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# REVIEWING AND EXTENDING SOME RECENT WORK ON STOCHASTIC ELECTRODYNAMICS

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**ABSTRACT**

A brief overview is given here on the theory of stochastic electrodynamics, which involves the behavior of classical charged particles interacting with classical electromagnetic thermal radiation and applied electromagnetic fields. Recent work by the author is then discussed on the thermodynamics of classical electric dipole harmonic oscillators in thermal equilibrium with classical electromagnetic random radiation. An outline of a similar analysis is then presented for classical charged point particles to partially expand upon and extend the previous work, and to indicate how in principle the previous analysis of thermodynamic operations for electric dipole oscillators can be applied to a system of charged particles.

## 1. Introduction and Brief Overview of SED

The purpose of this article is twofold: first, to bring to the attention of researchers who deal specifically with electromagnetic theory, that some of the age old problems with classical electrodynamics — such as atomic collapse and the unacceptable physical nature of the Rayleigh-Jeans thermal radiation spectrum — may be fundamental problems that have not yet been fully analyzed within the classical physics context. A body of work now often called stochastic electrodynamics[1–7] has attempted to reexamine these problems while utilizing one single common physical theme: namely, the behavior of the thermal radiation spectrum as temperature goes to zero. Second, I wish to mention some recent work in this regard on specific electrodynamics systems[8–11] and to indicate how this work can, in principle, be extended to the thermodynamics of a system of classical charged point particles. This last problem has not been solved, nor should it be inferred that a solution method is in sight; however, by looking at the formalism of this last problem, and by comparing it with the specific systems that have been solved,[8–11] a few interesting points can be made.

Stochastic electrodynamics (SED)[1–7] is now the name most often used for the following physical theory: specifically, the behavior of classical charged particles interacting with classical electromagnetic thermal radiation, but where the assumption is *not* made that the thermal radiation field must vanish at the temperature of $T = 0$. Several reasons exist for not imposing this assumption, but perhaps the most direct ones are that (1) this assumption of zero radiation at $T = 0$ is an unnecessary one, in the sense that it is not inherently included within the laws of thermodynamics,[8–10] and (2) physical observation clearly shows that fluctuating motion and fluctuating fields exist in nature at $T = 0$. Upon recognizing these two points, and upon taking into account a nonzero radiation spectrum at $T = 0$ that conforms with appropriate thermodynamic requirements at $T = 0$, the natural question arises as to what new predictions result from this classical theory?

Before answering this question, I should emphasize that SED is, indeed, an entirely classical theory, in the sense that Maxwell's equations apply, and charged particles obey the relativistic generalization of Newton's second law of motion. The usual form taken for this relativistic generalization is the Lorentz-Dirac equation of motion.[a,12–16] At least two interesting aspects exist for this SED field of study. Certainly the most eye-opening aspect is that quantum mechanical properties of simple electrodynamic systems are predicted,[1–7] despite the theory being an entirely classical one. Indeed, the tantalizing possibility exists that this theory may accurately describe the physics of atomic systems, and may even be extendable to the other regimes of physics (gravitation, as well as weak and strong interactions) by essentially the same principles existing at the atomic level. (I will briefly discuss this point more fully at the end of this introduction.) Without a doubt, however, although such thoughts are intriguing, such a claim at this point represents largely unsupported speculation; the physical predictions so far obtained from SED are not yet even *close* to yielding the wide predictions from quantum electrodynamics, never mind the other regimes of physics. To date, except for fairly simple systems, practical calculational methods have not yet been devised to fully test the theory of SED on the wide range of phenomena available in atomic physics.

A second interesting aspect of SED is that the theory again opens up the question of whether equilibrium can exist between classical charged particles

[a] For an important, yet small subset of this literature, see Refs. 12–16. In particular, see the excellent review article of Ref. 16. Also see the extensive list of references given in Refs. 13 and 16.



and classical electromagnetic random radiation, and, if so, what are the properties of this behavior? The old problem in classical physics of atomic collapse invites this question to be reexamined. If we consider a stationary positively charged nucleus with a negatively charged classical electron orbiting around it, then radiation will be emitted as the electron revolves around the nucleus. The orbit must then decay, resulting in the electron spiraling in towards the nucleus.

How much energy will be emitted? At first thought, one might respond that an infinite amount of energy will be emitted, since $-Z\frac{e^2}{r} \to -\infty$ as $r \to 0$, where $Z$ is the atomic number, $e$ is the electronic charge, and $r$ is the radius. At second thought, this infinite result clearly reveals the absurdity of this scenario actually occurring in nature. Hence, one might then respond that the particles will undoubtedly break down, in some form, before total collapse occurs, but that without more information on the structure of the particles, further discussion on a classical "annihilation" of the particles is not very helpful. Moreover, if we assume that this nucleus has an extended charge structure, then the binding potential will no longer be singular as $r \to 0$. Consequently, for this reason also, an infinite amount of energy will *not* be emitted during the orbital collapse. In any case, though, two points can be made from this discussion:(1) the radiation emitted will be fairly large, and (2) clearly an equilibrium situation does not exist for this example such that the electron resides largely outside the nucleus. This second point conflicts with the observed atomic stability found in nature.

Suppose we now consider the following hypothetical case: namely, an infinite universe filled with atomic systems of the type just described. If all the orbits of the electrons started to collapse, an enormous amount of radiation would be emitted, thereby effecting the behavior of neighboring orbiting electrons. The smaller the orbits, the more radiation would be emitted, so the larger the influence the radiation would have on the behavior of the electrons. The radiation would act to perturb the electrons from their spiraling path. Possibly the influence of the radiation could become extremely important. With this mechanism of radiation altering the straight decay of the orbit, is there a way by which a thermal equilibrium state can exist between the radiation and particles?

Somewhat different approaches to this idea of equilibrium have been discussed by others.[1,17,18,b] In particular, instead of just particles being present at time $t = -\infty$, and no radiation, suppose there exists some spectrum of radiation at $t = -\infty$, along with the charged particles. Again we can ask the question about equilibrium. Boyer in Ref. 1 presented an interesting analysis that equilibrium can exist between what is called classical electromagnetic zero-point (ZP) radiation, and a single classical model of the hydrogen atom. As the electron's orbit decays, the electron spirals faster, thereby making a higher frequency part of the ZP field act more effectively on the electron, in a sort of resonance effect. The higher the frequency for the ZP field, the stronger the fields. Hence, the likelihood is then increased that work will be done on the electron, thereby expanding it's orbit. Once the orbit becomes large, the lower frequency part of the ZP field becomes the most effective part. However, these fields are weaker. Hence, the electron is then likely to radiate more energy than it is to pick up energy from the ZP field, thereby increasing the probability that the orbit will decrease in size. The physical idea here is that this pattern of increasing and decreasing orbital size can continue indefinitely, with the electron following a fluctuating orbital motion, characterized by some average radius. In this way, atomic stability might arise. Indeed, by making the scale factor of the ZP radiation spectrum correctly correspond with Planck's constant, Ref. 1 estimates this average radius to be the Bohr atomic radius.[17]

A strong point to SED is that the physical assumptions are simple ones. Planck's constant only enters the theory at one point: namely, as a scale factor of the electromagnetic thermal radiation spectrum when $T \to 0$. Hence, the theory should be quite testable, since the basic ideas in the theory leave little room for alternative approaches. However, in general the calculations to date have been difficult enough that when conflicts with nature have been predicted, it is hard to ascertain whether the conflicts are fundamental ones or whether the calculations have simply not yet been carried out accurately enough, such as by relativistic corrections, or by more accurately characterizing the interaction between particles, or even the structure of the particles.

The most successful area of SED has been the nonrelativistic simple harmonic charged oscillator and the evaluation of the effects of free random fields with and without macroscopic boundary conditions imposed. For these two

[b] Rueda comments on Puthoff's work (Refs. 17 and 18) in footnote 3 of Ref. 19. Also see P. S. Wesson, *Phys. Rev.* **A44** (1991) 3379 and E. Santos, *Phys. Rev.* **A44** (1991) 3383, and the accompanying replies by H. E. Puthoff.



cases the mathematical analysis has been carried out in detail. In both instances, good agreement between quantum theory and classical theory has been obtained.[20,2,3,5] Also, good agreement has been obtained for the free particle.[21] For more complicated systems that are nonlinear in nature, agreement between the two has not been obtained.[22–26] However, important questions still remain on the appropriateness of these systems for making comparisons with real systems in nature.[27,28,8] A more critical test than these systems is the classical hydrogen atom described earlier. Nevertheless, despite some initial vigorous attempts,[29–33] this problem has still not been solved within SED.[28]

As for the harmonic oscillator, probably the most impressive application of this model has been for calculating the retarded van der Waals force between electric dipole oscillators at all distances.[34] The results agree with quantum theory predictions at $T = 0$.[35] Moreover, the calculation method of SED enables a very easy extension to be made that immediately generalizes the results to the case of nonzero temperatures.[36] This extension is more difficult using the traditional methods in quantum theory; indeed, I am not aware that this last calculation has yet been carried out by others using quantum theory.

These SED results were initially obtained in Refs. 34 and 36 for the case of two electric dipole oscillators. Section III in Ref. 37 extended this calculation to $N$ dipole oscillators, while Refs. 8, 9 and 11 analyzed the thermodynamic behavior of $N$ such oscillators due to slowly displacing the oscillators with respect to each other, and due to slowly changing the temperature of the thermal radiation. In particular, the thermodynamic analysis in Ref. 8 enabled the following result to be obtained. Aside from a proportionality constant, only one nonzero spectrum is suitable for establishing an equilibrium state with the electric dipole oscillators at a temperature of absolute zero: namely, an energy spectrum proportional to $\omega^3$, which is the spectral form of classical electromagnetic ZP radiation. Only this spectral form will yield the result of no net electromagnetic energy being radiated off to infinity upon slowly displacing the oscillators. Thus, this spectrum corresponds with the demand that at $T = 0$, no heat flow can take place during an isothermal, reversible thermodynamic operation.

This result represents a fundamental thermodynamic property of the classical electromagnetic ZP field in SED, which must hold for all physically realizable systems in nature if SED is to be an acceptable physical theory. Reference 10 showed that this property holds for at least one other system: specifically, no heat flow occurs during quasistatic displacements of conducting parallel plates immersed in classical electromagnetic ZP radiation.

Classical electromagnetic ZP radiation also has a number of other important physical properties that are discussed, for example, in Refs. 1 and 3. If we consider just radiation alone, then the strongest reasoning for the appropriate $T \to 0$ form of the classical electromagnetic thermal radiation spectrum is the demand that the $T = 0$ spectrum should appear the same in *all* inertial reference frames.[38,39] After all, if one inertial reference frame corresponds to a $T = 0$ thermodynamic state, then all inertial reference frames should possess the same state. If $T \neq 0$, the same statement cannot be made. In particular, if a perfectly conducting box contains thermal radiation at $T \neq 0$, the spectrum will appear as the expected thermal form for the reference frame in which the box is at rest, but it will appear different from this form in other reference frames. This demand of Lorentz invariance for radiation in the $T = 0$ thermodynamic state yields that the spectrum must be proportional to $\omega^3$, thereby also predicting the spectral form of classical electromagnetic ZP spectrum.[38,39]

Of course, this property only involves the radiation and does not address other requirements we must demand of a thermodynamic equilibrium state between radiation *and* acceptable systems of matter found in nature. In particular, an acceptable physical system of classical charged particles in thermal equilibrium with radiation should not alter the radiation spectrum via scattering.[c] This requirement should hold at *all* temperatures. Other thermodynamic demands must also be met, such as: (1) the requirement just mentioned that at $T = 0$ no heat flow should take place during reversible thermodynamic operations, (2) the requirement of finite specific heat for thermal radiation,[9] and (3) the demand that the second and third laws of thermodynamics should hold.[9,10] In particular, the second law of thermodynamics applied to the electric dipole oscillator system in Ref. 9 and the parallel plates in Ref. 10, resulted in the requirement that the appropriate classical thermal spectrum must obey a generalized Wien's displacement law, which holds even when the thermal

[c] This point apparently had its origin in the arguments by researchers around 1900 involving the effect of a black carbon particle on changing an arbitrary state of radiation over to a maximum entropy state of thermal radiation. In particular, see the clear discussions by Planck in Ref. 40. Boyer examined this point of view in more depth in Ref. 22, which apparently contains the first explicit calculations regarding a detailed balance condition for the scattering of random classical electromagnetic radiation by a specific nonlinear charged oscillator system. Since then, the point has been recognized that the form of the scattering system may be critical in determining which spectrum is appropriate to ensure thermal equilibrium, since all atomic systems in nature inherently possess interactions of electromagnetic origin. See Refs. 27 and 28.



spectrum does not vanish at $T = 0$.

The thermodynamic analysis followed in Refs. 8 and 9 will be discussed in the following sections for the case of classical charged point particles. Specific results were obtained in Refs. 8 and 9 that will not be obtained here, such as the caloric entropy. Nevertheless, the present analysis will enable the specific form of the radiated heat to be examined, as well as provide a basis for discussing examples of thermodynamic operations on this electrodynamic system. Section 2 deduces a set of energy-balance relationships for a system of $N$ charged point particles, and their electromagnetic fields, within a volume $V$ in space. The motion of the particles is treated by fully relativistic dynamics here, with mass renormalization taken into account. The resulting energy-balance relationships are then used in Sec. 3 to qualitatively discuss quasistatic thermodynamic operations on this electrodynamic system. An attempt is made to clarify possible conceptual problems with (1) the role of "incident radiation", (2) the microscopic picture of a "classical ideal gas" from the point of view of SED, as well as (3) the statistical meaning of the first and second laws of thermodynamics with thermal radiation and classical charged particles present. The appendix enables the analysis described here on charged point particles to be connected to the electric dipole oscillator model used in Refs. 8, 9, and 11. Finally, Sec. 4 contains some concluding remarks.

Before turning to this analysis, a few more introductory comments on SED are in order. First, it should be emphasized that the basic ideas involved in SED are not new ones. Early significant papers that laid the ground work for the present state of SED were in the 1960's by Marshall[20,38] and Boyer.[39,41] Some of these ideas were considered even earlier, as discussed in Refs. 3 and 5. References 1–7 serve as good reviews and research papers for an introduction to SED. Reference 7 presents a semipopular account, while Refs. 3, 5 and 6 contain an extensive list of references, with Ref. 5 being a concise overview up until 1980. Reference 6 concentrates on an acceleration effect that results from SED, but it also gives an overview of the theory and it contains recent references up through 1990. Reference 3 provides an extensive review through 1982.

Quantitative and qualitative physical predictions have been made within SED that certainly have the flavor of a physical world described by quantum theory. For example, in addition to the earlier topics, there exists advances in SED on the following subjects: classical derivations of blackbody radiation,[39,41–43] explanations for quantum optics effects,[44–48] a deeper understanding of excited states,[49] analysis of systems exhibiting thermal effects due to uniform acceleration through the vacuum,[43,50–53,37] diamagnetism,[20,2,54] paramagnetism,[55] a classical mechanism due to the ZP field that may provide a source for cosmic rays,[6] the Compton effect,[56,57] the Debye law for the specific heat of solids,[58] suppression of spontaneous decay in confined space,[59] a possible deeper understanding of Schrödinger's equation,[60,61] and even a proposed explanation for the source of gravity.[62] Qualitative ideas based on SED have been given for tunneling, diffraction, interference, and the Heisenberg uncertainty principle.[1] Still other effects are described in Refs. 1–6.

These predicted phenomena are certainly surprising within a classical theory of nature. Whether SED is simply an anomaly that makes close connections with quantum mechanics for simple systems, or whether it actually represents a more fundamental description of nature, is not yet known. We do know that both theories do not agree in all aspects, such as for Aharonov-Bohm type experiments,[63–65] for experiments testing Bell's inequalities,[44–48] and even in the detailed description of classical systems, such as continuous versus discrete energy levels for the simple harmonic oscillator. As for Aharanov-Bohm and Bell's inequality experiments, the differences between quantum mechanics and the classical physics of SED are certainly testable, so at some point we should be able to safely make some firmer conclusions about the applicability of SED to nature.

Moreover, the domain of physics where SED is most likely to accurately describe nature, if anywhere, is in the atomic and molecular domain. The binding forces are clearly electromagnetic in origin here. We can at least envision that equilibrium conditions can result from a balance between the electromagnetic energy radiated and the work done by electromagnetic forces on atomic and molecular systems. However, certainly for smaller domains of physics, other interactions besides electromagnetic ones are dominant. If classical fields and force concepts are applicable here, then work will be done by these fields on subatomic particles, and energy will be carried by these fields. Thus, these other interactions should play an important role in establishing equilibrium conditions in this subatomic domain. Since equilibrium must hold even at $T = 0$, then the "zero-point" stochastic fields of these subatomic interactions may be critical in establishing the appropriate equilibrium and thermodynamic behavior of these systems.

For the physical domain much larger than the molecular regime, gravity is dominant. Puthoff presented some very interesting ideas in Ref. 62 on gravitation that does not require introducing other classical forces. His analysis did not attempt to predict aspects of gravity beyond the first order description where gravity is treated as a Newtonian force in flat space-time. For a first major attack on the issue of gravity and the classical electromagnetic ZP field, this approach is certainly quite appropriate. However, due to this simplification, much work still remains to be done to determine the success and limitations of this approach in tackling other problems and questions in general relativity. Also, since Puthoff's work on gravitation had its origin in a classical description of parton motion, a detailed examination of the parton model appears to be critical. In Ref. 62, a nonrelativistic simple harmonic oscillator was used as a model for the behavior of a parton, where the *very* high frequency behavior of this model was taken quite seriously. Consequently, some points that should be addressed if this theory is to be pushed further are the issues of: (1) accounting for very fast fluctuating motion of the parton by relativistic kinetics, (2) examining the effects of nonlinear binding forces on the partons, (3) including the fact that all charged partons comprising a single nuclear particle should interact electromagnetically, as well as by nonelectromagnetic, subatomic binding forces (the nonlinear electromagnetic interaction alone will couple the motion of charged partons in a single nuclear particle, thereby requiring a deeper examination of the resulting behavior than what a simple harmonic oscillator description can offer), and (4) examining the condition that the parton itself must be in thermal equilibrium with its subatomic binding field.



Thus we see that a very large number of uncertainties exist for the success of SED in accurately describing the regimes of physics mentioned above. Despite these uncertainties, we also see that some very intriguing results have been obtained to date from a purely classical description of nature. This classical theory does not seek to describe different phenomena in nature with a wide range of unrelated models and explanations, but rather by one main motivating fact: specifically, if natural thermodynamic equilibrium conditions can exist between charged particles and electromagnetic radiation, then the equilibrium state must consist of fluctuating motion and fields, even at $T = 0$.[6] To what extent the results of SED are a mere curiosity, or the consequence of a fundamental truth concerning the description of nature, presently remains unknown. Research by workers in SED is aimed at addressing this question.

## 2. Energy Relationships for Relativistic Charged Point Particles Interacting with Radiation

### 2.1. *General remarks concerning the singular energy terms associated with each particle*

In this section we will obtain an energy-rate equation for a set of $N$ classical charged point particles interacting with incident electromagnetic radiation. In order to be explicit when discussing the electromagnetic energy, consider the traditional expression for the electromagnetic energy of $N$ charged point particles in a volume $V$, with incident fields $\mathbf{E}_{in}$ and $\mathbf{B}_{in}$ present:

$$\frac{1}{8\pi}\int_V d^3x\,(\mathbf{E}_{tot}^2+\mathbf{B}_{tot}^2)\ ,$$

where

$$\mathbf{E}_{tot}=\mathbf{E}_{in}+\sum_{A=1}^{N}\mathbf{E}_A\ , \tag{1}$$

$$\mathbf{B}_{tot}=\mathbf{B}_{in}+\sum_{A=1}^{N}\mathbf{B}_A\ . \tag{2}$$

Here $\mathbf{E}_A$ and $\mathbf{B}_A$ represent the full retarded electromagnetic fields associated with the particle labeled by $A$. The fields $\mathbf{E}_{in}$ and $\mathbf{B}_{in}$ will be treated later as representing the fields due to classical electromagnetic thermal radiation.

The terms in the total electromagnetic energy expression associated with a single particle are given by

$$\frac{1}{8\pi}\int_V d^3x\,(\mathbf{E}_A^2+\mathbf{B}_A^2)\ .$$

These terms are singular, thereby causing complications. Teitelboim's method[15,16] will be used here for separating each of these single-particle energy terms into a singular and nonsingular part, which he referred to, respectively, as bound and emitted electromagnetic energies. The bound energy contains a term that appears as the electromagnetic mass, which is singular for a charged point particle. Using Dirac's renormalization procedure, this singular electromagnetic mass will be assumed to combine with a singular bare mass in the kinetic energy so as to remove the singularity and result in the experimentally measurable finite mass. Also, in order to stabilize the charged point particle, the singular electromagnetic stress must cancel with the bare material stress.[d]

The combination of the bound electromagnetic four-momentum and the bare material four-momentum results in a net four-momentum that depends on the instantaneous value of the four-velocity and four-acceleration.[e] The energy part of this material plus electromagnetic bound four-momentum equals (1) the particle's kinetic energy, with a renormalized mass, plus (2) a term that's usually called the Schott energy. These two terms appear in the energy component of Dirac's relativistic equation.[f]

Besides the bound electromagnetic energy terms that appear in the total electromagnetic energy expression, two types of terms remain: namely, (1) the emitted (and nonsingular) part of each single-particle energy term,[15,16] and (2) the cross-term parts of the total electromagnetic energy due to $\mathbf{E}_A\cdot\mathbf{E}_B (A\neq B)$, $\mathbf{E}_A\cdot\mathbf{E}_{in}$, $\mathbf{B}_A\cdot\mathbf{B}_B\,(A\neq B)$, and $\mathbf{B}_A\cdot\mathbf{B}_{in}$. The following sections analyze these terms in more detail and establish appropriate "energy-rate" relationships for them. In Sec. 2.5, these different relationships will all be brought together into a single energy-balance equation that describes a system of $N$ charged point particles acted upon by external forces and incident radiation. To begin, a single charged point particle will be briefly discussed.

### 2.2. *Energy relationship for a single charged point particle*

The starting point here will be the Lorentz-Dirac equation[12–16,g]

$$m\frac{d^2z^\mu}{d\tau^2}=\frac{2q^2}{3c^3}\left[\frac{d^3z^\mu}{d\tau^3}-\frac{1}{c^2}\left(\frac{d^2z^\lambda}{d\tau^2}\frac{d^2z_\lambda}{d\tau^2}\right)\frac{dz^\mu}{d\tau}\right]+F^\mu\ , \tag{3}$$

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[d] For an explanation of this point of view, see Ref. 16, Sec. 8.1, as well as the comment at the end of Sec. 7.2. In particular, compare Eqs. (7.9) and (8.3) in Ref. 16. Note that this procedure of renormalization removes the dependence of the sum of the material four-momentum plus the bound electromagnetic four-momentum upon the orientation of the surface $\sigma(\tau)$ described in Ref. 16. Thus we are free to let $\sigma(\tau)$ be the laboratory inertial reference frame in which we will examine the usual forms for the first and second laws of thermodynamics.

[e] See Ref. 16, Sec. 7 (particularly Sec. 7.2) and Eq. (8.7).

[f] See Eq. (5) of this article; the Schott energy term appears in the first term on the right-hand side.

[g] The following notation is used here: Greek indices take on values 0, 1, 2, 3, a time-space point is denoted by $x^\mu=(ct;\mathbf{x})$, and the metric is $g^{\mu\nu}=\text{diag}\,(-1,1,1,1)$.

where $F^\mu$ represents the sum of all four-vector forces acting on the particle due to sources other than the particle's own fields. Let $\dot{\mathbf{z}} \equiv \frac{d\mathbf{z}}{dt}$, $\ddot{\mathbf{z}} \equiv \frac{d^2\mathbf{z}}{dt^2}$, etc. As usual, let $\gamma = \left[1 - \left(\frac{\dot{\mathbf{z}}}{c}\right)^2\right]^{-1/2}$. Expressing $F^\mu$ in terms of its three-vector form[h]

$$F^\mu = \left(\gamma\frac{\dot{\mathbf{z}}}{c}\cdot\mathbf{F};\gamma\mathbf{F}\right) , \tag{4}$$

multiplying Eq. (3) for $\mu = 0$ by $c/\gamma$, and recasting $\frac{d^n z^\mu}{d\tau^n}$ in terms of $\dot{\mathbf{z}}$, $\ddot{\mathbf{z}}$, etc.,[i] yields

$$\frac{d}{dt}(m\gamma c^2) = \frac{d^2}{dt^2}\left(\frac{q^2}{3c}\gamma^2\right) - R + \dot{\mathbf{z}}\cdot\mathbf{F} , \tag{5}$$

where

$$R \equiv \frac{2}{3}\frac{q^2}{c^3}\frac{d^2z^\lambda}{d\tau^2}\frac{d^2z_\lambda}{d\tau^2} = \frac{2}{3}\frac{q^2}{c^3}\left[\gamma^4\ddot{\mathbf{z}}^2 + \gamma^6\left(\frac{\dot{\mathbf{z}}\cdot\ddot{\mathbf{z}}}{c}\right)^2\right] . \tag{6}$$

Thus, the rate of change of the particle's kinetic energy[j] has been expressed in terms of the rate of work $\dot{\mathbf{z}}\cdot\mathbf{F}$ by $\mathbf{F}$ on the particle, plus two other terms. The first term, namely,

$$\frac{d^2}{dt^2}\left(\frac{q^2}{3c}\gamma^2\right) ,$$

equals the time derivative of the Schott energy and arises from the negative of the rate of change of the bound electromagnetic energy due to the particle's self-fields after a mass renormalization has been carried out.[15,16] More specifically, the combination of the term on the left-hand side and the first term on the right-hand side of Eq. (5), or

$$\frac{d}{dt}\left[m\gamma c^2 - \frac{d}{dt}\left(\frac{q^2}{3c}\gamma^2\right)\right] ,$$

equals the time rate of change of the particle's material plus bound electromagnetic energy, as evaluated within some selected inertial reference frame.

[h] See Eq. (A1–49) in Ref. 13.

[i] See Eqs. (5.25)–(5.27) in Ref. 13.

[j] The traditional definition of the kinetic energy is $m(\gamma - 1)c^2$ (See H. Goldstein, Classical Mechanics, 2nd ed. [Addison-Wesley, Reading, Mass., 1980], p. 307.) Since $m$ = constant in this article, then the left-hand side of Eq. (5) equals the time rate of change of the kinetic energy. Consequently, $m\gamma c^2$ will be loosely referred to here as being the kinetic energy.



Concerning the remaining term in Eq. (5), $R$ was shown by Teitelboim to be associated with the rate of electromagnetic energy radiated by a single particle.

### 2.3. *Radiation Energy for a Single Charged Point Particle*

Here, the quantity $R$ in Eq. (5) will be expressed in a form appropriate for calculating the energy flow and changes in internal energy associated with the emitted electromagnetic energy of a single point charge. From Eqs. (5.13) and (5.21) in Ref. 13,

$$\partial_\beta t^{\alpha\beta}_{(-2)}(x) = \int_{-\infty}^{\infty} d\tau R\frac{dz^\alpha}{d\tau}\frac{1}{c}\delta^{(4)}(x - z(\tau)) , \tag{7}$$

where $t^{\alpha\beta}_{(-2)}$ is obtained by a covariant splitting of the symmetric energy-momentum tensor,

$$t^{\alpha\beta} = \frac{1}{4\pi}\left(f^{\alpha\gamma}f^{\beta}{}_{\gamma} - \frac{1}{4}f^{\gamma\delta}f_{\gamma\delta}g^{\alpha\beta}\right) , \tag{8}$$

into two parts:[15,16,k] $t^{\alpha\beta} = t^{\alpha\beta}_{(-2)} + t^{\alpha\beta}_{(-3}$. Here, $f^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ and $A^\alpha$ is the four-potential. In any given reference frame, $t^{\alpha\beta}_{(-2)}$ is equal to the part of $t^{\alpha\beta}$ that is proportional to $|\mathbf{x} - \mathbf{z}(t_r)|^{-2}$, where $t_r$ is the retarded time, while $t^{\alpha\beta}_{(-3}$ contains terms proportional to $|\mathbf{x} - \mathbf{z}(t_r)|^{-n}$ for $n \geq 3$. Consequently, within a given inertial reference frame

$$t^{00}_{(-2)} = \frac{1}{8\pi}\left[(\mathbf{E}_a)^2 + (\mathbf{B}_a)^2\right] , \tag{9}$$

$$t^{0i}_{(-2)} = \frac{1}{4\pi}(\mathbf{E}_a \times \mathbf{B}_a)_i , \quad \text{for } i = 1,2,3 , \tag{10}$$

where $\mathbf{E}_a$ and $\mathbf{B}_a$ are proportional to $|\mathbf{x} - \mathbf{z}(t_r)|^{-1}$ and thus represent the usual acceleration fields due to a single point charge.[l]

Integrating Eq. (7), with $\alpha = 0$, over a volume $V$ that contains the particles at some instant of time, yields

[k] Please note that the notation of $(-2)$ and $(-3$ is intentional. Here, $(-2)$ indicates $n = 2$, while $(-3$ indicates $n \geq 3$ in, for example, the expression $|\mathbf{x} - \mathbf{z}(t_r)|^n$, as discussed here. See Refs. 15 and 16 for more detail.

[l] See, for example, p. 657 in Ref. 66.

$$R = \frac{d}{dt}\left\{\frac{1}{8\pi}\int_V d^3x\,[(\mathbf{E}_a)^2 + (\mathbf{B}_a)^2]\right\} + \frac{c}{4\pi}\oint_S d^2x\,\hat{\mathbf{n}}\cdot(\mathbf{E}_a \times \mathbf{B}_a)\,. \quad (11)$$

Thus, $R$ equals the time rate of change of electromagnetic energy within $V$ due to the acceleration fields of the charged particle, plus the rate of flow of this radiated energy out of $V$, through the enclosing surface $S$ of the volume $V$.

### 2.4. *Cross-term relationship for electromagnetic energy*

As will be seen more clearly in Sec. 2.5, the use of Eqs. (5) and (11) enables an energy-rate equation to be established for a single charged point particle that involves: (1) work done by external forces, (2) changes in kinetic and electromagnetic energies, and (3) electromagnetic energy flow. For a system with more than one charged particle, however, as is the interest here, the work done by one particle upon the others must be taken into account. Moreover, the effects of incident thermal radiation needs to be explicitly accounted for if an energy-rate equation is to be obtained that is appropriate for use in thermodynamic analysis. The present section addresses these points.

The last term in Eq. (5) must contain the rate of work $q\dot{\mathbf{z}}(t)\cdot\mathbf{E}(\mathbf{z}(t),t)$ that is done on the particle due to electromagnetic fields other than those of the particle. Poynting's theorem relates the rate of work $\int d^3x\mathbf{J}\cdot\mathbf{E}$ by electromagnetic forces to the rate of change of electromagnetic energy and to electromagnetic energy flow.[m] Such an energy-rate relationship is desired here. However, the traditional derivation of Poynting's theorem holds for continuous charge and current distributions rather than for point charges. Consequently, Poynting's theorem is not immediately helpful here. Of course, if we were to consider an extended charge model, then we could readily apply Poynting's theorem. Indeed, for any finite size of this extended charge, this theorem holds, no matter how small we choose the size.

However Poynting's theorem involves the total electric and magnetic field at each point in space, which means that we would need to revisit our renormalization procedure to extract out the experimentally relevant quantities in this relationship that do not become singular as the size of the particle becomes vanishingly small. For example, the quantity $\int d^3x\mathbf{J}\cdot\mathbf{E}$ in Poynting's theorem involves the total electric field and the total current density, so that this term includes not only (i) the work done on a particle due to the electric fields of other particles, but it also includes (ii) the work being done on itself due to its own electric field during the particle's trajectory. At this point in our analysis we only want the former quantity and not the latter, since the self-energy terms have already been treated via the renormalization procedure described in the preceding sections.

[m] See, for example, p. 236 in Ref. 66.



Nevertheless, we can use the general method of deriving Poynting's theorem to obtain an energy-rate equation that is similar in content to Poynting's theorem, but that holds for the time rate of change of the cross-term quantities associated with the electromagnetic energy of point charges. Let $\mathbf{E}_A$ and $\mathbf{B}_A$ be the electric and magnetic fields associated with a charge density $\rho_A(\mathbf{x},t)$ and current density $\mathbf{J}_A(\mathbf{x},t)$. Let $\mathbf{E}_A$ and $\mathbf{B}_A$ represent the full retarded fields associated with the sources $\rho_A$ and $\mathbf{J}_A$, provided that either $\rho_A$ or $\mathbf{J}_A$ are nonzero. If both $\rho_A$ and $\mathbf{J}_A$ equal zero, then let $\mathbf{E}_A$ and $\mathbf{B}_A$ represent free incident fields.

Now consider a second set of fields $\mathbf{E}_B$ and $\mathbf{B}_B$ that are attributed to the electromagnetic sources $\rho_B(\mathbf{x},t)$ and $\mathbf{J}_B(\mathbf{x},t)$. Following the same procedure as used in deriving Poynting's theorem,[m] one can show that

$$\int_V d^3x\,[\mathbf{J}_A(\mathbf{x},t)\cdot\mathbf{E}_B(\mathbf{x},t) + \mathbf{J}_B(\mathbf{x},t)\cdot\mathbf{E}_A(\mathbf{x},t)]$$
$$= -\frac{d}{dt}\left(\int_V d^3x\,u_{(A-B)}\right) - \oint_S d^2x\,\hat{\mathbf{n}}\cdot\mathbf{S}_{(A-B)}\,, \quad (12)$$

where

$$u_{(A-B)}(\mathbf{x},t) = \frac{1}{8\pi}[(\mathbf{E}_A\cdot\mathbf{E}_B + \mathbf{E}_B\cdot\mathbf{E}_A) + (\mathbf{B}_A\cdot\mathbf{B}_B + \mathbf{B}_B\cdot\mathbf{B}_A)]\,, \quad (13)$$

$$\mathbf{S}_{(A-B)}(\mathbf{x},t) = \frac{c}{4\pi}[\mathbf{E}_A\times\mathbf{B}_B + \mathbf{E}_B\times\mathbf{B}_A]\,. \quad (14)$$

Thus, the work per unit time $\int_V d^3x\mathbf{J}_A(\mathbf{x},t)\cdot\mathbf{E}_B(\mathbf{x},t)$ by the fields of source $B$ upon the charge distribution $A$ within volume $V$, plus the analogous quantity $\int_V d^3x\mathbf{J}_B(\mathbf{x},t)\cdot\mathbf{E}_A(\mathbf{x},t)$, equals the negative of the $A$-$B$ cross-terms of: (1) the rate of change of the electromagnetic energy within $V$, plus (2) the electromagnetic energy per unit time leaving $V$.

If the charge distribution labelled "$A$" is that of a point charge contained within $V$, then the first term on the left-hand side of Eq. (12) becomes $q_A\dot{\mathbf{z}}_A(t)\cdot\mathbf{E}_B(\mathbf{z}_A(t),t)$. This term will appear in the last term of Eq. (5).

### 2.5. *Energy relationships for a system of charged point particles*

Let us now combine the results from the previous three sections. Consider a system of $N$ charged point particles that follow trajectories within a volume $V$ between times $t_{\text{I}}$ and $t_{\text{II}}$. Let $A$ and $B$ be indices that label the physical properties of these particles when $A$ and $B$ take on the values $1, 2, \ldots, N$. Let the force $\mathbf{F}$ in Eq. (5) acting on particle $A$ be given by

$$\mathbf{F}_A(t) = q_A\left[\mathbf{E}_{\text{in}}(\mathbf{z}_A(t),t) + \sum_{\substack{B=1\\(B\neq A)}}^{N} \mathbf{E}_B(\mathbf{z}_A(t),t)\right] + q_A\frac{\dot{\mathbf{z}}_A(t)}{c}\times\left[\mathbf{B}_{\text{in}}(\mathbf{z}_A(t),t) + \sum_{\substack{B=1\\(B\neq A)}}^{N} \mathbf{B}_B(\mathbf{z}_A(t),t)\right] + \mathbf{F}_{\text{ext},A}(t)\,. \qquad (15)$$

Thus, $\mathbf{F}_A(t)$ includes the Lorentz force due to the full retarded fields of all the other $(N-1)$ particles plus the force due to the incident free fields. The quantity $\mathbf{F}_{\text{ext},A}$ represents the sum of all other forces acting on particle $A$. One restriction on $\mathbf{F}_{\text{ext},A}$ is that the four-vector force $F^{\mu}_{\text{ext},A}$ is related to the Newtonian (nonrelativistic) force $\mathbf{F}_{\text{ext},A}$ via Eq. (4).[h] Such forces might arise, for example, from (1) the applied electromagnetic fields that an experimenter controls, or (2) from current or charge sources that are present in addition to the $N$ charged particles explicitly considered here. Under the second category we could include a container wall that holds the $N$ particles. After all, any physical container, like a jar, a metal box, etc., is composed of atoms that are in turn composed of negatively charged electrons and positively charged nuclei. When any of the $N$ particles within the container come close to the walls, a force due to the walls is exerted upon them. Such a force, along with the other "external" forces mentioned above, we can describe via $\mathbf{F}_{\text{ext},A}$. We can think of these forces as acting to change or maintain the configuration of the $N$ particle system of charges.

Incidentally, here we should note that our description so far of the system of $N$ charged particles includes a classical physics model of a gas of neutral atoms within a container of volume $V$. For example, if $N$ is an even number, and half of the $N$ charges each have charge -e and a mass equal to that of an electron, while the other half have charge $+e$ and the mass of a proton, then we have the means for describing a gas of $N/2$ classical hydrogen atoms within a container. Of course, we do not yet know whether SED will indeed predict that a relatively stable classical model of a hydrogen atom, aside from some expected ionization probability at nonzero temperatures, will result from the above classical electron and proton charged particles interacting with the appropriate classical electromagnetic thermal radiation.[28] However, our purpose here is to include this possibility within our general description, with the realization that this problem has not yet been solved, but also recognizing that some insight can be gained by simply writing out the full problem.



By summing over Eq. (5) for all $N$ particles and by using Eqs. (11)–(15), then

$$\sum_{A=1}^{N}\dot{\mathbf{z}}_A\cdot\mathbf{F}_{\text{ext},A} = \frac{d}{dt}\left[\sum_{A=1}^{N}\left(m_A\gamma_A c^2 - \frac{q_A^2}{3c}\frac{d}{dt}(\gamma_A^2)\right) + \int_V d^3x\, u'_{\text{E\&M}}\right] + \oint_S d^2x\,\hat{\mathbf{n}}\cdot\mathbf{S}'_{\text{E\&M}}\,, \qquad (16)$$

where

$$u'_{\text{E\&M}} = \frac{1}{8\pi}\left[\sum_{A=1}^{N}(\mathbf{E}_{a,A}^2 + \mathbf{B}_{a,A}^2) + \sum_{\substack{A,B=1\\(A\neq B)}}^{N}(\mathbf{E}_A\cdot\mathbf{E}_B + \mathbf{B}_A\cdot\mathbf{B}_B) + 2\sum_{A=1}^{N}(\mathbf{E}_A\cdot\mathbf{E}_{\text{in}} + \mathbf{B}_A\cdot\mathbf{B}_{\text{in}})\right], \qquad (17)$$

$$\mathbf{S}'_{\text{E\&M}} = \frac{c}{4\pi}\left[\sum_{A=1}^{N}(\mathbf{E}_{a,A}\times\mathbf{B}_{a,A}) + \sum_{\substack{A,B=1\\(A\neq B)}}^{N}(\mathbf{E}_A\times\mathbf{B}_B) + \sum_{A=1}^{N}(\mathbf{E}_A\times\mathbf{B}_{\text{in}} + \mathbf{E}_{\text{in}}\times\mathbf{B}_A)\right]. \qquad (18)$$

The physical significance of $u'_{\text{E\&M}}$ and $\mathbf{S}'_{\text{E\&M}}$ can be understood more clearly by writing their expressions differently. Let

$$(\mathbf{E}_A^2)_{(-3} \equiv \mathbf{E}_A^2 - (\mathbf{E}_{a,A})^2\,, \qquad (19)$$

$$(\mathbf{B}_A^2)_{(-3} \equiv \mathbf{B}_A^2 - (\mathbf{B}_{a,A})^2\,, \qquad (20)$$

$$(\mathbf{E}_A\times\mathbf{B}_A)_{(-3} \equiv \mathbf{E}_A\times\mathbf{B}_A - \mathbf{E}_{a,A}\times\mathbf{B}_{a,A}\,. \qquad (21)$$

The notation of $(-3$ used above again follows the notation in Ref. 16: quantities labeled with $(-3$ contain terms proportional to $|\mathbf{x} - \mathbf{z}(t_r)|^{-n}$ for $n \geq 3$. From Eqs. (1), (2), and (17)–(21),

$$u'_{\text{E\&M}} = \frac{1}{8\pi}\left[(\mathbf{E}_{\text{tot}})^2 + (\mathbf{B}_{\text{tot}})^2 - \sum_{A=1}^{N}(\mathbf{E}_A^2 + \mathbf{B}_A^2)_{(-3} - (\mathbf{E}_{\text{in}}^2 + \mathbf{B}_{\text{in}}^2)\right], \tag{22}$$

$$\mathbf{S}'_{\text{E\&M}} = \frac{c}{4\pi}\left[\mathbf{E}_{\text{tot}} \times \mathbf{B}_{\text{tot}} - \sum_{A=1}^{N}(\mathbf{E}_A \times \mathbf{B}_A)_{(-3} - (\mathbf{E}_{\text{in}} \times \mathbf{B}_{\text{in}})\right]. \tag{23}$$

Thus, $u'_{\text{E\&M}}$ and $\mathbf{S}'_{\text{E\&M}}$ represent the total electromagnetic energy density and energy flow, respectively, minus (1) the terms associated with the incident fields and (2) the $(-3$ parts of the $N$ inhomogeneous particle fields. The reasons for these subtractions can be understood by the following arguments. First, regarding the free incident field terms in Eqs. (22) and (23), these terms could either be subtracted out or left in the total energy-rate equation because they separately satisfy another energy-rate equation:

$$\frac{d}{dt}\left[\frac{1}{8\pi}\int_V d^3x(\mathbf{E}_{\text{in}}^2 + \mathbf{B}_{\text{in}}^2)\right] + \frac{c}{4\pi}\oint_S d^2x\hat{\mathbf{n}}\cdot[\mathbf{E}_{\text{in}} \times \mathbf{B}_{\text{in}}] = 0. \tag{24}$$

Second, regarding the $(-3$ terms, these quantities are correctly subtracted out in Eqs. (22) and (23) because they represent the (singular) bound electromagnetic energy quantities that are closely tied in with the observable kinetic energy of the particles. These parts are taken into account in Eq. (16) by the first term on the right-hand side, which contains the Schott energy terms as well as the kinetic energies of the particles, with $m_A$ being the renormalized mass.

A somewhat subtle point remains, however, concerning the $(-3$ terms. The mass renormalization was assumed to be carried out by integrating each $(-3$ energy term over all of space, rather than simply the volume $V$, and by then combining each term with the corresponding bare kinetic energy term. Since an energy-rate equation is desired concerning the volume $V$, then a difficulty arises here. It should be noted, however, that since $\partial_\beta t^{\alpha\beta}_{(-3}(x) = 0$ for all points $\mathbf{x}$ lying outside of $V$,[15,16] then the flow of electromagnetic energy through $S$ due to each $(-3$ energy term will satisfy a separate energy-rate equation:

$$-\frac{d}{dt}\left[\frac{1}{8\pi}\oint_{\tilde{V}} d^3x(\mathbf{E}_A^2 + \mathbf{B}_A^2)_{(-3}\right] + \frac{c}{4\pi}\oint_S d^2x\hat{\mathbf{n}}\cdot[\mathbf{E}_A \times \mathbf{B}_A]_{(-3} = 0, \tag{25}$$



where $\tilde{V}$ represents all of space excluding the volume $V$, and $\hat{\mathbf{n}}$ represents the unit surface normal vector pointing out of $V$ and into $\tilde{V}$. (The missing surface term at infinity equals zero since the integrand varies as $|\mathbf{x}|^{-n}$ for $n \geq 3$; similarly, both terms in Eq. (25) go to zero as $V$ becomes large.)

In summary, three energy-rate equations have been identified: namely, Eqs. (16), (24), and (25). Only Eq. (16) involves work done by external forces; Eqs. (24) and (25) do not. Using these equations, energy-balance relationships will now be established.

By defining the following quantities:

$$U'_{\text{int}}(t) \equiv \sum_{A=1}^{N}\left[m_A\gamma_A c^2 - \frac{q_A^2}{3c}\frac{d}{dt}(\gamma_A^2)\right] + \int_V d^3x u'_{\text{E\&M}}, \tag{26a}$$

$$U_{\text{in}}(t) \equiv \frac{1}{8\pi}\int_V d^3x(\mathbf{E}_{\text{in}}^2 + \mathbf{B}_{\text{in}}^2), \tag{26b}$$

$$\tilde{U}_{A(-3}(t) \equiv \frac{1}{8\pi}\int_{\tilde{V}} d^3x(\mathbf{E}_A^2 + \mathbf{B}_A^2)_{(-3}, \tag{26c}$$

$$U_{\text{int}}(t) \equiv U'_{\text{int}}(t) + U_{\text{in}}(t) - \sum_{A=1}^{N}\tilde{U}_{A(-3}(t), \tag{26d}$$

$$W \equiv +\int_{t_I}^{t_{II}} dt \sum_{A=1}^{N}\dot{\mathbf{z}}\cdot\mathbf{F}_{\text{ext},A}, \tag{27}$$

$$Q' \equiv -\int_{t_I}^{t_{II}} dt \oint_S d^2x\hat{\mathbf{n}}\cdot\mathbf{S}'_{\text{E\&M}}, \tag{28a}$$

$$Q_{\text{in}} \equiv -\int_{t_I}^{t_{II}} dt \oint_S d^2x\hat{\mathbf{n}}\cdot\frac{c}{4\pi}(\mathbf{E}_{\text{in}} \times \mathbf{B}_{\text{in}}), \tag{28b}$$

$$Q_{A(-3} \equiv -\int_{t_I}^{t_{II}} dt \oint_S d^2x\hat{\mathbf{n}}\cdot\frac{c}{4\pi}(\mathbf{E}_A \times \mathbf{B}_A)_{(-3}, \tag{28c}$$

$$Q \equiv Q' + Q_{\text{in}} + \sum_{A=1}^{N}Q_{A(-3}, \tag{28d}$$

one obtains the three energy-balance relationships

$$U'_{\rm int}(t_{\rm II}) - U'_{\rm int}(t_{\rm I}) = Q' + W \ , \tag{29a}$$

$$U_{\rm in}(t_{\rm II}) - U_{\rm in}(t_{\rm I}) = Q_{\rm in} \ , \tag{29b}$$

$$-\left[\tilde{U}_{A(-3}(t_{\rm II}) - \tilde{U}_{A(-3}(t_{\rm I})\right] = Q_{A(-3} \ , \tag{29c}$$

as well as their sum,

$$U_{\rm int}(t_{\rm II}) - U_{\rm int}(t_{\rm I}) = Q + W \ . \tag{29d}$$

Equation (29d) is the main result we are interested in here, as this equation will form our basis for discussing the first and second laws of thermodynamics for this system of particles and electromagnetic fields. However, the other energy-rate equations are also of interest, since they give us an insight into the dynamics involved in this electrodynamic system.

As can be immediately verified from Eqs. (23) and (28a)–(28d), $Q$ in Eq. (28d) equals the electromagnetic energy that flows *into* $V$ due to $\mathbf{E}_{\rm tot}$ and $\mathbf{B}_{\rm tot}$ in Eqs. (1) and (2); i.e.,

$$Q = + \int_{t_{\rm I}}^{t_{\rm II}} dt \oint_S d^2\chi(-\hat{\mathbf{n}}) \cdot \frac{c}{4\pi}(\mathbf{E}_{\rm tot} \times \mathbf{B}_{\rm tot}) \ . \tag{30}$$

Moreover, from Eqs. (22) and (26a)–(26d), the internal energy $U_{\rm int}$ equals the total kinetic plus the electromagnetic energy of the system of $N$ particles and the incident fields, where the electromagnetic energy is calculated only over the volume $V$ that contains the particles. Consequently, the minus sign that appears with the $U_{A(-3}$ terms in Eq. (26d) accounts for the fact that $U'_{\rm int}$ was computed by integrating the (−3 energy terms over all of space rather than just over $V$. Due to mass renormalizations, the first and third terms in Eq. (26d) do not contain the singularities of the single particle energies

$$\frac{1}{8\pi}\int_V d^3x(\mathbf{E}_A^2 + \mathbf{B}_B^2) \ .$$

Consequently, Eq. (26d) represents a useful means for calculating changes in internal energy. A very similar expression to this one was used in Ref. 8 for calculating the changes in internal energy of $N$ electric dipole oscillators,



where each oscillator was modeled by a $+e$ point charge oscillating within a $-e$ uniform sphere of charge.

Although Eq. (26d) is useful for specific calculations, a somewhat more intuitive representation of $U_{\rm int}$ is to write it in the form

$$U_{\rm int} = \left\{ \sum_{A=1}^{N} \left[ m_A \gamma_A c^2 - \frac{q_A^2}{3c}\frac{d}{dt}(\gamma_A)^2 \right] \right\} + \left\{ \frac{1}{8\pi}\int_V d^3x \left[\mathbf{E}_{\rm tot}^2 + \mathbf{B}_{\rm tot}^2\right] - \sum_{A=1}^{N} \frac{1}{8\pi}\int_{\bar{V}+V} d^3x \left[\mathbf{E}_A^2 + \mathbf{B}_A^2\right]_{(-3} \right\} \ . \tag{31}$$

The last term above involving the (-3 terms is integrated over all space. This term is subtracted from the other terms since it represents the singular bound electromagnetic energy that has already been used to obtain the observable finite masses of the point charges, as well as the Schott energy terms. These last terms are contained in the first quantity in curly brackets in Eq. (31); this quantity represents the sum over all particles of the material energy plus bound electromagnetic energy for each particle.

Thus, the "bound" electromagnetic energy of a particle is not fully contained in any finite volume $V$ that the particle moves within; after all, electromagnetic energy due to the (−3 terms does indeed leave $V$ in accordance with Eq. (29c). If $V$ is taken to be quite large, however, so that the particles do not pass near the surface $S$ between times $t_{\rm I}$ and $t_{\rm II}$, then $\tilde{U}_{A(-3}$ and $Q_{A(-3}$ become negligible quantities and contribute insignificantly to $U_{\rm int}$ and $Q$, respectively. The (−3 electromagnetic energy may then be considered as being essentially contained totally within $V$. Of course, in order to make this approximation for the model of a gas of "neutral" atoms within a container, as described earlier, then we must take $V$ larger than the container, since some of the atoms will be colliding with and bouncing off the walls.[n]

Thus, Eq. (29d) yields the desired intuitive result that is expressed quantitatively here: the change in the sum of the kinetic and electromagnetic energies in volume $V$ equals (1) the electromagnetic energy that flows *into* $V$ plus (2)

[n] See the interesting comments in Ref. 41 by Boyer on the effect of a wall upon a particle's motion.

the work $W$ by the external forces on the $N$ particles. Moreover, what has also been identified is that the net energy-balance relationship expressed by Eq. (29d) is actually a sum of three separate energy-balance relationships: Eqs. (29a)–(29c). We will discuss the significance of these separate relationships in the following section. These results apply for charged point particles undergoing arbitrary relativistic motion within $V$. (It should be noted that if the particles leave the volume $V$ between times $t_I$ and $t_{II}$, then these results need to be generalized to take this fact into account.) These identities form a natural basis for examining thermodynamic properties, which we will now briefly discuss.

## 3. Thermodynamic Considerations and the Connection to Earlier Work

The first law of thermodynamics deals with the expectation value of the change in energy of a system. Consequently, Eq. (29d) should be used in obtaining a statistical interpretation of the first law of thermodynamics for a system of $N$ classical charged point particles interacting with classical electromagnetic radiation. In particular, if $\mathbf{E}_{in}$ and $\mathbf{B}_{in}$ are characterized by a stochastic process, as would occur for radiation in thermal equilibrium with the particles, then the interaction of these fields with the particles will result in the motion of the latter obeying a stochastic process. Upon taking the expectation value of Eq. (29d), the first law of thermodynamics is obtained, where[a] $\mathrm{Q} = \langle Q \rangle$ represents the heat flowing into $V$ in the form of random electromagnetic energy, $\mathrm{W} = \langle W \rangle$ equals the expectation value of the work by external forces on the $N$ particles, and $\Delta\ \mathrm{U}_{\mathrm{int}} = \langle U_{\mathrm{int}} \rangle|_{t_I}^{t_{II}}$ represents the change in internal energy associated with these particles and the thermal radiation. Here, probably the simplest way to think of the expectation value is via an ensemble average of similar systems.

We should note that although three separate energy rate equations, namely, Eqs. (29a)–(29c), plus their sum, Eq. (29d), have been identified, only the latter equation is in general appropriate for the first law of thermodynamics when both particles and radiation are present. For example, suppose that thermal radiation energy flows into the volume $V$, such as occurs when raising the temperature of the walls of a cavity of volume $V$. Equation (29b) describes this total flow of energy into $V$, providing no charged point particles are present in $V$. However, if particles are in $V$, they will also be radiating energy during this operation, which means that Eqs. (29a) and (29c) need to be included as well in the total energy description, as is expressed by Eq. (29d). Equations (28d) or (30) describe the total flow of energy into $V$ for this situation.

[a] The angular brackets indicate expectation value.



The relationships we have obtained so far complement and support the work in Refs. 8 and 9. Specifically, Refs. 8 and 9 calculated the expectation value of the work done and the change in internal energy due to slowly displacing a set of $N$ electric dipole oscillators, or due to slowly changing the temperature of the thermal radiation. From these two quantities the heat $\langle Q \rangle$, in the form of electromagnetic energy, was determined that flowed into a large volume $V$ containing the $N$ oscillators.

However, Refs. 8 and 9 did not explicitly show that the expressions used for the work done and the change in the internal energy were consistent with $Q$ as expressed in terms of Poynting's vector, and as given here in Eq. (30). The present analysis helps to remedy this situation. References 8 and 9 modeled an electric dipole oscillator as a $+e$ point charge oscillating within a $-e$ uniform sphere of charge, and considered the case where the $-e$ spheres of charge were quasistatically displaced. For a surface $S$ far from the oscillators, which was the condition examined in Refs. 8 and 9, these slowly moving spheres of charge make a negligible contribution to the average flow of electromagnetic energy out of or into $V$. Consequently, with only some small modifications to our present analysis to account for the presence of the $-e$ extended spheres of change, the present analysis can be made to carry over to the case of the dipole oscillators in Refs. 8 and 9. Further details on this connection are given in the Appendix.

Returning to our case of $N$ charged point particles and thermal radiation, suppose we want to repeat some of the thermodynamic analysis considered in Refs. 8–11 for our present system. For example, in Refs. 8, 10, and 11 the requirement of no heat flow at $T = 0$ during quasistatic displacement operations enabled the functional form of the $T \to 0$ thermal radiation spectrum to be obtained. In Refs. 9 and 10 the demand that the second law of thermodynamics must hold resulted in a generalized Wien's displacement law, where the "generalized" qualification means that this law was shown to hold under the condition where the thermal spectrum does *not* vanish as $T \to 0$.

If we wish to apply a similar set of analyses to the present system of $N$ charged point particles, what sort of thermodynamic operations might we consider? One natural example is a compression of a volume containing the model of the gas of neutral atoms considered in Sec. 2.5. Here, the atoms consist of positively charged "classical nuclei" and negatively charged "classical electrons" interacting with thermal radiation. Let us assume a container exists to confine the atoms to some region of space. Slowly compressing or expanding the container enables a thermodynamic operation to be considered that parallels those operations discussed in Refs. 8–11. Here, the atoms are free to move in widely varying trajectories within the container, while acted upon by the thermal radiation fields, the fields of the other particles, and by the electromagnetic forces due to the atoms within the walls of the container. Likewise, the charged point particles discussed in Refs. 8, 9 and 11 were free to follow oscillating trajectories within the $-e$ spheres of charge, while acted upon by the thermal radiation fields, the fields of the other dipole particles, and the binding harmonic force within each dipole oscillator. Although large fluctuating behavior occurs for both of these systems, namely, (1) the gas of neutral atoms discussed here and (2) the oscillating charges in the electric dipole oscillators, the connection to thermodynamics can be made by quasistatically varying one or more parameters that affect the trajectories of these particles. In Ref. 8, this parameter was the relative positions of the $-e$ spheres of charge; here, it could be the position of the walls of the container.

Many other thermodynamic operations can also be conceived of that involve the present system of classical charged point particles. For example, suppose we consider a simple model of a hydrogen atom, where: $N = 2$, one particle has charge $+e$, the other has charge $-e$, and the $+e$ charge is held fixed in some inertial frame. Slowly turning on an electric field will change the average position of the $-e$ oscillating particle with respect to the $+e$ charge. Thus, a net average amount of work will have been done on the $+e$ charge. Likewise, we can expect that the average internal energy of the system will change, and that a net average amount of energy will flow into or out of some large volume containing the atom.

Since the examples above involved heat flow, then in principle the second law of thermodynamics could be used to obtain the change in entropy between two thermodynamic states $i$ and $f$. If the fields and particles in $V$ are in near thermal equilibrium and characterized by a temperature $T$, then the change in caloric entropy will be given by

$$S_{\text{cal}\,f} - S_{\text{cal}\,i} = \int_{i\atop R}^{f} \frac{dQ}{T}\,. \tag{32}$$



Here, $R$ indicates that a reversible thermodynamic process is followed that connects the two states.

Nevertheless, unlike the work in Refs. 8–11, we cannot at present carry out the above suggestions for thermodynamic analysis. The problem has to do with the fact that the statistical equilibrium configuration of the above classical atoms has not yet been solved;[28] e.g. we do not yet know if such a classical hydrogen atom can exist without the electron (1) "spiraling" into the nucleus, or (2) "spiraling" off to infinity (ionizing). In contrast, for a nonrelativistic charged particle acted upon by a binding force due to a simple harmonic oscillator potential, a statistical equilibrium configuration does exist between thermal radiation and the particle. Hence, here quasistatic changes to the system can be investigated to examine thermodynamic properties associated with the oscillator. At present, we are not yet at this stage of analysis for the physically more realistic problem of the classical hydrogen atom.

Consequently, we are restricted to making only a few qualitative comments about the thermodynamics of such systems as mentioned above, if they do indeed form dynamical equilibrium states. Perhaps these comments will help to remove possible confusion on the thermodynamics being discussed here, which is similar to, yet so different from the *usual* discussion of the thermodynamics of classical physical systems, where one assumes that at $T = 0$ all fluctuations in fields and motion vanish.

To clarify some points, we should note that in connection with the gas of neutral classical atoms mentioned above, Refs. 39, 41, and 67 contain relevant analysis on the important thermodynamic role of thermal radiation that is nonzero as $T \to 0$, as well as the important role of the walls of the container for such a system. The neutrally charged classical atoms discussed in these references are again modeled by electric dipole harmonic oscillators, so that this aspect of the analysis is not an accurate description of a true gas of atoms found in nature. However, the role of the ZP radiation, and the role of the walls of the container, are discussed in some detail there.

Also, we can certainly comment on the meaning of the "incident" radiation. The conventional definition for incident radiation is that radiation which satisfies the homogeneous Maxwell's equations, and is present at time $t \to -\infty$.[p] For much of the discussed thermodynamic operations, such as the quasistatic

[p] See, for example, Ref. 66, pp. 225–226, or for more discussion, see the article by S. Coleman in Chap. 6 of *Electromagnetism: Paths to Research*, edited by D. Teplitz (Plenum, New York, 1982). Also see the initial discussion in Ref. 1.

displacement operations in Ref. 8, for individual dipole oscillators, this meaning is perfectly fine. Here, the incident radiation is just the thermal radiation. However, when changes in temperature are made, then this meaning needs to be clarified. For example, in Ref. 9 the change in entropy due to changing the temperature was calculated by following the usual prescription given in thermodynamics:[f] specifically, the temperature of a thermodynamic system can be changed quasistatically by imagining the system to be placed in contact with an infinite series of heat reservoirs ranging in temperature from $T_i$ to $T_f$. For the system of electric dipoles in Ref. 9, the thermal radiation acted as the heat reservoir. Hence, a quasistatic change in temperature of the system of dipole oscillators was carried out by slowly varying the temperature of the thermal radiation.

Now, clearly this procedure makes no sense if we think of the thermal radiation as being incident radiation, since once incident radiation is specified in the distant past, it can then be determined for all time. If incident radiation is also thermal radiation, it's statistical properties will be governed by the single temperature $T$, which we are then not at liberty to arbitrarily change. Of course we can add radiation to the incident radiation by having charges radiate energy while accelerating, but we cannot alter the *incident* radiation *itself* in this way.

To make sense of this situation, one must realize that the procedure of quasistatically changing the temperature of a thermodynamic system by placing the system in contact with an infinite series of heat reservoirs, is but a thought experiment. If one attempted to mimic this procedure in practice, such as by changing the temperature $T$ of a system to $T + \Delta T$, a considerable amount of physics would be involved in the procedure that is typically not discussed, such as the flow of heat within the system before equilibrium sets in, and the length of time before all parts of the system are essentially at the same temperature $T + \Delta T$. Although these questions are certainly legitimate ones to ask, they are not typically addressed in discussing such "quasistatic" thermodynamic operations.

Just as the physics of "an infinite series of heat reservoirs" are not examined in the standard practices of thermodynamics, in Refs. 9 and 10 we do not pretend to examine the detailed physics involved with changing the properties

[f] See, for example, F. W. Sears and G. L. Salinger, *Thermodynamics, Kinetic Theory, and Statistical Thermodynamics*, 3rd ed. (Addison Wesley, Reading, Massachusetts, 1975). p. 131.



of the thermal radiation. When the temperature of a system, such as in Ref. 9, is changed by a small amount, we are simply approximating what one would do in the laboratory to achieve this result, by imagining the problem to begin all over again from the start. The new "incident" thermal radiation has similar stochastic properties to the old "incident" radiation, but it is now parameterized by $T + \Delta T$, rather than $T$. In this sense the term "incident" is perhaps unfortunate, but providing we understand precisely the assumptions involved in this procedure, there should be no confusion. In changing the temperature by $\Delta T$, we are not examining the time period of our system when it is slightly out of equilibrium. Instead, we are only examining the initial and final points of this operation, when all parts of the system are essentially in equilibrium with each other, and when the system can be approximately described by, yes, incident radiation in thermal equilibrium with the classical electrodynamic system in question. By then treating $\Delta T$ as being infinitesimally small, and by considering an infinite series of distinct thermal equilibrium states, properties can then be deduced from these changes in equilibrium, such as the change in entropy obtained in Refs. 9 and 10.

## 4. Concluding Remarks

The present article first presented a brief overview of the theory of SED and then discussed some recent work in SED involving the thermodynamics of classical electrodynamic systems. This work was then partially extended to aid in eventually addressing questions concerning the behavior of a possible thermodynamic equilibrium between a system of classical charged point particles and classical electromagnetic thermal radiation. Such a system of charged particles is a more realistic one to find in nature than the electric dipole harmonic and anharmonic oscillator models typically discussed in SED; however, a solution to this more realistic problem is still missing. Some insight into the dynamics of such a system was obtained by setting up the problem in a similar form to the analysis in Refs. 8–11. The calculations presented in these last references for the change in internal energy, work done, heat flow, and change in entropy, represent the type of analysis that is essential in deducing the appropriate thermodynamic behavior of classical electrodynamic systems in general.

## Appendix: Connection to Earlier Work

Here a connection will be made between the energy rate relationship obtained in Sec. 2 for charged point particles, and the analysis carried out in Refs. 8, 9, and 11 for $N$ electric dipole oscillators. The connection involves simply applying the conservation of energy. First, in terms of changes in kinetic energy, the $-e$ spheres of charge make no contribution, since they are displaced quasistatically. However, in terms of electromagnetic energy, these spheres do make a contribution. As the oscillator models are displaced from each other, the average position of the oscillating $+e$ charges within each $-e$ sphere will change, thereby probing a different effective potential energy region due to each $-e$ sphere. Consequently, the total internal energy is just

$$U_{\text{int}} = \sum_{A=1}^{N} (m^* \gamma_{+A} c^2) + \frac{1}{8\pi} \int_V d^3x (\mathbf{E}_{\text{tot}}^2 + \mathbf{B}_{\text{tot}}^2) , \tag{A1}$$

where $m^*$ is the bare mass of an oscillating $+e$ charge, and where now

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{in}} + \sum_{A=1}^{N} (\mathbf{E}_{+A} + \mathbf{E}_{-A}) , \tag{A2}$$

$$\mathbf{B}_{\text{tot}} = \mathbf{B}_{\text{in}} + \sum_{A=1}^{N} (\mathbf{B}_{+A} + \mathbf{B}_{-A}) . \tag{A3}$$

Here, $\mathbf{E}_{+A}$ and $\mathbf{E}_{-A}$ are the retarded electric fields due to the $+e$ point charge and the $-e$ extended spherical charge of the $A$ labelled electric dipole model, respectively, and similarly for $\mathbf{B}_{+A}$ and $\mathbf{B}_{-A}$.

The analysis in Sec. II of Ref. 8 uses the same renormalization procedure for the mass of the point charges as discussed in the present article. Following the analysis in this reference enables the internal energy in Eq. (A1) to be put into a form suitable for explicit calculations of changes in expectation values. Thus, we can write $U_{\text{int}}$ as in Eq. (31) in the present article, but where $\mathbf{E}_{\text{tot}}$ and $\mathbf{B}_{\text{tot}}$ are now given by Eqs. (A2) and (A3). The simple harmonic oscillator potential that each $(+e)$-$A$ point charge experiences due to the $(-e)$-$A$ spheres, is obtained via Eq. (16) in Ref. 8.

Turning to the work done during quasistatic displacements of the electric dipole oscillator models in Refs. 8 and 9, there were no applied external forces acting on the $+e$ point charges. Instead, the external forces were applied to the



$-e$ spheres to hold them steady, and to slowly displace them. Consequently, the work done on this system of $N$ dipole oscillator models was due entirely to

$$W = \int_{t_I}^{t_{II}} dt \sum_{A=1}^{N} \dot{\mathbf{Z}}_A(t) \cdot \mathbf{F}_{\text{ext},A}(t) , \tag{A4}$$

where $\dot{\mathbf{Z}}_A(t)$ represents the velocity of the quasistatically displaced $(-e)$-$A$ sphere of charge, and $\mathbf{F}_{\text{ext},A}$ is the external force applied to this sphere. Here, all points in each $(-e)$-$A$ sphere of charge are assumed to move at the same velocity $\dot{\mathbf{Z}}_A(t)$ in the reference frame at which the spheres start and end at rest at times $t_I$ and $t_{II}$.

Finally, as for the electromagnetic energy that flows into or out of $V$ during a quasistatic displacement, the fields of the quasistatically displaced $(-e)$-$A$ spheres will contribute negligibly to the average radiated energy for a surface $S$ taken to be sufficiently far removed from any of the dipole oscillators. Consequently, our net energy rate equation for the dipole oscillators is not very different from the net energy rate equation obtained here for charged point particles: namely, Eq. (29d). Specifically, for the dipole oscillators we obtain

$$U_{\text{int}}(t_{II}) - U_{\text{int}}(t_I) = \int_{t_I}^{t_{II}} dt \oint_S d^2x(-\hat{\mathbf{n}}) \cdot \frac{c}{4\pi}(\mathbf{E}_{\text{tot}} \times \mathbf{B}_{\text{tot}}) + \int_{t_I}^{t_{II}} dt \sum_{A=1}^{N} \dot{\mathbf{Z}}_A \cdot \mathbf{F}_{\text{ext},A}(t) , \tag{A5}$$

where

$$U_{\text{int}} = \left\{ \sum_{A=1}^{N} \left[ m\gamma_{+A}c^2 - \frac{e^2}{3c}\frac{d}{dt}(\gamma_{+A})^2 \right] \right\} + \left\{ \frac{1}{8\pi} \int_V d^3x \left[\mathbf{E}_{\text{tot}}^2 + \mathbf{B}_{\text{tot}}^2\right] - \sum_{A=1}^{N} \frac{1}{8\pi} \int_{V+\hat{V}} d^3x \left[\mathbf{E}_{+A}^2 + \mathbf{B}_{+A}^2\right]_{(-3)} \right\} . \tag{A6}$$

Here, $m$ is the experimentally measurable mass of the $+e$ oscillating charges, and $\mathbf{E}_{\text{tot}}$ and $\mathbf{B}_{\text{tot}}$ are given in Eqs. (A2) and (A3). In the first term on the right-hand side of Eq. (A5), the fields from the $(-e)$-$A$ spheres can be ignored when calculating the expectation value of the electromagnetic energy flowing into $V$ through a surface $S$ far removed from the oscillators. Equation (A6) agrees with Eq. (10) in Ref. 8. Section II in Ref. 8 reduces this result further to enable a calculation of changes in the expectation value of $U_{\rm int}$ to be carried out.

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