B(t)$. Consequently, let us replace $\mathbf{E}_{{\rm osc},B}$ and $\mathbf{B}_{{\rm osc},B}$ in all subsequent calculations by the electric dipole fields $\mathbf{E}_{\mathcal{D},B}$ and $\mathbf{B}_{\mathcal{D},B}$, where

$$[\mathbf{E}_{\mathcal{D},B}(\mathbf{x},t)]_i=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\left[\sum_{j=1}^{3}\eta_{ij}^{\mathcal{D}}(\mathbf{x}-\mathbf{Z}_B,\omega)\tilde{p}_{Bj}(\omega)\right], \tag{38}$$

$$[\mathbf{B}_{\mathcal{D},B}(\mathbf{x},t)]_i=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\left[\sum_{j=1}^{3}\rho_{ij}^{\mathcal{D}}(\mathbf{x}-\mathbf{Z}_B,\omega)\tilde{p}_{Bj}(\omega)\right], \tag{39}$$

and

$$\eta_{ij}^{\mathcal{D}}(\mathbf{x}-\mathbf{Z}_B,\omega)=(\nabla_i\nabla_j+k^2\delta_{ij})\left[\frac{e^{ik|\mathbf{x}-\mathbf{Z}_B|}}{|\mathbf{x}-\mathbf{Z}_B|}\right], \tag{40}$$

$$\rho_{ij}^{\mathcal{D}}(\mathbf{x}-\mathbf{Z}_B,\omega)=ik\sum_{l=1}^{3}\epsilon_{ijl}\nabla_l\left[\frac{e^{ik|\mathbf{x}-\mathbf{Z}_B|}}{|\mathbf{x}-\mathbf{Z}_B|}\right]. \tag{41}$$

Here, $\nabla_i=\partial/\partial_i$, and $\tilde{\mathbf{p}}_B(\omega)$ is the Fourier transform of $\mathbf{p}_B(t)$. For $|\mathbf{x}-\mathbf{Z}_B|\neq 0$, Eqs. (40) and (41) are equivalent to the expressions of Eqs. (16) and (22) in Ref. 31; however, the above expressions also contain the correct singular properties of the dipole fields in the limit of $|\mathbf{x}-\mathbf{Z}_B|\to 0$.[51]

Equation (37) is a linear stochastic differential equation in the coordinates $\delta\mathbf{z}_A$, $A=1,2,\ldots,N$, with $e\mathbf{E}_{\text{in}}$ acting as the driving term. [Alternatively, Eq. (37) can be written in terms of the electric dipoles $\mathbf{p}_A=e\delta\mathbf{z}_A$.] We will assume that the steady-state behavior is given by the usual Fourier transform solution. Let

$$\delta\mathbf{z}_A(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t}\delta\tilde{\mathbf{z}}_A(\omega), \tag{42}$$

$$E_{\text{in}}(\mathbf{x},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t}\tilde{\mathbf{E}}_{\text{in}}(\mathbf{x},\omega). \tag{43}$$

Hence

$$\delta\tilde{z}_{Ai}(\omega)=\frac{e}{m}\sum_{B=1}^{N}\sum_{j=1}^{3}[\underline{M}^{-1}(\omega)]_{Ai;Bj}\frac{E_{\text{in}\,j}(\mathbf{Z}_B,\omega)}{C(\omega)}, \tag{44}$$

where

$$C(\omega)=-\omega^2+\omega_0^2-i\Gamma\omega^3, \tag{45}$$

$$M_{Ai;Bj}(\omega)=\delta_{AB}\delta_{ij}-(1-\delta_{AB})\frac{e^2}{m}\frac{\eta_{ij}^{\mathcal{D}}(\mathbf{Z}_A-\mathbf{Z}_B,\omega)}{C(\omega)}. \tag{46}$$

IV. EXPECTATION VALUE OF $\Delta\mathcal{U}_{\text{int}}$

Let us now compute the expectation value of Eq. (20), assuming that Eqs. (42) and (44) yield the correct steady-state behavior for times $t\leq t_{\text{I}}$ and $t_{\text{II}}\leq t$. The first two terms in Eq. (20) can then be simplified immediately.

Since we are evaluating Eq. (20) at times where we assume the oscillators follow a stationary stochastic process in time, we may drop the Schott energy term and the surface term in Eq. (20). For example, at t_{I} and t_{II},

$$\left\langle\frac{d}{dt}[(\gamma_{+A})^2]\right\rangle=\frac{d}{dt}\langle(\gamma_{+A})^2\rangle=0. \tag{47}$$

Here, the expectation value is computed by taking an ensemble average over identical oscillator systems, but with different realizations for the θ's in the incident fields in Eqs. (22) and (23). Likewise, the expectation value of the surface term in Eq. (20) drops out due to Eq. (18).

In keeping with our nonrelativistic equation of motion for the $+e$ oscillating charges [i.e., Eq. (37)], where we have assumed that $|\delta\dot{\mathbf{z}}_A|/c\ll 1$, then

$$(m\gamma_{+A}c^2)|_{t_{\text{I}}}^{t_{\text{II}}}\approx\left[\frac{m}{2}|\delta\dot{\mathbf{z}}_A|^2\right]\Bigg|_{t_{\text{I}}}^{t_{\text{II}}}. \tag{48}$$

The expectation values of the energy terms in Eq. (20) are listed below. Evaluating the potential and kinetic energy terms below of U_{PE} and U_{KE} follows the same procedure as in Ref. 31. Also, evaluating the electromagnetic energy term $U_{\text{EM},\mathcal{D}a\text{-}\mathcal{D}a}$ is not too difficult. However, obtaining the two remaining electromagnetic energy terms is considerably more involved. Due to the length of the calculations, this work has been placed in Appendix A and reserved in AIP's Physics Auxiliary Publication Service (PAPS).[52] The final results are

$$U_{\text{PE}}\equiv\sum_{A=1}^{N}\left\langle\frac{m\omega_0^2}{2}|\delta\mathbf{z}_A|^2\right\rangle=\pi\int_0^{\infty}d\omega\, h_{\text{in}}^2\,\text{Im}\left[\frac{1}{C}\sum_{A,B=1}^{N}\sum_{i,j=1}^{3}(\underline{M}^{-1})_{Ai;Bj}\left[\delta_{AB}\delta_{ij}\frac{\omega_0^2}{\omega}\right]\right], \tag{49}$$

$$U_{\text{KE}}\equiv\sum_{A=1}^{N}\left\langle\frac{m}{2}|\delta\dot{\mathbf{z}}_A|^2\right\rangle=\pi\int_0^{\infty}d\omega\, h_{\text{in}}^2\,\text{Im}\left[\frac{1}{C}\sum_{A,B=1}^{N}\sum_{i,j=1}^{3}(\underline{M}^{-1})_{Ai;Bj}(\delta_{AB}\delta_{ij}\omega)\right], \tag{50}$$

$$\begin{aligned}U_{\text{EM},\mathcal{D}a\text{-}\mathcal{D}a}&\equiv\sum_{A=1}^{N}\frac{1}{8\pi}\int_{\mathcal{V}}d^3x\,\langle\mathbf{E}_{\mathcal{D}a,A}\cdot\mathbf{E}_{\mathcal{D}a,A}+\mathbf{B}_{\mathcal{D}a,A}\cdot\mathbf{B}_{\mathcal{D}a,A}\rangle\\&=\pi\int_0^{\infty}d\omega\, h_{\text{in}}^2\,\text{Im}\left[\frac{1}{C}\sum_{A,B=1}^{N}\sum_{i,j=1}^{3}(\underline{M}^{-1})_{Ai;Bj}\left[\delta_{AB}\frac{e^2}{m\omega}\mathcal{S}_{Aij}\right]\right],\end{aligned} \tag{51}$$

$$U_{\mathrm{EM},\mathcal{D}\text{-}\mathcal{D}} \equiv \sum_{\substack{A,B=1\\A\neq B}}^{N} \frac{1}{8\pi}\int_{\mathcal{V}} d^3x\,\langle \mathbf{E}_{\mathcal{D},A}\cdot\mathbf{E}_{\mathcal{D},B}+\mathbf{B}_{\mathcal{D},A}\cdot\mathbf{B}_{\mathcal{D},B}\rangle$$

$$=\pi\int_0^\infty d\omega\, h_{\mathrm{in}}^2 \,\mathrm{Im}\left[\frac{1}{C}\sum_{A,B=1}^{N}\sum_{i,j=1}^{3}(\underline{M}^{-1})_{Ai;Bj}\left[+(1-\delta_{AB})\frac{e^2}{m\omega}\mathrm{Re}(E_{Ai;Bj})\right]\right], \tag{52}$$

$$U_{\mathrm{EM},\mathcal{D}\text{-in}} \equiv \sum_{A=1}^{N} \frac{2}{8\pi}\int_{\mathcal{V}} d^3x\,\langle \mathbf{E}_{\mathcal{D},A}\cdot\mathbf{E}_{\mathrm{in}}+\mathbf{B}_{\mathcal{D},A}\cdot\mathbf{B}_{\mathrm{in}}\rangle$$

$$=\pi\int_0^\infty d\omega\, h_{\mathrm{in}}^2 \,\mathrm{Im}\left\{\frac{1}{C}\sum_{A,B=1}^{N}\sum_{i,j=1}^{3}(\underline{M}^{-1})_{Ai;Bj}\left[\delta_{AB}\frac{e^2}{m\omega}(-\mathcal{S}_{Aij}+i\delta_{ij}\tfrac{4}{3}k^3)\right.\right.$$

$$\left.\left.+\frac{e^2}{m\omega}(1-\delta_{AB})\left[-\mathrm{Re}(E_{Ai;Bj})-\eta_{ij}^{\mathcal{D}}(\mathbf{Z}_A-\mathbf{Z}_B,\omega)+\omega\frac{d}{d\omega}\eta_{ij}^{\mathcal{D}}(\mathbf{Z}_A-\mathbf{Z}_B,\omega)\right]\right]\right\}. \tag{53}$$

The energy term in Eq. (53) was obtained by assuming that the surface $\mathcal{S}$ of the volume $\mathcal{V}$ enclosing all the particles was far removed from any of the particles.

Two quantities appear in Eqs. (51)–(53) that have not been discussed yet: $\mathrm{Re}(E_{Ai;Bj})$ and $\mathcal{S}_{Aij}$. Both terms depend upon the precise shape and size of $\mathcal{V}$. However, these quantities drop out completely upon adding Eqs. (51)–(53). Hence we obtain the interesting result that the sum of the electromagnetic energies in Eqs. (51)–(53) is essentially independent of $\mathcal{V}$, at least for $\mathcal{V}$ sufficiently large.

Appendix A analyzes the fairly complicated quantity $E_{Ai;Bj}$ in some detail, so it need not be mentioned further here. However, the less complicated quantity of $\mathcal{S}_{Aij}$ deserves a few remarks, as it has an interesting property associated with it. Here,

$$\mathcal{S}_{Aij}\equiv\frac{k^4}{2\pi}\int_{\mathcal{V}} d^3x\frac{\left[\delta_{ij}-\dfrac{(\mathbf{x}-\mathbf{Z}_A)_i(\mathbf{x}-\mathbf{Z}_A)_j}{|\mathbf{x}-\mathbf{Z}_A|^2}\right]}{|\mathbf{x}-\mathbf{Z}_A|^2}. \tag{54}$$

In the limit where $\mathcal{V}$ becomes infinitely large, then $\mathcal{S}_{Aii}\rightarrow\infty$, so that both $U_{\mathrm{EM},\mathcal{D}a\text{-}\mathcal{D}a}$ and $U_{\mathrm{EM},\mathcal{D}\text{-in}}$ become singular. This result should not be too surprising for $U_{\mathrm{EM},\mathcal{D}a\text{-}\mathcal{D}a}$, which is probably a more familiar quantity than $U_{\mathrm{EM},\mathcal{D}\text{-in}}$. Here, $U_{\mathrm{EM},\mathcal{D}\text{-in}}$ equals the electromagnetic energy cross terms of (1) the oscillator fields and (2) the incident fields driving the oscillatory motion, while $U_{\mathrm{EM},\mathcal{D}a\text{-}\mathcal{D}a}$ represents the electromagnetic energy radiated by the N oscillators into $\mathcal{V}$, if we treated each oscillator as being a separate energy source and ignored the interference between fields. The electromagnetic energy radiated by a single fluctuating dipole will fill all of space if it is kept in a steady-state motion for all of time. The energy density of this radiated energy falls off as $1/R^2$ at a distance R from an electric dipole. Hence, when integrated over all of space, a singular result is obtained, which is precisely what we should expect for an oscillator that radiates out energy for all of time. Of course, for a finite volume $\mathcal{V}$, $\mathcal{S}_{Aij}$ is indeed finite. For example, when $\mathcal{V}$ is approximately a sphere of radius R, with oscillator A at the center, then we explicitly have that $\mathcal{S}_{Aij}=\frac{4}{3}\delta_{ij}k^4R$. Consequently, $U_{\mathrm{EM},\mathcal{D}a\text{-}\mathcal{D}a}$ in Eq. (51) is then directly proportional to the radius of $\mathcal{V}$.

However, what is satisfying about the above results in Eqs. (51)–(53) is that the sum of these electromagnetic energy terms is finite, even when $\mathcal{V}\rightarrow\infty$, so that the singularities in the individual terms precisely cancel. This result should be expected, since the sum of the energy terms in Eqs. (49)–(53) must be equal to (1) the work required by the incident radiation fields to change $N+e$ charged particles from being at rest to following the steady-state oscillatory motion of Eqs. (42) and (44), minus (2) the electromagnetic energy that will flow out of $\mathcal{V}$ during this process. We of course expect this work to be finite, as well as the net energy that flows out of $\mathcal{V}$.

Summing the electromagnetic energy terms in Eqs. (51)–(53) gives, with $\Gamma=\frac{2}{3}(e^2/mc^3)$,

$$U_{\mathrm{EM|in}} \equiv U_{\mathrm{EM},\mathcal{D}a\text{-}\mathcal{D}a} + U_{\mathrm{EM},\mathcal{D}-\mathcal{D}} + U_{\mathrm{EM},\mathcal{D}\text{-in}}$$
$$= \pi \int_0^\infty d\omega\, h_{\mathrm{in}}^2 \,\mathrm{Im}\left\{ \frac{1}{C} \sum_{A,B=1}^{N} \sum_{i,j=1}^{3} (\underline{M}^{-1})_{Ai;Bj} \left[\delta_{AB}\delta_{ij} 2i\Gamma\omega^2 + (1-\delta_{AB})\frac{e^2}{m\omega} \left[-\eta_{ij}^{\mathcal{D}}(\mathbf{Z}_A - \mathbf{Z}_B,\omega) + \omega \frac{d\eta_{ij}^{\mathcal{D}}(\mathbf{Z}_A - \mathbf{Z}_B,\omega)}{d\omega} \right] \right] \right\} . \quad (55)$$

The subscript EM|in on $U_{\mathrm{EM|in}}$ is meant to indicate that $U_{\mathrm{EM,in}}$ in Eq. (11) has *not* been included, since it drops out in $\langle \Delta \mathcal{U}_{\mathrm{int}} \rangle$. However, this term will need to be included for the processes considered in the planned, separate paper (Ref. 43) mentioned earlier, which analyzes situations involving a nonzero change in $U_{\mathrm{EM,in}}$.

Adding Eqs. (49), (50), and (55), and using Eqs. (45) and (46), yields

$$U_{\mathrm{KE}} + U_{\mathrm{PE}} + U_{\mathrm{EM|in}} = \pi \int_0^\infty d\omega\, h_{\mathrm{in}}^2 \,\mathrm{Im}\left\{ \frac{1}{C} \sum_{A,B=1}^{N} \sum_{i,j=1}^{3} (\underline{M}^{-1})_{Ai;Bj} \left[-C\frac{d}{d\omega} M_{Ai;Bj} + M_{Ai;Bj}\left[\frac{1}{\omega}C - \frac{dC}{d\omega} \right] \right] \right\} . \quad (56)$$

This expression can be simplified by using the matrix identity

$$(\underline{M}^{-1})_{Ai;Bj} = \frac{\partial}{\partial M_{Bj;Ai}} \ln \det(\underline{M}) , \quad (57)$$

plus the symmetry property $M_{Ai;Bj} = M_{Bj;Ai}$, to obtain

$$U_{\mathrm{KE}} + U_{\mathrm{PE}} + U_{\mathrm{EM|in}} = \pi \int_0^\infty d\omega\, h_{\mathrm{in}}^2 \,\mathrm{Im} \left[-\frac{d}{d\omega} \ln \det(\underline{M}) + \frac{3N}{\omega} - 3N \frac{1}{C}\frac{dC}{d\omega} \right]$$
$$= -\pi \int_0^\infty d\omega (h_{\mathrm{in}})^2 \frac{d}{d\omega} \mathrm{Im}[\ln \det(\underline{M}) + 3N \ln C] . \quad (58)$$

Alternatively, we can rewrite the term above in parentheses concisely as $\ln[C^{3N}\det(\underline{M})]$. However, the form of Eq. (58) is helpful since it explicitly shows the simple additive parts of the net energy: namely, (1) the part associated with $3N \ln C$, which is due to N electric dipoles fluctuating independently in three-dimensional space, plus (2) the term $\ln[\det(\underline{M})]$, which contains the entire electromagnetic interaction energy between the fluctuating electric dipoles. To see that this last statement is true, note that $\ln[\det(\underline{M})]$ is the only part of Eq. (58) that depends on the positions $\mathbf{Z}_A$ of the electric dipoles, via $M_{Ai;Bj}$ in Eq. (46). Moreover, as $|\mathbf{Z}_B - \mathbf{Z}_A| \to \infty$ for all $A \neq B$, so that the dipoles become infinitely far apart from each other, then $M_{Ai;Bj} \to \delta_{ij}$, so $\ln[\det(\underline{M})] \to 0$, via Eqs. (40) and (46).

The expectation value of Eq. (20) is then given by

$$\Delta U_{\mathrm{int}} = -\pi \int_0^\infty d\omega (h_{\mathrm{in}})^2 \frac{d}{d\omega} \mathrm{Im}[\ln \det(\underline{M})] \Big|_{\mathrm{I}}^{\mathrm{II}} , \quad (59)$$

where $|_{\mathrm{I}}^{\mathrm{II}}$ means to take the difference in the quantity indicated above, when evaluated at the electric dipole final positions $\mathbf{Z}_{A,\mathrm{II}}$, $A = 1, \ldots, N$, and the initial positions $\mathbf{Z}_{A,\mathrm{I}}$.

V. EXPECTATION VALUE OF W DURING A QUASISTATIC PROCESS

Since we have found the expectation value of $\Delta\mathcal{U}_{\mathrm{int}}$ in Eq. (1), we now need to turn to finding $\langle \mathcal{W} \rangle$, in order to be able to deduce $\langle \mathcal{Q} \rangle$ and the change in entropy. Fortunately, most of the calculation for $\langle \mathcal{W} \rangle$ has essentially already been done for us in Sec. III of Ref. 31. There, the expectation value was found for

$$\mathbf{F}_{\mathrm{Lor},A}(t) = (\mathbf{p}_A \cdot \nabla) \left[\mathbf{E}_{\mathrm{in}}(\mathbf{Z}_A,t) + \sum_{\substack{B=1 \\ B\neq A}}^{N} \mathbf{E}_{\mathcal{D},B}(\mathbf{Z}_A,t) \right] + \frac{1}{c}\dot{\mathbf{p}}_A \times \left[\mathbf{B}_{\mathrm{in}}(\mathbf{Z}_A,t) + \sum_{\substack{B=1 \\ B\neq A}}^{N} \mathbf{B}_{\mathcal{D},B}(\mathbf{Z}_A,t) \right] , \quad (60)$$

which represents the Lorentz force acting on electric dipole A due to the incident radiation and due to the electromagnetic force from the other $N-1$ electric dipoles. This calculation was carried out for the case where all N simple harmonic electric dipole oscillators were held fixed at positions $\mathbf{Z}_A$, $A = 1, 2, \ldots, N$. [In terms of our simple oscillator dipole model described in Sec. II, the $(+e)$ spheres should be pictured as being held fixed while the $(-e)$ point charges are free to oscillate.]

From Eqs. (78) and (79) in Ref. 31,

$$\langle \mathbf{F}_{\mathrm{Lor},A}(t) \rangle = -\nabla_{\mathbf{Z}_A} U_0 , \quad (61)$$

$$U_0 = \pi \int_0^\infty d\omega \frac{[h_{\mathrm{in}}(\omega)]^2}{\omega} \mathrm{Im}\{\ln \det[\underline{M}(\omega)]\} , \quad (62)$$

where $\nabla_{\mathbf{Z}_A}$ in Eq. (61) means to differentiate with respect to $\mathbf{Z}_A$. Thus $U_0 = U_0(\mathbf{Z}_1, \ldots, \mathbf{Z}_N)$ is a function of the positions $\mathbf{Z}_A$ of the dipole particles via $M_{Ai;Bj}$ in Eq. (46), just as was the case for the interaction energy part in Eqs. (58) and (59), which also arose from the $\ln[\det(\underline{M})]$ term.

External forces must be applied to each of the N electric dipoles in order to hold them at fixed positions. Let $\mathbf{F}_{\mathrm{ext},A}(t)$ be the external force acting on electric dipole $\mathbf{p}_A(t)$ that prevents the electromagnetic force in Eq. (60) from displacing the center of the electric dipole. Clearly we expect that

$$\langle \mathbf{F}_{\mathrm{ext},A}(t) = -\langle \mathbf{F}_{\mathrm{Lor},A}(t) \rangle \ . \tag{63}$$

In evaluating the expectation value of the work done by applied external forces in displacing the electric dipoles, our analysis will be considerably simplified due to our restriction that the electric dipoles are moved at an infinitesimally slow rate. The "fast" time fluctuations of the electric dipoles $\mathbf{p}_A$, $A = 1, \ldots, N$, are then not coupled to the very slow motion of the positions $\mathbf{Z}_A(t)$ of the dipole particles. Consequently, we can make the following quasistatic approximation:

$$\begin{aligned} W &= +\left\langle \int_{t_{\mathrm{I}}}^{t_{\mathrm{II}}} dt \sum_{A=1}^{N} \dot{\mathbf{Z}}_A(t) \cdot \mathbf{F}_{\mathrm{ext},A}(t) \right\rangle \\ &\approx \int_{t_{\mathrm{I}}}^{t_{\mathrm{II}}} dt \sum_{A=1}^{N} \dot{\mathbf{Z}}_A(t) \cdot \langle \mathbf{F}_{\mathrm{ext},A}(t) \rangle \\ &= \int_{t_{\mathrm{I}}}^{t_{\mathrm{II}}} dt \sum_{A=1}^{N} \dot{\mathbf{Z}}_A(t) \cdot \nabla_{\mathbf{Z}_A} U_0(\mathbf{Z}_1(t), \mathbf{Z}_2(t), \ldots, \mathbf{Z}_N(t)) \\ &= U_0(\mathbf{Z}_1(t), \mathbf{Z}_2(t), \ldots, \mathbf{Z}_N(t)) \Big|_{t_{\mathrm{I}}}^{t_{\mathrm{II}}} \ . \end{aligned} \tag{64}$$

Thus the work done by the external forces only depends on the initial and final positions of the dipole particles when they are displaced at an infinitesimally slow rate from one point to another, just as was the case for ΔU_{int} in Eq. (59). Consequently, our quasistatic operation of moving the electric dipoles is a reversible operation in the thermodynamic sense, since all of the work done in displacing the N dipole particles can be returned, or, rather, this work is equal to the negative of the work done during the reverse displacement (see p. 163 of Ref. 6).

VI. HEAT FLOW

A. Calculation of Q and demonstration of the special status of ZP radiation

From Eqs. (1), (59), and (64), we have that

$$\begin{aligned} Q &= \Delta U_{\mathrm{int}} - W \\ &= -\pi \int_0^\infty d\omega \left[(h_{\mathrm{in}})^2 \frac{d}{d\omega} [\mathrm{Im} \ln \det(\underline{M})] + \frac{(h_{\mathrm{in}})^2}{\omega} \mathrm{Im} \ln \det(\underline{M}) \right] \Bigg|_{\mathbf{Z}_{\mathrm{I}}}^{\mathbf{Z}_{\mathrm{II}}} \ . \end{aligned} \tag{65}$$

We are now in a position to prove that classical electromagnetic ZP radiation satisfies a very special property: namely, that $Q = 0$ for a quasistatic displacement of the dipole particles. To show this result, let us consider all incident radiation spectra that satisfy

$$[h_{\mathrm{in}}(\omega)]^2 \mathrm{Im} \ln \det(\underline{M}) \Big|_{\omega=0}^{\omega=\infty} = 0 \ . \tag{66}$$

As is shown in Appendix B,[52] ZP radiation satisfies this property, as does ZPP and RJ radiation. Since $\mathrm{Im}[\ln \det(\underline{M})]$ does indeed equal zero at $\omega = 0$ and $\omega = \infty$, then our natural expectation should be that Eq. (66) holds for any other reasonable candidate of h_{in} as a thermal equilibrium spectrum. Indeed, if other factors could be taken into account for the oscillators, such as relativistic effects for the high-frequency motion of the oscillating particles, and particle instability at high energies (particle annihilation and creation), then it seems reasonable to expect that such cutoff mechanisms will provide even stronger reasons for being able to ignore the high-frequency component contributions of the above product. In proving Eq. (66) for ZP and ZPP radiation, the $\omega = \infty$ case was the part that needed to be checked carefully, since h_{ZP}^2 and h_{ZPP}^2 both go to ∞ as $\omega \rightarrow \infty$. Nevertheless, Eq. (66) was indeed found to hold for these two cases, even without taking the effects just mentioned into account. (The planned work of Ref. 43 will prove more generally that Eq. (66) *must* hold true for classical electromagnetic thermal radiation.)

Integrating the first term in Eq. (65) by parts, and using Eq. (66), yields

$$\begin{aligned} Q = -\pi \int_0^\infty d\omega \left[-\frac{d h_{\mathrm{in}}^2}{d\omega} + \frac{h_{\mathrm{in}}^2}{\omega} \right] \\ \times [\mathrm{Im} \ln \det(\underline{M})|_{\mathbf{Z}_{\mathrm{II}}} - \mathrm{Im} \ln \det(\underline{M})|_{\mathbf{Z}_{\mathrm{I}}}] \ . \end{aligned} \tag{67}$$

Hence, when

$$[h_{\mathrm{in}}(\omega)]^2 = \kappa\omega \ , \tag{68}$$

or $\rho_{\mathrm{in}} = \kappa\omega^3/c^3$ from Eq. (35), then we have the very special result that $Q = 0$. Classical electromagnetic ZP radiation corresponds precisely to this spectral form, where κ is equal to $\hbar/(2\pi^2)$.

B. Discussion on heat flow

When the incident radiation corresponds to thermal radiation, then it can be characterized by a single temperature T. The dipole particles are treated here as a small system that interacts with the rest of the universe, which we have in turn approximated as being a heat reservoir at a temperature T. Slowly displacing the fluctuating dipole particles allows heat to flow in the form of electromagnetic radiation. This heat flows either into or out of the environment surrounding the particles, but the temperature remains fixed because of the heat reservoir approximation. Thus the process we have considered is a *reversible isothermal* one.

For the two candidates that are usually considered for constituting classical thermal radiation, namely, RJ and ZPP radiation [Eqs. (27) and (28)], Q in Eq. (67) does not in general equal zero. This result is certainly what we expect for nonzero temperature situations, since heat typically flows between two interacting systems for a reversible isothermal operation. However, when $T=0$ for two interacting systems, then no thermal energy is available to be exchanged between the two systems, via the definition of the temperature of absolute zero.[6]

We have just seen the interesting result that $Q=0$ for ZP radiation, thereby satisfying the definition of absolute zero. What about for RJ radiation? This case works also, but in a rather trivial way: when $T=0$, $h_{\rm RJ}(\omega)=0$ from Eq. (27), so from Eq. (65) or (67), $Q=0$. Indeed, we can view the RJ case at $T=0$ as being but a special case of Eq. (68), namely, where $\kappa=0$.

Thus, although ZPP and RJ radiation both satisfy $Q=0$ at $T=0$, in the RJ case no radiation is present at $T=0$. Hence, from Eq. (44), $\mathbf{p}_A=e\delta\mathbf{z}_A=\mathbf{0}$, so that no particle fluctuating motion is present either. Each of the energy terms in Eqs. (49)–(53) is trivially equal to zero. Indeed, any candidate for thermal radiation that reduces to no radiation at $T=0$ will also yield $Q=0$ [see Eq. (67)].

In contrast to this situation, when $T=0$ for ZPP radiation we have the very nontrivial case of nonzero incident radiation and fluctuating motion for the oscillating particles. For this very special case where $(h_{\rm in})^2=\kappa\omega$ and $\kappa\neq0$, there exists a zero-point energy associated with the electric dipole oscillators, as seen by the nonzero values of the energy terms in Eqs. (49)–(53). The total change in internal energy upon displacing the dipole particles, as given by Eq. (59), is also nonzero at $T=0$. However, it is precisely matched by the work done in displacing the particles, as given by Eqs. (64) and (62). Thus, despite the nonzero radiation and fluctuating motion at $T=0$, still heat does not flow for the reversible isothermal process we have examined, which must indeed be the situation for any candidate of radiation that is to qualify as thermal radiation.

Consequently, for a brief moment we find that the RJ and ZPP spectra are acceptable as thermal radiation spectrums. This situation will change in Sec. VII when we examine the third law of thermodynamics. Moreover, the planned work of Ref. 43 will discuss much more obvious reasons why RJ radiation fails quite badly as a thermal radiation candidate: for example, an infinite amount of energy is required to change its temperature in a finite region of space.

Of course, if we are willing to demand that a nonzero fluctuating behavior *must* exist at $T=0$ for our electric dipole systems, just as is observed for atomic and molecular systems in nature, then we already have enough information to rule out RJ radiation as a thermal radiation candidate, as well as any other spectrum that reduces to zero at $T=0$. By combining this demand with the result of the next section (i.e., Sec. VI C), we obtain an important restriction on the appropriate spectral characteristics of thermal radiation: namely, that at $T=0$ the thermal spectrum *must* reduce to a zero-temperature radiation spectrum of the form $\rho_{\rm in}=\kappa\omega^3/c^3$, $\kappa\neq0$, as this zero-temperature spectrum not only satisfies $Q=0$, but it is the *only* nonzero spectrum that can in general yield this result.

C. Uniqueness of $\rho_{\rm in}=\kappa\omega^3/c^3$ result

Intuitively it seems fairly clear by looking at Eq. (67) that in order for $Q=0$ for *all* possible choices of the initial and final positions of the N dipole particles, and for any value of N, then we must have that $(h_{\rm in})^2=\kappa\omega$. To quantify this observation, we should first note that, clearly, $(h_{\rm in})^2=\kappa\omega$ is a sufficient condition to achieve this requirement. To prove that $(h_{\rm in})^2=\kappa\omega$ is also a necessary condition, let us consider one special case where its necessity can easily be recognized: namely, by restricting our attention to the case where $N=2$, and by using a resonance approximation.

Let $\mathbf{Z}_B-\mathbf{Z}_A=\hat{\mathbf{z}}R$, so that the two dipole particles lie on a line in the $\hat{\mathbf{z}}$ direction, separated by a distance R. One can then show that

$$\eta^{\mathcal{D}}_{ij}(\mathbf{Z}_B-\mathbf{Z}_A,\omega)=\delta_{ij}\eta^{\mathcal{D}}_{ii}(\mathbf{Z}_B-\mathbf{Z}_A,\omega)\ , \tag{69}$$

and

$$\begin{aligned}\ln\det(\underline{M})&=\sum_{i=1}^{3}\ln\left[1-\left[\frac{e^2}{m}\frac{\eta^{\mathcal{D}}_{ii}}{C}\right]^2\right]\\&=\sum_{i=1}^{3}\ln\left[1-\frac{e^2}{m}\frac{\eta^{\mathcal{D}}_{ii}}{C}\right]+\sum_{i=1}^{3}\ln\left[1+\frac{e^2}{m}\frac{\eta^{\mathcal{D}}_{ii}}{C}\right]\ .\end{aligned} \tag{70}$$

For a small displacement δR,

$$\operatorname{Im}\ln\det(\underline{M})\big|_{\rm I}^{\rm II}=\delta R\frac{\partial}{\partial R}\operatorname{Im}\ln\det(\underline{M})\ . \tag{71}$$

Here we can make use of the algebraic identity

$$\frac{\partial}{\partial R}\operatorname{Im}\ln\left[1\pm\frac{e^2}{m}\frac{\eta^{\mathcal{D}}_{ii}}{C}\right]=\frac{\operatorname{Re}\left[C\pm\frac{e^2}{m}\eta^{\mathcal{D}}_{ii}\right]\frac{\partial}{\partial R}\operatorname{Im}\left[C\pm\frac{e^2}{m}\eta^{\mathcal{D}}_{ii}\right]-\operatorname{Im}\left[C\pm\frac{e^2}{m}\eta^{\mathcal{D}}_{ii}\right]\frac{\partial}{\partial R}\operatorname{Re}\left[C\pm\frac{e^2}{m}\eta^{\mathcal{D}}_{ii}\right]}{\left|C\pm\frac{e^2}{m}\eta^{\mathcal{D}}_{ii}\right|^2}\ , \tag{72}$$

in order to extract the denominators $|C\pm(e^2/m)\eta_{ii}^{\mathcal{D}}|$, which are what enable the resonant calculation to be carried out. Following the steps on pp. 569 and 570 of Ref. 32, or pp. 1653 and 1654 of Ref. 30, then for the unretarded van der Waals condition of $\omega_0 R/c \ll 1$, we obtain

$$Q \approx \delta R \frac{\pi^2}{2} \sum_{i=1}^{3} \left[-\frac{dh_{\rm in}^2}{d\omega} + \frac{h_{\rm in}^2}{\omega} \right] \frac{\dfrac{\partial}{\partial R} \mathrm{Re}\left[\dfrac{e^2}{m} n_{ii}^{\mathcal{D}} \right]}{\omega} \Bigg|_{\omega=\omega_{i-}}^{\omega=\omega_{i+}} , \tag{73}$$

where

$$\omega_{i\pm} = \left[\omega_0^2 \pm \frac{e^2}{m} n_{ii}^{\mathcal{D}} \right]^{1/2} . \tag{74}$$

Thus we have extracted the important quantity

$$\left[-\frac{d(h_{\rm in})^2}{d\omega} + \frac{(h_{\rm in})^2}{\omega} \right] ,$$

evaluated at the six resonant frequencies for the system: $\omega_{i\pm}$, $i=1,2,3$. We could reduce Eq. (73) further by expanding each of the six terms about ω_0, but we need not bother. At this point it should be clear that the only way for Q to equal zero, no matter what frequency ω_0 we choose for our pair of oscillators, is when $(h_{\rm in})^2 = \kappa\omega$, or $\rho_{\rm in} = \kappa\omega^3/c^3$.

Thus we have found an interesting and fundamental property for ZP radiation that can be added to the list of other properties that have already been discovered by others: for example, Lorentz invariance of its stochastic properties,[10,11] and the fact that it does not give rise to velocity-dependent forces.[11,12,53] We will return to these properties, and others, in Sec. VIII. Now let us examine the restriction imposed on a thermal radiation spectrum by the third law of thermodynamics.

VII. FURTHER RESTRICTION: THIRD LAW OF THERMODYNAMICS

According to the Nernst-Simon form of the third law of thermodynamics,[7] "The entropy change associated with any isothermal, reversible process of a condensed system approaches zero as the temperature approaches zero." This form of the third law is ideally suited for our purposes, since we have carried out calculations for an isothermal, reversible process.

For this situation, the change in caloric entropy $S_{\rm cal}$ is given by

$$\Delta S_{\rm cal} = \frac{Q}{T} , \tag{75}$$

where Q has been calculated in Eq. (67). Hence, to satisfy the third law of thermodynamics, we must have that

$$\lim_{T\to 0} (\Delta S_{\rm cal}) = \lim_{T\to 0} \left[\frac{Q}{T} \right] = 0 . \tag{76}$$

Here we should note that we are dealing directly with the thermodynamic definition of entropy, where $\Delta S_{\rm cal}$ is related to temperature and heat flow as in Eq. (75). Thus, following Boyer's arguments,[12] we are *not* making the additional assumption usually made in statistical mechanics that $S_{\rm cal}$ is equal to $S_{\rm prob}$, where $S_{\rm prob} \equiv (S_{\rm prob})_0 + k_B \ln\Omega$, Ω equals the number of microstates (or complexions) of a system that result in the same macrostate, and Ω is deduced by the usual methods in statistical mechanics.[54] Nevertheless, despite the fact that we are not attempting to equate $S_{\rm cal}$ and $S_{\rm prob}$, $S_{\rm cal}$ most certainly does have a statistical basis here, as it was calculated by finding the expectation value of the electromagnetic energy that flows out of a volume $\mathcal{V}$ enclosing the dipole particles, when the particles undergo a quasistatic displacement.

In Eq. (67), the only part of the expression for Q that depends on temperature arises from the two $h_{\rm in}^2$ terms. Consequently, the following condition on $h_{\rm in}^2$ of

$$\lim_{T\to 0} \frac{\left[-\dfrac{dh_{\rm in}^2}{d\omega} + \dfrac{h_{\rm in}^2}{\omega} \right]}{T} = 0 \tag{77}$$

is certainly sufficient to satisfy the third law of thermodynamics. Using the resonance approximation, as in Sec. VI C, we can argue that Eq. (77) is also a necessary condition.

Let us now test the above thermodynamic restriction for the situations of RJ and ZPP radiation. From Eqs. (27) and (77),

$$\frac{1}{T} \left[-\frac{d(h_{\rm RJ})^2}{d\omega} + \frac{(h_{\rm RJ})^2}{\omega} \right] = \frac{k_B}{\pi^2\omega} , \tag{78}$$

which does *not* satisfy Eq. (77). Indeed, a direct substitution of Eq. (27) into Eqs. (75) and (67) confirms that Eq. (76) cannot in general be satisfied by the RJ radiation spectrum.

From Eqs. (28) and (67), for ZPP radiation,

$$\frac{1}{T} \left[-\frac{d(h_{\rm ZPP})^2}{d\omega} + \frac{(h_{\rm ZPP})^2}{\omega} \right] = \frac{1}{T} \frac{\hbar\omega}{2\pi^2} \left[\frac{\hbar}{2k_B T} \right] \frac{1}{\sinh^2\left[\dfrac{\hbar\omega}{2k_B T} \right]} . \tag{79}$$

For small T,

$$\frac{1}{T^2 \sinh^2\left[\dfrac{\hbar\omega}{2k_B T} \right]} \approx \frac{(1/T^2)}{\left[\frac{1}{2}\exp\left[\dfrac{\hbar\omega}{2k_B T} \right] \right]^2} . \tag{80}$$

As $T\to 0$, the denominator in Eq. (80) goes to ∞ much faster than does the numerator of $(1/T^2)$. Hence Eq. (79) has the limiting value of zero as $T\to 0$, so that Eq. (77) *is* satisfied.

Thus we see that for the system of fluctuating electric dipoles considered here, RJ radiation violates the third law of thermodynamics, while ZPP radiation obeys it. Boyer was the first to recognize that ZPP radiation might satisfy this thermodynamic property.[55] To verify this point he needed to introduce some assumptions about how energy fluctuations were related to caloric entropy. These additional assumptions were not required in the present calculations, yet we indeed confirmed Boyer's prediction. Here, we obtained an explicit evaluation of $\Delta S_{\rm cal}$, which enabled the third law of thermodynamics to be tested directly for an incident radiation spectrum $\rho_{\rm in}=\omega^2 h_{\rm in}^2/c^3$.

VIII. CONCLUDING REMARKS

Contrary to what one might intuitively guess, the laws of thermodynamics do *not* inherently require zero radiation and zero fluctuating motion at $T=0$ for *classical* electrodynamic systems. However, the appropriate equilibrium radiation spectrum at $T=0$ must be restricted in form. In this paper we saw that to have an isothermal quasistatic displacement of electric dipole oscillators with no heat being radiated off into space, which is the condition of absolute zero temperature, requires that the incident radiation spectrum be given by $\rho_{\rm in}=\kappa\omega^3/c^3$. This spectral form agrees precisely with the spectrum of classical electromagnetic ZP radiation.

Thus we now have a fundamental *thermodynamic* property for classical electromagnetic ZP radiation that can be added to the list of properties for classical electromagnetic ZP radiation discovered by others. In particular, the stochastic properties of ZP radiation are that it is homogeneous and isotropic in space, stationary in time, and Lorentz invariant.[1] The demand of Lorentz invariance[10,11] yields the same requirement of $\rho_{\rm in}=\kappa\omega^3/c^3$ as found here. Other properties of ZP radiation are that (1) it does not yield velocity-dependent forces,[11,12,53] (2) it has an invariance property under scattering by an electric dipole SHO moving at an arbitrary velocity,[1] (3) its spectrum is invariant for an adiabatic compression of a cavity,[12] (4) it preserves the adiabatic invariants of mechanical systems without harmonics,[56] (5) it gives rise to thermal effects associated with acceleration,[25,45,31,32,57] and (6) the expectation values of products of its fields agree with the quantum-mechanical expectation values of the symmetrized product of corresponding quantum field operators.[2]

Of course, for the properties above such as (a) $Q=0$ at $T=0$, (b) Lorentz invariance, and (c) properties (1)–(4) above, a null radiation field (such as RJ radiation at $T=0$) also satisfies these same properties, but in a very trivial way. For example, a null radiation field is indeed Lorentz invariant since observers in different inertial frames all see the same thing: no radiation. Such a null field is but a special case of the required spectrum we derived at $T=0$ of $\rho_{\rm in}=\kappa\omega^3/c^3$, but where $\kappa=0$. However, this choice of κ does not yield the $T=0$ physical properties of electrodynamic systems we observe in nature, such as their $T=0$ fluctuating motion, and the existence of Casimir and van der Waals forces at $T=0$.

Choosing the correct value of κ for the $T=0$ thermal equilibrium radiation spectrum, so as to obtain agreement with physical systems at $T=0$, is also of crucial importance in deducing the physical properties of classical electromagnetic thermal radiation and mechanical systems at *nonzero* temperatures. After all, the thermodynamic analysis of classical systems changes considerably once one allows nonzero radiation to be present at $T=0$. As noted in Ref. 11, this point can clearly be seen by comparing the derivations presented in Refs. 8 and 11 for the classical electromagnetic thermal radiation spectrum. The two derivations differed in their assumptions about the $T=0$ radiation. Assuming null radiation, or $\kappa=0$, for the $T=0$ radiation, led to the conclusion that RJ radiation[8] was the appropriate classical electromagnetic thermal radiation spectrum at nonzero temperatures; in contrast, assuming that the $T=0$ spectrum was given by $\rho_{\rm in}=\kappa(\omega/c)^3$ with $\kappa=\hbar/2\pi^2$, led to the deduction of ZPP radiation[11] as thermal radiation.[58]

Thus the assumption of null radiation for the $T=0$ spectrum was critically important in the conclusion formed by early researchers that RJ radiation resulted in thermal equilibrium conditions with classical electrodynamic systems.[8,11,12] This particular apparent inconsistency between classical physics and observation was one of the important factors that helped to persuade early researchers that classical physics was simply inadequate for describing the behavior of atomic systems.

However, we have seen that $\kappa=0$ is an unnecessary thermodynamic restriction for classical electromagnetic thermal radiation. Indeed, as shown here, this choice of κ results in a violation of the third law of thermodynamics under the following condition: namely, when the thermal radiation spectrum reduces to zero as $T\to 0$ by being proportional to T. This situation is precisely the one that occurs for RJ radiation.

Finally, we have noted in this paper that there are a number of other suggestive reasons why we should expect that κ should *not* be assumed to equal zero for classical thermal radiation. In particular, if an equilibrium situation is possible for classical electromagnetic radiation and classical charged particles, then on account of Earnshaw's theorem, the only possible mechanism for attaining equilibrium is via the existence of nonzero radiation and nonzero fluctuating motion, even at $T=0$. Combining this reasoning with the deduction of the thermodynamic analysis presented here that the spectrum must be of the form $\rho_{\rm in}=\kappa(\omega/c)^3$, yields that κ must not equal zero for classical electrodynamic systems. The nonzero choice of $\kappa=\hbar/2\pi^2$ then brings us the closest to experimental observation.

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[40]As far as I know, this "derivation" of the classical electromagnetic ZP radiation spectrum has not previously been recognized, where this spectrum has been deduced as a direct consequence of the fundamental thermodynamic definition of $T=0$ (see Ref. 6). However, one point should probably be clarified here. Evidently, some researchers are under the impression that Wien's thermodynamic derivation of the relationship $\rho_{\text{in}}(\omega,T)=\omega^3 f_{\text{in}}(\omega/T)$ already contains the result that $\rho_{\text{in}}(\omega,T)\rightarrow\kappa\omega^3/c^3$ as $T\rightarrow 0$, where $\kappa/c^3=\lim_{\theta\rightarrow\infty}f_{\text{in}}(\theta)$, and κ is some constant greater than or equal to zero. [Indeed, the review article of Ref. 3 explicitly states this idea in material associated with Eq. (2.3).] Clearly, if this reasoning was correct, then the analysis Wien carried out a century ago would already contain the thermodynamic proof for the possible existence of classical electromagnetic ZP radiation. However, by going through Wien's thermodynamic derivation (see, for example, Planck's clear discussion of Wien's analysis in Ref. 18), one can readily show that the implicit assumption is made that no radiation is present at $T=0$. [See, for example, the paragraph following the second line after Eq. (96) in Ref. 18.] Hence $\kappa=0$ is an implicit assumption in this analysis. Thus Wien's thermodynamic derivation of $\rho_{\text{in}}(\omega,T)=\omega^3 f_{\text{in}}(\omega/T)$ is not sufficiently general to account for the possible presence of nonzero radiation at $T=0$. I have found that this original derivation involving the moving wall of a cylinder can be generalized to account for ZP radiation, but the task is nontrivial, since one needs to take into account the thermodynamics of van der Waals forces acting between the macroscopic boundaries of the cavity enclosing the radiation. Reference 43 is planned to clarify these points and to provide a generalized derivation of $\rho_{\text{in}}(\omega,T)=\omega^3 f_{\text{in}}(\omega/T)$ that includes the case of nonzero radiation being present at $T=0$. This derivation is carried out by further analyzing the thermodynamics associated with van der Waals forces between the microscopic atomic oscillator models treated here (as opposed to Wien's derivation involving macroscopic boundary conditions).

[41]As we have noted, a statistically stationary equilibrium configuration has not yet been shown to exist (or not exist) for a system of classical charged particles, such as for the following simple model of the classical hydrogen atom: i.e., an atom consisting of two classical charged point particles with charges $-e$ and $+e$, and with masses of finite value m, and an infinite mass, respectively (see Ref. 37). The case where both particles have a finite mass is even more difficult. One then needs to analyze the possibility of this "atom" continually picking up energy from the ZP field. If a confinement mechanism is present, then some of the difficulty is removed, since the atom will be prevented from continually accelerating up to velocities near the speed of light. See Ref. 11, as well as T. H. Boyer, Phys. Rev. A **20**, 1246 (1979). Further analysis on this problem enters us into discussions on a possible mechanism for cosmic rays, which has been proposed and analyzed by Rueda. See, in particular, A. Rueda, Nuovo Cimento A **48**, 155 (1978); Phys. Rev. A **23**, 2020 (1981); Nuovo Cimento C **6**, 523 (1983); A. Rueda and G. Cavalleri, Nuovo Cimento C **6**, 239 (1983); A. Rueda, Phys. Rev. A **30**, 2221 (1984).

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[46]C. Teitelboim, Phys. Rev. D **1**, 1572 (1970); **2**, 1763(E) (1970).

[47]C. Teitelboim, D. Villarroel, and Ch. G. van Weert, Riv. del Nuovo Cimento **3**, 1 (1980).

[48]See, for example, J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 657.

[49]Here, we have assumed the renormalization procedure holds that is described in Sec. 8.1 of Ref. 47. (Also see the comment at the end of Sec. 7.2 in Ref. 47.) It should be noted that this procedure of renormalization removes the dependence of the sum of the material four-momentum plus the bound electromagnetic four-momentum upon the orientation of the surface $\sigma(\tau)$ described in Ref. 47. Thus we are free to let $\sigma(\tau)$ be the inertial reference frame chosen here for evaluating Eq. (2): namely, the frame where the electric dipoles in the present paper start and end at rest, but are slowly moved apart between these times.

[50]An example where $\Delta U_{\mathrm{EM,in}}$ is nonzero, but the temperature can be held fixed, occurs in the case of the Casimir force between conducting plates at $T=0$. This force can be calculated directly from $\Delta U_{\mathrm{EM,in}}$, which will be nonzero upon a small quasistatic displacement of the plates, since a change will occur in the normal modes of the radiation both between and outside the plates. This change in normal modes results in a change in the radiation's statistical properties. (See Ref. 3, Sec. 4.1, for more information and references.)

[51]The singular behavior of the dipole fields as $R \rightarrow 0$ did not need to be considered in Ref. 31, so that the use of Eqs. (16) and (22) in Ref. 31, rather than Eqs. (40) and (41) given here, is justifiable. More specifically, Ref. 31 related $\mathrm{Im}(\eta_{ij}^{\mathcal{D}})$ and $\mathrm{Re}(\rho_{ij}^{\mathcal{D}})$ to the two-point correlation functions of the Gaussian incident electromagnetic fields, via Eqs. (14)–(16) and (20)–(22) in Ref. 31. The quantities $\mathrm{Im}(\eta_{ij}^{\mathcal{D}})$ and $\mathrm{Re}(\rho_{ij}^{\mathcal{D}})$ are not singular as $R \rightarrow 0$. Consequently, the imaginary part of Eq. (16) in Ref. 31, and the real part of Eq. (22) in Ref. 31, equals the imaginary and real parts, respectively, of Eqs. (40) and (41) given here.

[52]See AIP document no. PAPS PLRAA-42-1847-42 for 42 pages of Appendixes A and B. Appendix A contains the calculations of the expectation value of the energy terms in Eqs. (49)–(53). Appendix B proves that Eq. (66) is satisified by ZPP and RJ radiation. Order by PAPS number and journal reference from American Institute of Physics, Physics Auxiliary Publication Service, 335 E. 45 St., New York, NY 10017. The prepaid price is $1.50 for a microfiche or $5.00 for a photocopy. Airmail is additional.

[53]See the following article, plus the other relevant references cited therein: C. Díaz-Salamanca and A. Rueda, Phys. Rev. D **29**, 648 (1984).

[54]See, for example, Ref. 6, Chap. 11.

[55]T. H. Boyer, Phys. Rev. D **1**, 1526 (1970).

[56]T. H. Boyer, Phys. Rev. D **18**, 1238 (1978).

[57]T. H. Boyer, Phys. Rev. D **30**, 1228 (1984).

[58]It is interesting to note the following: not only does the required $T=0$ spectrum of $\rho_{\mathrm{in}}=\kappa(\omega/c)^3$ yield the $T=0$ RJ and ZPP spectrums when κ equals 0 and $\hbar/2\pi^2$, respectively, but the spectrum $\kappa(\omega/c)^3\coth(\pi^2\kappa\omega/k_B T)$ yields the RJ and ZPP spectra for arbitrary values of T when $\kappa \rightarrow 0$ and when $\kappa=\hbar/2\pi^2$, respectively.

[59]See, for example, G. Arfken, *Mathematical Methods for Physicists*, 2nd ed. (Academic, New York, 1970), Sec. 2.10.

[60]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).