Liquidity Hoarding[1]

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29 August, 2011

[1]The views expressed here are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of New York or the Federal Reserve System. We would like to thank Franklin Allen, Gadi Barlevy, Marco Bassetto, Markus Brunnermeier, Xavier Gabaix, Stephen Morris, Hyun Shin, Jeremy Stein, Anjan Thakor, Dimitri Vayanos, Vish Viswanathan, Wei Xiong and seminar participants at the New York Fed, Board of Governors, London School of Economics, Chicago Fed, European Summer Symposium in Economic Theory (ESSET) in Gerzensee, Summer Workshop on Money, Banking, Payments and Finance at the Chicago Fed, Sabanci University, Seoul National University, Southern Methodist University and the University of Texas at Dallas. Peter Hull provided excellent research assistance. All errors remain our own.

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Abstract

Banks hold liquid and illiquid assets. An illiquid bank that receives a liquidity shock sells assets to liquid banks in exchange for cash. We characterize the constrained efficient allocation as the solution to a planner's problem and show that the market equilibrium is constrained inefficient, with too little liquidity and inefficient hoarding. Our model features a precautionary as well as a speculative motive for hoarding liquidity, but the inefficiency of liquidity provision can be traced to the incompleteness of markets (due to private information) and the increased price volatility that results from trading assets for cash.

J.E.L. Classification: G12, G21, G24, G32, G33, D8.

Keywords: Interbank market, fire sale, market freeze, cash-in-the-market pricing.

1 Introduction

One of the most interesting phenomena marking the recent financial crisis was the "freezing" of the interbank market. As early as the fall of 2007, following the collapse of the market for asset backed commercial paper, European banks reported difficulty borrowing in the interbank market. At the same time, interbank borrowing rates reached record levels. Difficulty obtaining liquidity in interbank markets was subsequently experienced in many countries. As a result, central bank borrowing facilities became an essential source of liquidity for financial institutions.[1]

One possible explanation for this phenomenon is counter-party risk. Because of the widespread exposure to sub-prime asset-backed securities, banks had good reason to be wary of lending to any bank that might be a credit risk because of this or any other exposure. In this paper we explore a second possible explanation, that banks were *hoarding* liquidity because of fears that their own future access to liquidity might be compromised. There is substantial evidence that banks reduced their lending to other banks in order to build up cash positions (Acharya and Merrouche, 2009; Heider, Hoerova and Holthausen, 2008; Ashcraft, McAndrews and Skeie, 2008). The two possible explanations are not unrelated. Even if the interbank market initially froze because of counter-party risk, liquidity hoarding would still be a rational response to fears of future lack of access to liquidity.

In this paper, we use a simple model of liquidity management to analyze the efficiency of laisser faire equilibrium. Our model assumes a large number of "bankers," who can hold two types of assets, a liquid asset and an illiquid asset. We refer to the liquid asset as "cash" and refer to the illiquid asset simply as "the asset." Bankers are subject to stochastic liquidity shocks that we interpret as unanticipated demands for repayment of a senior debt. If a banker receives a liquidity shock and lacks the cash to meet the claim, he is forced to sell some of his holdings of the asset. If the demand for cash is high, the price of the asset may be correspondingly low. In equilibrium, bankers weigh the cost of holding cash against the

[1] See footnote 5 in Section 4.2 for a discussion of the liquidity facilities introduced by the Federal Reserve in the recent crisis.

cost of having to sell the asset at a "fire sale" price.

We begin our analysis by solving the problem of a planner who determines how much cash to hold and when to distribute it. The solution to the planner's problem takes a very simple form: after determining the efficient amount to hold at the first date, the planner supplies cash to every banker who needs it at a given date until the supply runs out. Even though there may be a future need for the cash, the planner never carries forward a positive balance as long as there is a banker who needs cash to meet a liquidity demand today.

The simple form of the solution to the planner's problem makes it easy to identify inefficient hoarding. Hoarding liquidity is inefficient if and only if it occurs, at some date, when there are still bankers who need liquidity. Our second result is to show that, in a laisser-faire market equilibrium, there is always (inefficient) hoarding. More precisely, when the demand for cash is sufficiently high some bankers will be priced out of the market for cash, while liquid bankers are hoarding cash rather than supplying it to the market.

A liquid banker has two reasons for hoarding cash. One is a *precautionary* motive. The banker may himself receive a liquidity shock in the future. If he uses his cash today and then receives a liquidity shock tomorrow, he can obtain cash by selling the illiquid asset, but the price may be very high. There is also a *speculative* motive. If the future demand for cash is very high, asset prices will be low. If he does not receive a liquidity shock, a hoarder may profit from buying assets at fire sale prices. Clearly, these two motives cannot be separated: the same cash holdings serve both motives.

The incentive to hoard cash come from the expected volatility of future asset prices and, in a laisser-faire equilibrium, the incentives to hoard are simply too high. Asset-price volatility results from the use of the asset market as a source of liquidity. When liquid bankers first supply cash in exchange for assets, they create an imbalance in the system. If these bankers are subsequently hit by a liquidity shock, they have even more assets to dump on the market, producing a greater fire sale and reducing asset prices further. This build up in volatility increases both the precautionary and speculative motives, which are responsible for inefficient hoarding.

As a thought experiment, we consider an alternative version of the model in which the assets purchased in exchange for cash at one date can be spun off into a special purpose vehicle (SPV) that is completely independent of the bank holding company. If the bank is threatened with default in a future period, it has no reason to dump the assets in the SPV, since these will survive the bank's default. If all banks pursue this strategy, future volatility will be lower than in the baseline model, the incentive to hoard is also lower, and there is no inefficient hoarding in equilibrium. Our thought experiment shows that the imbalance in bank portfolios only results in inefficient hoarding if banks become too big as well as too illiquid.

Our third result characterizes the optimal intervention by a central bank. A central bank is subject to more constrains than a central planner. A central planner has exclusive control of the allocation of liquidity. A central bank, by contrast, has to compete with markets in which cash and assets are exchanged. Generally speaking, the existence of markets is a problem because it provides bankers with arbitrage opportunities that might undermine the central bank's effort to improve welfare. In this case, however, the central bank can successfully implement the planner's solution. Because the central bank is a large player, it can influence the prices at which markets clear. The optimal strategy is for the central bank to accumulate and supply so much liquidity that private bankers are forced out of the market entirely. More precisely, the central bank makes liquidity cheap enough that none of the bankers wants to supply liquidity in competition with the central bank. In equilibrium no one, apart from the central bank, holds cash and every one relies for liquidity on the lender of last resort, who becomes in effect the lender of first resort.

We also explore a number of smaller interventions in the market for liquidity. One of these allows the central bank to control the total quantity of the liquid asset in the system, but leaves it up to the market to determine when and at what price this cash is used to purchase assets. We show that it is always optimal to increase the quantity of the liquid asset above the equilibrium level. A similar experiment allows the central bank to control the amount of liquidity hoarded while allowing bankers to determine freely the amount of liquidity in the system. We show that the central bank can always improve welfare by reducing the amount of inefficient hoarding, while allowing markets to clear at other dates. These results confirm our intuition about the sources of inefficiency in laisser-faire equilibrium, specifically, the inadequate incentive for banks to hold cash initially and the excessive incentive to hoard liquidity once liquidity shocks are realized.

The fundamental reason for the inefficiency of the laisser-faire equilibrium is the incompleteness of markets. Illiquid bankers are forced to acquire the liquid asset ex post by selling the illiquid asset on a spot market rather than entering into contingent contracts for the provision of liquidity ex ante. We argue that contingent contracts cannot improve on equilibrium welfare in the presence of asymmetric information. More precisely, if bankers cannot be forced to deliver the liquid asset when they have received a liquidity shock or, conversely, cannot be forced to receive the liquid asset when they have not received a liquidity shock, the possibility of arbitrage in spot markets plus private information about the liquidity shock rule out any gains from trade.

The rest of this paper is organized as follows. We begin our analysis in Section 2 by studying the constrained-efficient allocation chosen by a central planner who accumulates a stock of liquid assets and distributes them to the banks that report a need for liquidity. Then, in Section 3, we analyze a laisser-faire economy in which banks make their own decisions about liquidity accumulation and liquidity provision. In Section 4, we investigate the constrained (in)efficiency of the laisser-faire economy, and show that there are several simple interventions that can improve on the laisser-faire allocation. We conclude by discussing some variants of the model to shed more light on various sources of inefficiency in Section 5.

2 Constrained efficiency

In this section, we characterize the constrained-efficient allocation as the solution to a planner's problem in which the planner accumulates and distributes the liquid asset. The resulting allocation serves as a benchmark in our welfare analysis.

2.1 Primitives

Time: Time is divided into four dates, indexed by $t = 0, 1, 2, 3$. At the first date, bankers choose the amount of liquidity they hold as part of their portfolio. At the second and third dates, bankers receive liquidity shocks and trade assets in order to obtain the liquidity they need. At the final date, asset returns are realized.

Assets: There are two assets, a liquid asset that we refer to as *cash*, and an illiquid asset that we will refer to simply as the *asset*. Cash can be used to discharge debts and can be stored from period to period. *One unit of cash can be converted into one unit of consumption at any date*. The asset cannot be used to discharge debts (unless it is first exchanged for cash). The asset can be stored from period to period. One unit of the asset has a return of $R > 1$ units of consumption at date 3.

Bankers: There is a continuum of identical, risk neutral agents, indexed by $i \in [0, 1]$, whom we call *bankers*. Each bank has an initial endowment consisting of unit of the asset and one unit of cash at date 0, denoted by the vector $(1, 1)$, where the first and the second components represent the quantity of the asset and cash in bank's portfolio, respectively. The banker's utility function is

$$U(c_0, c_3) = \rho c_0 + c_3,$$

where c_0 denotes consumption at date 0 and c_3 denotes consumption at date 3 and $\rho > 1$ is a parameter. The interpretation of this utility function is the following: bankers prefer consumption at date 0 to consumption at date 3, so holding cash after date 0 (instead of converting it into consumption immediately) involves a cost $\rho - 1 > 0$.

Creditors: There is a continuum of identical, risk neutral agents, indexed by $j \in [0, 1]$, whom we call *creditors*. Each creditor j is owed a debt by bank $i = j$ that is payable on demand. The face value of the debt is one unit of cash. Creditors are uncertain about their time preferences. More precisely, they want to consume at precisely one of the dates $t = 1, 2, 3$ but uncertain which date they prefer. A typical creditor wants to consume at date

1 with probability θ_1, at date 2 with probability $(1-\theta_1)\,\theta_2$, and at date 3 with probability $(1-\theta_1)(1-\theta_2)$. The shocks θ_1 and θ_2 are random variables with density functions $f_1(\theta_1)$ and $f_2(\theta_2)$ and c.d.f. $F_1(\theta_1)$ and $F_2(\theta_2)$, respectively. We assume that θ_1 and θ_2 are *iid* with support $[0,1]$.

The creditor's expected utility function is given by

$$V(c_1,c_2,c_3) = E\left[\theta_1 c_1 + (1-\theta_1)\,\theta_2 c_2 + (1-\theta_1)(1-\theta_2)\,c_3\right],$$

where c_t denotes consumption at date $t = 1, 2, 3$.

Liquidity shocks: Bankers are said to receive a *liquidity shock* if the banker's creditor demands repayment at date 1 or date 2. If a banker is not hit by one of these shocks, he pays off his debt at $t = 3$, after the return from the asset is realized. A banker who receives a shock must immediately deliver one unit of cash to discharge the existing debt; otherwise he will be forced to default. If the banker becomes bankrupt, we assume that all his assets are immediately liquidated and, for simplicity, we assume that the liquidation costs consume the entire value of the assets. This assumption can be relaxed, but it greatly simplifies the analysis and does not appear to affect the qualitative results too much. In order to obtain cash, a banker can sell some or all of his holdings of the asset. Bankers who receive a liquidity shock at date 1 will not receive a liquidity shock at date 2.

2.2 The planner's problem

There are two groups of economic agents, bankers and creditors, but each group consists of ex ante identical agents at date 0. Since it is possible to make transfers between the two groups at date 3, we can redistribute the total surplus any way we like between the groups. So, in order to maximize ex ante welfare, it is necessary and sufficient to maximize total expected surplus. In what follows, we take this as the planner's objective function. In addition to the usual feasibility constraints, the planner operates subject to the constraint that he cannot transfer assets between bankers. If the planner were able to transfer assets, he would assign all assets at date 1 to bankers who had already received a liquidity shock, thus rendering the liquidity shocks at date 2 irrelevant. To avoid this trivial solution, we restrict the planner's actions to accumulating cash at date 0, distributing cash at dates 1 and 2, and redistributing the consumption good at date 3.

Suppose that the planner has m_1 units of cash at the beginning of date 2 and the state is (θ_1, θ_2). There are $(1-\theta_1)\,\theta_2$ bankers who receive a liquidity shock in this period. The optimal strategy is to supply the lesser of $(1-\theta_1)\,\theta_2$ and m_1 to the bankers in need of cash to discharge their debts. Each unit of cash is worth one unit of consumption, whether it is held by the planner or paid to a creditor and, in addition, each unit distributed to a banker with a liquidity need saves an asset worth R at date 3. So it is optimal to save as many assets as possible.

Now suppose the planner has m_0 units of cash at the beginning of date 1 and the state is θ_1. There are θ_1 bankers who receive a liquidity shock in this period. Each unit of cash distributed to these bankers is worth $1+R$, because one unit of cash always produces a return of one unit of consumption and it is worth an additional R units if it saves an asset. On the other hand, the expected value of a marginal unit of cash held until date 2 must be less than $1+R$. We have seen before that the value of cash is at most $1+R$ and it will be only 1 if the amount carried forward is greater than $(1-\theta_1)\,\theta_2$, which happens with positive probability if the amount carried forward is positive. So it is optimal to save as many assets as possible at date 1 and the optimal strategy is to distribute the lesser of m_0 and θ_1 at date 1.

At date 0, the choice of how much liquidity to hold is determined by equating the marginal cost of cash, ρ, to the marginal value of cash. As usual, a unit of cash held at the end of date 0 is always worth one unit of consumption but it is worth an additional R units if it can be used to save an asset. The probability that the marginal unit of cash is used to save an asset is simply the probability that m_0 is less than $\theta_1 + (1-\theta_1)\,\theta_2$. This probability is calculated to be

$$\Pr\left[\theta_1 + (1-\theta_1)\,\theta_2 > m_0\right] = 1 - \int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1,$$

so the marginal value of cash carried forward at date 0 is

$$R\left(1-\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1\right)+1.$$

The solution to the planner's problem is characterized by an array $(m_0, m_1(\theta_1), m_2(\theta_1,\theta_2))$, where $m_0 \geq 0$ is the amount of cash carried from date 0, $m_1(\theta_1)$ is the amount of cash carried forward from date 1 in state θ_1 and $m_2(\theta_1,\theta_2)$ is the amount of cash carried forward from date 2 in state (θ_1,θ_2). The previous argument leads to the following proposition.

Proposition 1 *The planner's optimal strategy is characterized by an array* $(m_0, m_1(\theta_1), m_2(\theta_1,\theta_2))$ *defined by the following conditions:*

$$m_2(\theta_1,\theta_2) = \max\{m_1(\theta_1) - (1-\theta_1)\theta_2, 0\};$$

$$m_1(\theta_1) = \max\{m_0 - \theta_1, 0\}$$

and

$$R\left(1-\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1(\theta_1)\, d\theta_1\right)+1=\rho.$$

Proof. See Appendix. ■

We have assumed so far that the planner has complete information about the banker's types. That is, he observes the realizations of θ_1 and θ_2 and knows which bankers have received a liquidity shock at each date. In the case where liquidity shocks are private information, the planner needs to use an incentive-compatible mechanism in order to extract information from the bankers.

A *direct mechanism* is defined by an array $(\mu_1(\theta_1), p_1(\theta_1), \mu_2(\theta_1,\theta_2), p_2(\theta_1,\theta_2))$, where $\mu_1(\theta_1)$ is the probability that an agent who reports a liquidity shock at date 1 in state θ_1 receives one unit of cash and $p_1(\theta_1)$ is the price he pays for it and $\mu_2(\theta_1,\theta_2)$ is the probability that an agent who reports a liquidity shock at date 2 in state (θ_1,θ_2) receives a unit of cash and $p_2(\theta_1,\theta_2)$ is the price he pays for it. An agent who reports no liquidity shock is assumed without loss of generality to receive no cash and make no payment.

We can show that the constrained efficient allocation that solves the planner's problem can be implemented as a truth-telling equilibrium of a direct mechanism. We postpone this exercise until Section 4.1, where it appears as a corollary of another, stronger result.

3 A laisser-faire economy

In this section, we provide an informal account of equilibrium in a laisser-faire economy. The formal definition and analysis are contained in Appendix A. The time line illustrated in Figure 1 shows the activities that occur in each of the four dates $t = 0, 1, 2, 3$. We describe these activities in more detail below.

—Figure 1 about here—

Date 0 Bankers are initially endowed with one unit of the asset and one unit of cash. At date 0, bankers choose whether to consume their cash immediately or retain one unit in their portfolios for future use. We call the bankers who retain the cash *liquid* and those who do not *illiquid*. Let $0 \leq \alpha \leq 1$ denote the measure of illiquid bankers. The α illiquid bankers end the period with a portfolio consisting of one unit of the asset and no cash. The $1 - \alpha$ liquid bankers end the period with a portfolio consisting of one unit of cash and one unit of the asset.

Date 1 At the beginning of date 1, a fraction θ_1 of bankers receive the liquidity shock. The liquid bankers who receive the shock can discharge their debt using their cash holdings and end the period with a portfolio consisting of one unit of the asset and no cash. The alternative is to default and lose everything. The illiquid bankers who receive a liquidity shock sell part of their asset holdings in exchange for cash to discharge their debt and end the period with a portfolio consisting of $1 - p_1$ units of the asset and no cash, where $p_1 \leqslant 1$ denotes the price of one unit of cash.[2] If some of these bankers cannot obtain cash to discharge their debt, they must be indifferent between obtaining cash and default. This will be the case if $p_1 = 1$.

[2] Our results do not change if we allow for forced sale of assets when banks cannot generate the one of unit cash needed to pay creditors, where the price of one unit of cash at date 1 and 2 can take arbitrarily high values. To keep the analysis short we do not report these results but they are available from the authors.

An alternative to asset sales is secured borrowing, in which illiquid bankers who receive a shock put up p_1 units of the asset as collateral against a loan of one unit of cash. The loan matures at date 3, at which point the banker either repays $p_1 R$ units of cash or forfeits the collateral. Notice that the interest rate on the loan is $p_1 R - 1$, which is high enough to make the banker indifferent whether he reclaims the collateral or not. Clearly, under these assumptions, secured lending is equivalent to asset sales.

The illiquid bankers who do not receive a shock do not trade and end the period with their initial portfolio consisting of one unit of the asset and no cash.[3]

The liquid bankers who do not receive a liquidity shock have the option of acquiring p_1 units of the asset using their one unit of cash. Liquid bankers who use their cash to purchase the asset are called *buyers*; those who do not are called *hoarders*. We denote by λ the fraction of these bankers that become buyers and end the period with a portfolio of $1+p_1$ units of the asset and no cash. The complementary fraction $1-\lambda$ become hoarders and end the period with their initial portfolio consisting of one unit of the asset and one unit of cash.

Date 2 Some of the bankers at date 2 have nothing to trade and remain inactive. The bankers who received a liquidity shock at date 1 have no cash and have no motive to trade the asset for cash since they cannot receive another liquidity shock. Similarly, the illiquid bankers who did not receive a liquidity shock at date 1 have no cash and have no motive to trade the asset for cash if they do not receive a liquidity shock. For the same reason, the buyers who do not receive a liquidity shock at date 2 have no cash and no motive to trade the asset for cash. Finally, the hoarders who receive a liquidity shock at date 2 will use their cash to discharge their debt and then have no gains from trade. This leaves three types of agents who can actively trade at date 2, the hoarders who do not receive a liquidity shock and the buyers and illiquid bankers who receive a liquidity shock at date 2. These agents trade cash for the asset at the market-clearing price p_2. The hoarders are willing to

[3] We will show that, in equilibrium, the price of cash at date 1 is equal to the expected price of cash at date 2. This is sufficient to prove that an illiquid banker cannot improve his payoff by purchasing cash at date 1.

supply all of their cash at any price $p_2 \geq R^{-1}$. The illiquid bankers, who hold one unit of the asset, are willing to supply the asset for one unit of cash at any price $p_2 \leq 1$ (because the alternative is to default). Similarly, the buyers, who hold $1 + p_1$ units of the asset, are willing to supply the asset for one unit of cash at any price $p_2 \leq 1 + p_1$.

Again, an alternative of the model is that banker's in need of liquidity engage in secured lending at date 2. In order to obtain one unit of cash, the banker has to put up p_2 units of collateral. At date 3, he is obliged to pay $p_2 R$ units of cash to discharge the debt and reclaim the collateral.

The allocation of assets in the first two dates is illustrated in Figure 2 and the allocation of assets at the end of date 2 is illustrated in Figure 3.

—Figure 2 about here—

—Figure 3 about here—

Date 3 At the last date, bankers receive the payoffs from the portfolios of cash and the asset carried forward from date 2. Bankers who have not already discharged their debts must pay their creditors one unit of cash.

The terminal payoffs, which are easily calculated from the terminal allocation, are illustrated in Figure 4.

—Figure 4 about here—

Throughout, we assume that the liquid asset is indivisible. However, all our results go through when we allow the liquid asset to be divisible. In particular, we can allow bankers to hold a fraction $\beta \in (0, 1)$ units of liquidity and consume the rest $1 - \beta$ at $t = 0$, and, if not hit by the liquidity shock at $t = 1$, use a fraction $\gamma \in (0, 1)$ of his liquidity to purchase assets at $t = 1$ while hoarding the rest, a fraction $1 - \gamma$ of his liquidity. We can show that such a strategy is not a profitable deviation from the equilibrium we construct below where we restrict $\beta \in \{0,1\}$ and $\gamma \in \{0,1\}$. [4]

3.1 Market clearing

In this section, we identify the market clearing prices p_1 and p_2, beginning at date 2 and working back to date 1. The price at date 1 will be a function of the state θ_1 at date 1 and the price at date 2 will be a function of the state (θ_1, θ_2) at date 2, but for the most part this notation will be suppressed as we take the state as given.

3.1.1 Market clearing at date 2

Suppose that the state of the economy at date 2 is (θ_1, θ_2). As we explained above, the demand for cash comes from the buyers and illiquid bankers who receive a liquidity shock at date 2. The supply of cash comes from the hoarders who do *not* receive a liquidity shock at date 2. There are three regimes in the market for cash and assets at date 2, defined by two critical values of θ_2, denoted by θ_2^* and θ_2^{**} and defined by

$$\theta_2^* = (1-\alpha)(1-\lambda) \text{ and } \theta_2^{**} = 1-\lambda.$$

(i) **Low demand for liquidity** $\theta_2 < \theta_2^*$. When the value of θ_2 is low enough, the among of cash held by the hoarders is more than enough to supply the buyers and illiquid bankers, so at the margin some hoarders have to be willing to hold cash. This means that they are indifferent between holding cash and the asset, which will only be true if the price of liquidity satisfies $p_2 = R^{-1}$.

(ii) **Intermediate demand for liquidity** $\theta_2^* < \theta_2 < \theta_2^{**}$. When the value of θ_2 is in an intermediate range, the hoarders have enough cash to supply the buyers and some but not all illiquid bankers. Then the illiquid bankers must be indifferent between selling their assets for cash and defaulting. This will be true if and only if $p_2 = 1$.

[4] To keep the analysis short and simple we do not report these results that are available from the authors.

(iii) **High demand for liquidity** $\theta_2 > \theta_2^{**}$. Finally, when demand for cash is high, the hoarders have only enough cash to supply some but not all buyers, so the buyers must be indifferent between selling assets to obtain liquidity and defaulting. This occurs if $p_2 = 1 + p_1$.

We summarize the preceding discussion in the following proposition, which is illustrated in Figure 5.

Proposition 2 *The market-clearing price at date* 2 *is denoted by* $p_2(\theta_1, \theta_2)$ *and defined by*

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{for } 0 \leq \theta_2 < \theta_2^*; \\ 1 & \text{for } \theta_2^* < \theta_2 < \theta_2^{**}; \\ 1 + p_1(\theta_1) & \text{for } \theta_2^{**} < \theta_2 \leq 1; \end{cases}$$

where $\theta_2^* = (1-\alpha)(1-\lambda(\theta_1))$ *and* $\theta_2^{**} = 1 - \lambda(\theta_1)$.

3.1.2 Market clearing at date 1

The analysis of market clearing at date 1 is a bit more complicated, because bankers' decisions depend on expectations about date 2. The first step is to show that, in equilibrium, there will always be some bankers who buy assets and some who hoard cash at date 1. This requires that the bankers with spare cash are indifferent between buying and hoarding. We can show that it is optimal to hoard if and only if $p_1 \leq E[p_2]$ and, conversely, it is optimal to buy if and only if $p_1 \geq E[p_2]$. Thus, indifference is equivalent to $p_1 = E[p_2]$. Now consider what will happen if there are no buyers, that is, $\lambda = 0$. The excess demand for cash at date 1 implies that $p_1 = 1$, but at date 2 the price p_2 must be less than or equal to one (since there are no buyers) and will sometimes be less than one (when θ_2 is sufficiently small). Then $E[p_2] < 1 = p_1$ contradicting the optimality of hoarding. Conversely, if $\lambda = 1$, the price at date 2 must satisfy $p_2 = 1 + p_1$ because there will be excess demand for cash with probability one, at least from the buyers that get the liquidity shock at $t = 2$, but this violates the optimality condition for buying. Hence, we get the following proposition.

Proposition 3 *For every value of* θ_1*,*

$$0 < \lambda(\theta_1) < 1$$

in equilibrium at date 1*. Thus, bankers holding unneeded cash at date* 1 *are indifferent between hoarding cash and buying the asset in equilibrium, which holds if and only if*

$$p_1(\theta_1) = E[p_2(\theta_1, \theta_2)|\theta_1].$$

Proof. See Appendix. ■

From Proposition 3, we know that $p_1 = E[p_2]$ and from Proposition 2 we know the distribution of p_2 as a function of λ, which allows us to calculate the value of $E[p_2]$ as a function of λ. Let $\tilde{p}(\lambda)$ denote this value for each value of λ. There is a unique value of λ, call it $\bar{\lambda} \in (0,1)$, such that $\tilde{p}(\bar{\lambda}) = 1$ and $\tilde{p}(\lambda) < 1$ if and only if $\lambda < \bar{\lambda}$. If $p_1 < 1$, then the market-clearing condition tells us that

$$(1-\alpha)(1-\theta_1)\lambda = \alpha\theta_1$$

or

$$\lambda = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

On the other hand, $\tilde{p}(\lambda) = 1$ implies that $\lambda = \bar{\lambda}$. Putting these facts together, we can characterize the equilibrium values of p_1 and λ in the following result.

Proposition 4 *The market clears at date* 1 *if and only if the equilibrium values of* λ *and* p_1 *are given by*

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\}$$

and

$$p_1(\theta_1) = \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right), 1\right\},$$

for every value of $0 \leq \theta_1 \leq 1$*, where*

$$\tilde{p}(\lambda) = \frac{1 - F_2((1-\alpha)(1-\lambda))(1-R^{-1})}{F_2(1-\lambda)}$$

for every value of $0 \leq \lambda \leq 1$ *and* $\bar{\lambda}$ *is the unique value of* $\lambda \in (0,1)$ *satisfying* $\tilde{p}(\lambda) = 1$*.*

Proof. See Appendix. ■

Note that $\lambda(\theta_1)$ is weakly increasing in θ_1. Hence, there is a unique value of θ_1, denoted by $\bar{\theta}_1$, where $\lambda(\theta_1) = \bar{\lambda}$ for $\theta_1 \geqslant \bar{\theta}_1$. Thus, we observe rationing in the market for $\theta_1 > \bar{\theta}_1$, where some banks cannot get liquidity and their assets need to be liquidated prematurely. Figure 6a illustrates this point. Similarly, we can see that $\tilde{p}(\lambda(\theta_1))$ is weakly increasing in θ_1 since $\tilde{p}(\lambda)$ is increasing in λ. Hence, at $\theta_1 = \bar{\theta}_1$, price $\tilde{p}(\lambda)$ reaches 1, the market clearing price $p_1(\theta_1)$ cannot increase any further and stays at 1 for $\theta_1 > \bar{\theta}_1$. The market clearing price $p_1(\theta_1)$ is illustrated in Figure 6b.

—Figure 6 about here—

Example 5 *Let θ_2 be uniformly distributed over the unit interval $[0,1]$. In this case, we can show that*

$$\bar{\lambda} = \frac{\alpha\bar{\theta}_1}{(1-\alpha)\left(1-\bar{\theta}_1\right)},$$

where

$$\bar{\theta}_1 = \frac{(1-\alpha)^2 (R-1)}{\alpha R + (1-\alpha)(R-1)}.$$

Furthermore, we obtain

$$p_1(\theta_1) = \min\left\{(1-\alpha)\left(\frac{1}{R} + \frac{\alpha}{1-\theta_1-\alpha}\right), 1\right\}.$$

3.1.3 Market clearing at date 0

Just as we showed that buyers and the hoarders have the same expected return at date 1, we can show that $0 < \alpha < 1$ in equilibrium at date 0 and that bankers must therefore be indifferent between acquiring liquidity and not acquiring it.

Note that the cost of holding liquidity is ρ. We can derive the benefit from holding liquidity by comparing the expected return of a liquid bank (that becomes a hoarder) with that of an illiquid bank.[5] We have three cases:

[5]Note that in equilibrium buyers and hoarders have the same expected return. For simplicity, we focus on a liquid bank that decides to become a hoarder.

(i) **Shock at $t = 1$:** In this case, a liquid bank uses her own liquidity to pay creditors and saves her project, whereas an illiquid bank needs to sell a fraction p_1 of her assets. Hence, a liquid bank, in expectation, saves $\theta_1 p_1 R$ compared to an illiquid bank.

(ii) **Shock at $t = 2$:** In this case, a liquid bank can use her own liquidity to pay creditors. However, an illiquid bank needs to sell assets at $t = 2$. For $p_2 \leqslant 1$, the illiquid bank can get the needed liquidity by selling p_2 units of her asset but for $p_2 > 1$ she has to default. Hence, a liquid bank, in expectation, saves $(1 - \theta_1)\theta_2 R \min\{1, p_2\}$ compared to an illiquid bank.

(iii) **No shock:** In this case, a liquid bank can acquire p_2 units of the asset at $t = 2$, which results in an expected return of $(1 - \theta_1)(1 - \theta_2)p_2 R$ for a liquid bank compared to that of an illiquid bank.

When we combine these three cases and use the equilibrium condition $p_1 = E[p_2]$, we get the following formal proposition.

Proposition 6 *In equilibrium,* $0 < \alpha < 1$*, which implies that bankers will be indifferent at date* 0 *between holding liquidity and not holding it. Agents are indifferent if and only if*

$$R \int_0^1 p_1 \left\{1 - (1 - \theta_1)(1 - F_2(\theta_2^{**}))E\left[\theta_2 \left|\theta_2 > \theta_2^{**}\right.\right]\right\} \; f_1(\theta_1) d\theta_1 = \rho.$$

Proof. See Appendix. ■

The first-order condition in Proposition 6 differs from the first-order condition for the planner's choice of m_0. The right-hand side, the cost of liquidity, is the same in both conditions but the left-hand sides differ. First, the planner internalizes the benefit of the payment to the creditors whereas banks do not. Second, liquid banks that receive a shock at date 2 use their own cash to avoid default, even though there are illiquid banks that value the liquidity more highly. The planner, by contrast, minimizes welfare losses by supplying liquidity to the largest banks first. These two effects explain the difference between the two first-order conditions.

3.2 Equilibrium

An equilibrium is described by the endogenous variables α, $\lambda(\theta_1)$, $p_1(\theta_1)$, and $p_2(\theta_1, \theta_2)$ satisfying the following conditions. Define $\tilde{p}(\lambda)$ by putting

$$\tilde{p}(\lambda) = \frac{1 - F_2\left(\left(1-\alpha\right)\left(1-\lambda\right)\right)\left(1 - R^{-1}\right)}{F_2\left(1-\lambda\right)}$$

for every $0 \leq \lambda \leq 1$ and let $\bar{\lambda}$ be the unique value of $0 < \lambda < 0$ satisfying $\tilde{p}(\lambda) = 1$. Then the equilibrium functions $p_1(\theta_1)$ and $\lambda(\theta_1)$ satisfy

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{\left(1-\alpha\right)\left(1-\theta_1\right)}, \bar{\lambda}\right\}$$

and

$$p_1(\theta_1) = \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{\left(1-\alpha\right)\left(1-\theta_1\right)}\right), 1\right\},$$

for every value of $0 \leq \theta_1 \leq 1$.

The equilibrium price function $p_2(\theta_2)$ must satisfy

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{for } 0 \leq \theta_2 < \theta_2^*(\theta_1), \\ 1 & \text{for } \theta_2^*(\theta_1) < \theta_2 < \theta_2^{**}, \\ 1 + p_1(\theta_1) & \text{for } \theta_2^{**} < \theta_2 \leq 1, \end{cases}$$

where

$$\theta_2^*(\theta_1) = \left(1-\alpha\right)\left(1 - \lambda(\theta_1)\right) \text{ and } \theta_2^{**} = 1 - \lambda(\theta_1).$$

Finally, at date 0, market-clearing requires indifference between acquiring and not acquiring liquidity:

$$\int_0^1 p_1\left\{1 - (1-\theta_1)(1 - F_2(\theta_2^{**}))E\left[\theta_2 \left|\theta_2 > \theta_2^{**}\right.\right]\right\} f_1(\theta_1)d\theta_1 = \frac{\rho}{R}.$$

4 Policy analysis

In this section, we provide an analysis of various policies aimed at improving liquidity and its allocation in markets.

4.1 Central bank as sole lender

In this section, we introduce a Central Bank (CB) into the model. We describe an equilibrium in which the CB acts as the sole supplier of liquidity, all bankers choose to be illiquid, and the constrained efficient policy characterized in Proposition 1 can be implemented.

Our approach is constructive. We assume that $\alpha = 1$, that is, all banks choose to be illiquid, and that the CB chooses as its policy the constrained efficient policy (m_0, m_1, m_2) given in Proposition 1. We define an equilibrium with the CB acting as a LoLR along the lines of the laisser-faire equilibrium. At date 2, there are no buyers, so the demand for liquidity comes from the $(1-\theta_1)\theta_2$ bankers who have received a liquidity shock at date 2. Since the supply of money is $\max\{m_0 - \theta_1, 0\}$, the market clearing price $p_2(\theta_1, \theta_2)$ is defined by

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{if } (1-\theta_1)\theta_2 < \max\{m_0 - \theta_1, 0\}, \\ 1 & \text{if } (1-\theta_1)\theta_2 > \max\{m_0 - \theta_1, 0\}. \end{cases} \quad (1)$$

Similarly, at date 1, the demand for liquidity comes from the θ_1 bankers who receive a liquidity shock at date 1 and the supply is at most m_0. If $\theta_1 > m_0$ the market clearing price must be $p_1(\theta_1) = 1$, but when $\theta_1 < m_0$ the price may lie anywhere between $E[p_2(\theta_1, \theta_2) \mid \theta_1]$ and 1. Since the CB can control the price we assume that it sets $p_1(\theta_1) = E[p_2(\theta_1, \theta_2) \mid \theta_1]$, so that the $1-\theta_1$ bankers who did not receive a shock are indifferent between hoarding and buying. Then the market clearing price is

$$p_1(\theta_1) = \begin{cases} E[p_2(\theta_1, \theta_2) \mid \theta_1] & \text{if } \theta_1 < m_0, \\ 1 & \text{if } \theta_1 > m_0. \end{cases} \quad (2)$$

Market clearing at date 0 requires that it is optimal for bankers to choose $\alpha = 1$. We can show that this is the case, which gives us the following proposition.

Proposition 7 *In an equilibrium where the CB acts as the sole provider of liquidity, all bankers choose to become illiquid, that is, $\alpha = 1$; market-clearing prices at date 1 and 2 are given in equations (2) and (1), respectively; and the constrained efficient policy (m_0, m_1, m_2) given in Proposition 1 can be implemented.*

Proof. See Appendix. ■

Hence, in equilibrium, the CB by acting as the sole provider of liquidity can implement the constrained efficient allocation from the planner's problem in Section 2. This results in the CB liquidity crowding out private liquidity. Next, we look at some simpler ex ante (date 0) and ex post (date 1) policies that can be used to improve welfare.

4.2 Policy analysis with private liquidity

In this section, we analyze two different policies that aim at maximizing the expected total output that restrict (one at a time): (i) the portfolio choice (namely α) at date 0; and (ii) the level of lending (namely λ) at date 1. Other than the date we impose the restriction, we assume that the markets will function as in Section 3.1 where we characterize the equilibrium. Since the planner in Section 2.2 is already restricted the resulting outcome from the planner's problem is constrained efficient, say second best. The policies we analyze in this section constrain the policy maker more compared to the planner in Section 2.2. Hence, the resulting outcomes qualify for a third best and, for simplicity, we use the term *socially optimal* in this section.

First, we try to find the socially optimal level of lending at $t = 1$, denoted by λ^{soc}, that maximizes the expected output assuming that the market for asset sales at $t = 2$ will function as in Section 3.1.

At $t = 1$, the liquidity shock θ_1 is realized and we can find the expected output for each realization of θ_1. Then we can find λ^{soc} and compare it with the privately optimal level of lending given in Proposition (4). In calculating the expected output at $t = 1$, we need to consider three different regions for θ_2:

(i) For $\theta_2 < \theta_2^*$, there is enough liquidity for all agents that got hit by the liquidity shock at $t = 2$. Hence, no asset needs to be liquidated.

(ii) For $\theta_2^* < \theta_2 < \theta_2^{**}$, there is enough liquidity for all buyers that got hit by the liquidity shock but not enough for all illiquid agents that got hit. Hence, some of the assets held by illiquid agents that got hit need to be liquidated prematurely.

(iii) For $\theta_2 > \theta_2^{**}$, there is not enough liquidity even for all buyers that got hit by the liquidity shock. Hence, some of the assets held by buyers that got hit and all the assets held by illiquid agents that got hit need to be liquidated prematurely.

Using these we can calculate the total expected output and find the level of lending λ^{soc} that maximizes the expected output. The following proposition characterizes the socially optimal level of lending at $t = 1$ and compares it with the equilibrium level of lending at $t = 1$ characterized in Proposition (4).

Proposition 8 *We can characterize the socially optimal level of lending λ^{soc} as follows:*

$$\lambda^{soc}(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \tilde{\lambda}\right\},$$

where $\tilde{\lambda}$ is determined implicitly by the condition

$$F_2((1-\alpha)(1-\tilde{\lambda})) + F_2(1-\tilde{\lambda}) = 1.$$

Furthermore, we obtain $\tilde{\lambda} > \bar{\lambda}$.

Proof. See Appendix. ■

The socially optimal level of lending has the same structure as the equilibrium level of lending. In particular, as in the equilibrium, the socially optimal level of lending requires that the liquidity need of all illiquid agents that got hit by the shock at $t = 1$ be satisfied up to the threshold $\tilde{\lambda}$, which is higher than the threshold $\bar{\lambda}$ in equilibrium. Hence, in equilibrium there is inefficiently low level of lending, that is, equilibrium is characterized by an inefficiently high level of hoarding at $t = 1$.

Thus, a policy that aims at facilitating lending at $t = 1$ or lending directly to banks can improve efficiency. One possibility is that the central banks can provide liquidity to markets in general through open market operations (OMO), which then is transferred to institutions in need through the interbank market. Goodfriend and King (1988) argue that with efficient interbank markets central banks can provide sufficient liquidity via OMOs and the interbank market will allocate the liquidity among banks so that the activities of central banks should be limited to monetary policy and they should not lend to banks on an individual basis. The current crisis provides us evidence that OMOs can have limited effect in channeling liquidity to institutions that need it in the presence of uncertainty about future liquidity shocks and hoarding incentives.

For example, Governor of the Bank of England Mervyn King and the Chancellor of the Exchequer Alistair Darling, during the hearings about the Northern Rock episode in the Fall of 2007, pointed out the difficulties with OMOs in channeling liquidity to needy banks as the primary reason for lending directly to individual institutions. In particular, they pointed out that to channel the £14 billion that Northern Rock borrowed from the Bank of England to that institution would have required many more billions of pounds to be injected through the OMOs. In the same hearing, William Buiter suggested: "That would take an enormous amount of money injections. We know for instance that despite all the money that the Fed and especially the ECB have put into these longer term markets, the actual spreads of three months LIBOR and the euro equivalent and the dollar equivalent over the expected policy rate is no smaller in euro land today than it is here, so it really may take a large injection of liquidity to get an appreciable result if the market is really fearful."

Early in the crisis of 2007-09, the Federal Reserve used OMOs to ease the strain in money markets. While OMOs had some success in stabilizing the overnight rate, the rates on term loans continued to rise leading to the introduction of several new liquidity facilities. These new facilities have extended maturities to include up to 90-day loans, maturities at which money markets have dried up in the aftermath of sub-prime losses; extended eligible collateral to include investment-grade debt securities (including high-rated but illiquid mortgage-backed securities); and extended these privileges not only to banks but also to securities dealers since they are also affected by funding problems caused by the drying up of liquidity extension from banks.[6]

[6]In particular, in addition to the traditional tools the Fed uses to implement monetary policy (e.g., Open

Next, we show that the private choice of bankers to hold liquidity at $t = 0$ does not correspond to the level of liquidity that maximizes the expected output. To show that we calculate the expected output as we did in the analysis of the social optimum at $t = 1$. Then we show that at the equilibrium level, the expected output is decreasing in α so that, at the equilibrium, by increasing the proportion of liquid agents, we can increase expected output. This gives us the following formal proposition.

Proposition 9 *In equilibrium, expected output increases as the fraction of illiquid bankers* α *decreases.*

Our results show that equilibrium is characterized by bankers choosing an inefficiently low level of liquidity in their portfolio. One policy measure to address this issue can be liquidity requirements for banks. While some countries already have liquidity requirements, like the UK, others do not have such requirements and there is no international standard on

Market Operations, Discount Window, and Securities Lending program), new programs have been implemented since August 2007: 1) Term Discount Window Program (announced August 17, 2007) - extended the length of discount window loans available to institutions eligible for primary credit from overnight to a maximum of 90 days; 2) Term Auction Facility (TAF) (announced December 12, 2007) - provides funds to primary credit eligible institutions through an auction for a term of 28 days; 3) Single-Tranche OMO (Open Market Operations) Program (announced March 7, 2008) - allows primary dealers to secure funds for a term of 28 days. These operations are intended to augment the single day repurchase agreements (repos) that are typically conducted; 4) Term Securities Lending Facility (TSLF) (announced March 11, 2008) - allows primary dealers to pledge a broader range of collateral than is accepted with the Securities Lending program, and also to borrow for a longer term — 28 days versus overnight; and, 5) Primary Dealer Credit Facility (PDCF) (announced March 16, 2008) - is an overnight loan facility that provides funds directly to primary dealers in exchange for a range of eligible collateral; 6) Commercial Paper Funding Facility (CPFF) (announced November 7, 2008) - is designed to provide a liquidity backstop to U.S. issuers of commercial paper; 7) Money Market Investor Funding Facility (MMIFF) (announced November 21, 2008) - is aimed to support a private-sector initiative designed to provide liquidity to U.S. money market investors; 8) Term Asset-Backed Securities Loan Facility (TALF) (announced November 25, 2008) - is designed to help market participants meet the credit needs of households and small businesses by supporting the issuance of asset-backed securities (ABS) collateralized by auto loans, student loans, credit card loans etc.

liquidity regulation like the Basel requirements for bank capital. The Basel III regulatory requirements that are being designed propose two such measures for liquidity requirements, namely, the Liquidity Coverage Ratio (LCR) and the Net Stable Funding Ratio (NSFR).[7]

Below, we provide simulation results that illustrate the wedge between the equilibrium and the socially optimal levels of λ and α. We use the parameter values $R = 3$, $\rho = 2$ and assume that θ_1 and θ_2 are iid and $U[0,1]$. We find that in equilibrium a fraction $\alpha = 0.14$ of agents choose to become illiquid at $t = 0$, whereas the socially optimal level of α is 0.067. We also find that in the equilibrium $\bar{\lambda} = 0.364$, whereas in the social optimum $\tilde{\lambda} = 0.462$. We provide the simulation results for the equilibrium and socially optimal levels of λ as a function of θ_1 (Figure 7a) and α as a function of ρ (Figure 7b).

—Figure 7 about here—

4.2.1 Comparative statics

In this section, we provide comparative statics analysis for lending at $t = 1$. In particular, we analyze how the equilibrium and socially optimal levels of λ, and the wedge between the two, are affected by the expectations of future liquidity shocks and increased uncertainty and volatility of such shocks.

First, we focus on the case when higher liquidity shocks are more likely at $t = 2$. To capture the likelihood of liquidity shocks at $t = 2$, we use two different probability distributions, f_2 and g_2, for θ_2, where g_2 first-order stochastically dominates f_2. Hence, higher proportions of the liquidity shock at $t = 2$ are more likely under the probability distribution g_2.

From the equilibrium condition we have

$$F_2\left(1-\bar{\lambda}_f\right)+F_2\left((1-\alpha)\left(1-\bar{\lambda}_f\right)\right)\left(1-R^{-1}\right)=1.$$

[7] LCR requires banks to hold a minimum level of liquid assets that can cover a net cash outflow during a 30 day stress period, whereas NSFR establishes a minimum acceptable amount of stable funding based on the liquidity characteristics of an institution's assets and activities over a one year period. For more detail see: BCBS (2010) "Basel III: International framework for liquidity risk measurement, standards and monitoring."

Since g_2 first-order stochastically dominates f_2, we obtain

$$G_2\left(1-\bar{\lambda}_f\right)+G_2\left((1-\alpha)\left(1-\bar{\lambda}_f\right)\right)\left(1-R^{-1}\right)<1.$$

Note that the LHS of the above inequality is decreasing in λ so that we obtain $\bar{\lambda}_f > \bar{\lambda}_g$. We can use a similar argument to show that $\tilde{\lambda}_f > \tilde{\lambda}_g$. This gives us the following formal proposition.

Proposition 10 *Let f_2 and g_2 be two probability distributions over θ_2, where g_2 first-order stochastically dominates f_2. Let $\bar{\lambda}_f, \tilde{\lambda}_f$ and $\bar{\lambda}_g, \tilde{\lambda}_g$ be characterized as in Propositions (4) and (8) under probability distributions f_2 and g_2, respectively. We obtain $\bar{\lambda}_f > \bar{\lambda}_g$ and $\tilde{\lambda}_f > \tilde{\lambda}_g$.*

Hence, when expectations about high liquidity shocks in the future become stronger, both the equilibrium and socially optimal levels of lending are lower, resulting in higher levels of cash carried into the future.

Next, we analyze the wedge between the equilibrium and the socially optimal levels of lending when liquidity shocks at $t = 2$ become more likely and the volatility of liquidity shocks increases.

First, we focus on the effect of the likelihood of liquidity shocks. Let θ_2 be distributed uniformly according to the probability distribution $f_2^b = \frac{1}{b-a}$ over the interval $[a, b]$, with $0 \leqslant a < b \leqslant 1$. Note that for a fixed a, for $b' > b$, $f_2^{b'}$ first-order stochastically dominates f_2^b. Using the characterization in Proposition 8, we obtain $\tilde{\lambda} = 1 - \frac{b+a}{2-\alpha}$. Using the equilibrium condition in Proposition 4, we obtain

$$\bar{\lambda}=1-\frac{bR+a\left(R-1\right)}{R+\left(1-\alpha\right)\left(R-1\right)},$$

Furthermore,

$$\frac{d(\tilde{\lambda}-\bar{\lambda})}{db}=\frac{1-\alpha}{\left(2-\alpha\right)\left(R+\left(1-\alpha\right)\left(R-1\right)\right)}>0.$$

Hence, as higher shocks become more likely, in the first-order stochastic sense, the wedge between the socially optimal level of lending λ^{soc} and its equilibrium level λ increases. This suggests that during periods where expectations of high liquidity shocks in the future become stronger, even though liquidity management requires hoarding from a social welfare point of view as well, hoarding becomes a more serious problem as the wedge between the socially and privately optimal levels of lending widens.

Next, we look at how the wedge between the equilibrium and the socially optimal levels of lending change with the volatility of shocks. Let θ_2 be distributed uniformly according to the probability distribution $f_2 = \frac{1}{b-a}$ over the interval $[a, b]$ with $a + b = 1$ so that the distribution is symmetric around $\frac{1}{2}$. Note that for $b' > b$, $f_2^{b'}$ is a mean-preserving spread of f_2^b. From the equilibrium condition, we obtain

$$\bar{\lambda} = 1 - \frac{R - 1 + b}{R + (1 - \alpha)(R - 1)}.$$

Note that $\bar{\lambda}$ is decreasing in b. Furthermore, in this case, we have $\tilde{\lambda} = \frac{1-\alpha}{2-\alpha}$ so that the socially optimal level of lending is not affected by mean-preserving spreads. Hence, during periods of heightened uncertainty about future liquidity shocks, modelled by a probability distribution that is a mean-preserving spread, the wedge between the socially optimal and the equilibrium levels of lending increases, and hoarding becomes a more serious problem.

This result is related to recent papers in the literature that explain breakdown in markets using different frameworks. For example, Morris and Shin (2008) show that even small amounts of adverse selection in an asset market can lead to the total breakdown of trade due to the failure of market confidence, defined as approximate common knowledge of an upper bound on expected losses. Even though we use the expected utility theory framework in our analysis, our result is consistent with the literature that uses the notion of Knightian uncertainty (see Knight, 1921) and agents' overcautious behavior towards such uncertainty to generate hoarding and market freezes. Routledge and Zin (2004) and Easley and O'Hara (2009, 2010) use Knightian uncertainty and agents that use maxmin strategies to generate widening bid-ask spreads and freeze in financial markets. Caballero and Krishnamurthy (2008) build a model to show that during periods of increased Knightian uncertainty, agents refrain from making risky investments and hoard liquidity, leading to flight to quality and freezes in markets for risky assets.

5 Robustness and discussion

In this paper, we have tried to investigate the welfare implications of liquidity hoarding when markets are incomplete. The attempt is complicated by the fact that hoarding is not the only source of inefficiency in the model. In this section, we conclude by discussing some variants of the model to shed more light on these sources of inefficiency.

5.1 A model without hoarding

We begin by considering a benchmark model in which there is no role for hoarding. Suppose there are only three dates, indexed by $t = 0, 1, 2$. As before, bankers choose their portfolios (more precisely, the amount of liquidity in their portfolios) at date 0. At date 1, they observe the liquidity shock θ_1 and, at date 2, the asset returns are realized. The specification of the rest of the model is the same as before, mutatis mutandis. We solve for equilibrium backwards, beginning with the second period. If a fraction $1 - \alpha$ of the bankers hold cash at date 0 and the state is θ_1 at date 1, a fraction $(1 - \alpha)\,\theta_1$ of the bankers can supply their own cash needs and a fraction $(1 - \alpha)\,(1 - \theta_1)$ of the bankers have spare cash that they can supply to the market. The measure of illiquid bankers who need cash is $\alpha\theta_1$ and it is clear that the market for cash will clear at a price defined by

$$p_1\,(\theta_1) = \begin{cases} 1 & \text{if } \theta_1 > 1 - \alpha, \\ R^{-1} & \text{if } \theta_1 < 1 - \alpha. \end{cases} \qquad (3)$$

The allocation of cash at date 1 is efficient, since the number of bankers who can discharge their debts is $\min\{\theta_1, 1 - \alpha\}$, that is, every banker who receives a liquidity shock gets the cash she needs, unless the number of bankers receiving a shock exceeds the supply of cash.

The equilibrium allocation is not efficient, however, because the liquidity decision at date 0 is not constrained optimal. Bankers choose to hold too little cash at date 0 because they do not internalize the value of the cash provided to creditors. To see this, we need to compare the level of cash held in equilibrium with the level chosen by the planner. In equilibrium, bankers must be indifferent between being liquid and illiquid at date 0, that is,

$$\int_0^1 [R - \theta_1 p_1(\theta_1) R - (1-\theta_1)] f_1(\theta_1) d\theta_1 = \int_0^1 [R + (1-\theta_1)(p_1(\theta_1) R - 1)] f_1(\theta_1) d\theta_1 - \rho,$$

where the RHS and the LHS are the payoffs for a liquid and an illiquid banker, respectively. This, in turn, yields the equilibrium condition $E[p_1] = \rho/R$, which gives us

$$F_1(1-\alpha) = \frac{R-\rho}{R-1}. \tag{4}$$

In the planner's problem, the marginal cost of cash is ρ and the marginal value of cash is 1, if $\theta_1 < m_0$, and $R+1$, if $\theta_1 > m_0$. So the planner's first-order condition is $R(1-F_1(m_0))+1 = \rho$, that is,

$$F_1(m_0) = \frac{R+1-\rho}{R}. \tag{5}$$

Now we have to compare the equilibrium condition with the planner's first-order condition. Note that

$$F_1(m_0) - F_1(1-\alpha) = \frac{\rho - 1}{R(R-1)} > 0.$$

The fact that the difference is positive implies that $m_0 > 1-\alpha$. In other words, there is too little liquidity in equilibrium.

The analysis of this simplified version of the benchmark model provides us with some useful insights. In particular, when there is no possibility of hoarding, the only source of inefficiency is the fact that bankers do not receive any benefit from the cash they pay to creditors. Bankers have a lower marginal value of cash than the planner and therefore hold too little liquidity at date 0 in equilibrium. In other respects, equilibrium is efficient.

5.2 Price volatility

In this section, to show that the exchange of assets for cash at date 1 causes inefficient hoarding, we consider a variant of the model in which default costs consume only the bankers' original assets and not the assets acquired at date 1.

Consider the model described in Section 3 with the following change. The debts that come due randomly are considered to be *non-recourse* loans. That is, if the banker receives a liquidity shock and is unable or unwilling to discharge his debt, the creditor can seize the asset that serves as security but cannot seize any other assets owned by the banker. As before, the default costs consume the entire asset.

A buyer who acquires $p_1(\theta_1)$ units of the asset in exchange for its one unit of liquidity at date 1 is guaranteed to have a return of at least $p_1(\theta_1)R$ at date 3. Even if the buyer defaults on his loan, he only loses the unit of the asset originally pledged as security for the loan and retains the rest of his portfolio. Since only one unit of the asset is at risk, the buyer will only be willing to give up one unit of the asset in exchange for one unit of cash. Then the market-clearing price $p_2(\theta_1, \theta_2)$ has the distribution

$$p_2(\theta_1, \theta_2) = \begin{cases} R^{-1} & \text{w. pr. } F_2((1-\alpha)(1-\lambda(\theta_1))), \\ 1 & \text{w. pr. } 1 - F_2((1-\alpha)(1-\lambda(\theta_1))). \end{cases}$$

and the expected value of $p_2(\theta_1, \theta_2)$ is

$$E[p_2(\theta_1, \theta_2)|\theta_1] = F_2((1-\alpha)(1-\lambda(\theta_1)))R^{-1} + 1 - F_2((1-\alpha)(1-\lambda(\theta_1))).$$

A fraction θ_2 of the buyers receive a liquidity shock and have a payoff $(1 + p_1(\theta_1) - p_2(\theta_1, \theta_2))R$; a fraction $(1-\theta_2)$ do not receive a shock and have a payoff $(1 + p_1(\theta_1))R - 1$. Thus, the buyers' expected payoff at date 1 is

$$\int_0^1 \{(1 + p_1(\theta_1) - \theta_2 p_2(\theta_1, \theta_2))R - (1-\theta_2)\} f_2(\theta_2)\, d\theta_2.$$

Now consider the hoarders. A fraction θ_2 of the hoarders receive a liquidity shock and have a payoff R and a fraction $(1-\theta_2)$ do not receive a shock and have a payoff $(1 + p_2(\theta_1, \theta_2))R - 1$. Thus, the hoarders' payoff at date 1 is

$$\int_0^1 \{(1 + (1-\theta_2)p_2(\theta_1, \theta_2))R - (1-\theta_2)\} f_2(\theta_2)\, d\theta_2.$$

It is optimal to buy if and only if the buyers' payoff is at least as great as the hoarders, that is, $p_1(\theta_1) \geq E[p_2(\theta_1, \theta_2) \mid \theta_1]$. Similarly, it will be optimal to hoard if and only if $p_1(\theta_1) \leq E[p_2(\theta_1, \theta_2) \mid \theta_1]$.

Suppose that, in equilibrium, there is inefficient hoarding, that is, $\lambda(\theta_1) < \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}$. In that case, there are illiquid bankers hit by the shock that are willing to give all their asset for one unit of liquidity, which means $p_1(\theta_1) = 1$. But in equilibrium we have $E[p_2(\theta_1, \theta_2) \mid \theta_1] \geq p_1(\theta_1) = 1$, which requires that $F_2((1-\alpha)(1-\lambda(\theta_1))) = 0$, that is, $\alpha = 1$ or $\lambda(\theta_1) = 1$. We can rule out $\alpha = 1$ when ρ is not too high. And, $\lambda(\theta_1) = 1$ means there is no hoarding, which is a contradiction. Hence, when shocks affect only assets, rather than the entire bank, equilibrium is characterized by no hoarding.

The intuition for this result is quite clear. Inefficient hoarding at date 1 requires that $p_1(\theta_1) = 1$. However, the maximum number of assets that can be acquired by a hoarder (or saved when hit by the shock) is 1. Hence, liquid agents prefer to buy one unit of the asset at $t = 1$, rather than hoard.

The preceding analysis would not be changed if α were a fixed but arbitrary value. We have shown that the equilibrium is efficient conditional on that fixed value of α. If the planner sets α equal to the constrained-efficient level, the corresponding equilibrium would be constrained efficient.

5.3 Markets for liquidity insurance

In this section, we show that opening a forwards market for liquidity at date 0 cannot improve upon the allocation provided by the laisser-faire equilibrium with only spot markets. In particular, we consider a market formed at date 0 in which some bankers enter into a contract to acquire liquidity and supply it under certain conditions and other bankers simultaneously enter into a contract to supply the asset under certain conditions. The suppliers of liquidity are required to report their type, that is, whether or not they have received a liquidity shock at date 1 and date 2. In the event that they have not reported a shock, they may be required to supply one unit of liquidity, if they have not already done so, in exchange for a specified amount of the asset. The demanders of liquidity similarly are required to report their type, that is, whether or not they have received a liquidity shock at date 1 and date 2. In the event that they have reported a shock, they may be supplied with one unit of cash, if they have not already received it, in exchange for a specified amount of the asset. We let $\hat{p}_1(\theta_1)$ denote the price of cash at date 1 in state θ_1 and let $\hat{p}_2(\theta_1, \theta_2)$ denote the price of cash at date 2 in state (θ_1, θ_2). Suppose that there exists an equilibrium $\{\alpha, \lambda(\theta_1), p_1(\theta_1), p_2(\theta_1, \theta_2)\}$ and consider the effect of opening a market for liquidity at date 0. The market must satisfy an incentive compatibility constraint to ensure that bankers report their types truthfully. At date 1 in state θ_1, one unit of cash can be traded for $p_1(\theta_1)$ units of cash on the spot market. If $p_1(\theta_1) > \hat{p}_1(\theta_1)$, a banker with cash who has not received a liquidity shock is better off reporting a liquidity shock since he could always sell his unit of cash on the spot market for the higher price. Likewise, if $p_1(\theta_1) < \hat{p}_1(\theta_1)$, a banker without cash who has received a liquidity shock would be better off reporting no liquidity shock since he can always buy cash at the lower price. Thus, incentive compatibility at date 1 requires

$$\hat{p}_1(\theta_1) = p_1(\theta_1),$$

for every value of θ_1. A similar argument implies that

$$\hat{p}_2(\theta_1, \theta_2) = p_2(\theta_1, \theta_2),$$

for every value of (θ_1, θ_2). Since the prices are the same, it is clear that the market mechanism cannot improve on the allocation provided by the spot markets.

5.4 Limitations of the LoLR

Goodfriend and King (1986) argue that it is sufficient to provide adequate liquidity to the system as a whole when interbank markets function efficiently. We have shown that constrained efficiency can be achieved in our model if the central bank is the sole provider of liquidity. Is this realistic? What are the limits on the role of the Lender of Last Resort?

In recent discussions, several concerns have been raised about the liquidity facilities recently rolled out by the Federal Reserve System. One concern is the possibility that the increase in the Fed's balance sheet as a result of the increase in reserves and the secured lending facilities set up by the Fed will result in inflation. Another is the possibility that the Fed can make losses as a result of counterparty risk because it is willing to extend potentially loss-making loans in order to achieve policy objectives such as financial stability. Finally, there is the problem of unwinding its position as conditions change in the economy. Some writers doubt that the Fed will be able to shrink its balance sheet quickly enough when signs of inflation appear or that the attempt to do so will upset the securities market. These and other concerns should temper any enthusiasm for the possibility of achieved constrained efficient liquidity provision by having the Fed become the first and sole provider.

5.5 Related literature

Some recent papers provide empirical evidence for and discuss liquidity hoarding in interbank markets. Acharya and Merrouche (2009) document that the U.K. banks' liquidity buffers experienced an almost permanent upward shift of 30% in August 2007 (relative to their pre-August levels) and the result was a rise in borrowing costs between banks and an almost complete drying up of liquidity in interbank markets beyond the very short maturities. Heider et al. (2008) provide evidence of liquidity hoarding in the unsecured euro interbank market. They document that until August 9, 2007, the unsecured euro interbank market is characterized by a very low spread and infinitesimal amounts of excess reserves with the European Central Bank (ECB) since, in normal times, banks prefer to lend out excess cash as the interest rate on excess reserves is punitive relative to rates available in interbank markets. They document that the period between August 9, 2007 and the last weekend of September 2008 is characterized by a significantly higher spread, yet excess reserves remain virtually nil. As of September 28, 2008, the spread increases even further to a maximum of 186 basis points. More importantly, we observe a dramatic increase in excess reserves, where the average daily volume in the overnight unsecured interbank market halved. Ashcraft et al. (2008) use data on intraday account balances held by banks at the Federal Reserve and Fedwire interbank transactions for a sample of approximately 700 banks that ever lend or borrow during the period September 2007 through August 2008 to estimate all overnight fed funds trades. They present empirical evidence on banks' precautionary hoarding of reserves, their reluctance to lend, and extreme fed funds rate volatility. Afonso, Kovner and Schoar (2011) examine the response of the US Fed Funds market to the bankruptcy of Lehman Brothers and documents that while rates spiked and loan terms became more sensitive to borrower risk, mean borrowing amounts remained stable on aggregate. They argue that it is likely that the market did not expand to meet additional demand for funds, which is consistent with our result on rationing in the interbank market when demand for liquidity is high. Ivashina and Scharfstein (2010) show that new loans to large borrowers fell by 47% during the peak period of the financial crisis. After the failure of Lehman Brothers in September 2008, there was a run by short-term bank creditors accompanied by a simultaneous run by borrowers who drew down their credit lines. They show that banks cut their lending more the more reliant on short-term debt they were and the more vulnerable they were to credit-line drawdowns.

At a general level, our paper is related to Shleifer and Vishny (1992) and Allen and Gale (1994, 1998) that show that when potential buyers of assets are themselves financially constrained, the price of the assets may fall below their fundamental value and be determined by the available liquidity in market, that is, we observe cash-in-the-market prices.[8]

Our paper is related to the literature on portfolio choice of banks and how the level of liquidity is determined endogenously (e.g. Allen and Gale (2004a,b), Gorton and Huang (2004), Diamond and Rajan (2005), and Acharya, Shin and Yorulmazer (2011)). The recent work by Diamond and Rajan (2011) build a model, where banks in anticipation of future fire-sales have high expected returns from holding cash. Acharya and Skeie (2010) build a model where banks' decision to provide term lending depends on leverage and rollover risk over the term of the loan. Malherbe (2010) builds a model where liquidity hoarding can arise due to adverse selection as one of the multiple equilibria of the model.[9] Our paper differs from these papers in various aspects. First, in our paper bankers hold liquidity for protecting themselves against future liquidity shocks (precautionary motive) as well as taking advantage of potential sales (strategic motive). Second, in our paper, bankers make a portfolio choice initially as well as a choice to lend to needy bankers or hoard liquidity for future periods.

[8] Also, see Allen and Gale (2005) for a review of the literature that explores the relation between asset-price volatility and financial fragility when markets and contracts are incomplete.

[9] Also see Chapter 7 of Holmstrom and Tirole (2010) that uses the model built in Malharbe (2010).

This adds richness to our model and allow us to analyze the interaction between bankers' two choices. Furthermore, this allows us to analyze a rich set of policies such as ex ante liquidity requirements and various ex post lending facilities.

Our paper is related to the literature on interbank markets (e.g. Rochet and Tirole (1996) and Allen and Gale (2000)), and the failure of such markets to transfer liquidity efficiently that justifies regulatory intervention.[10] Goodfriend and King (1988) argue that with efficient interbank markets, central banks should not lend to individual banks, but instead provide liquidity via open market operations, which the interbank market would then allocate among banks. Others, however, argue that interbank markets may fail to allocate liquidity efficiently due to frictions such as asymmetric information about banks' assets (Flannery (1996), Freixas and Jorge (2007)), banks' free-riding on each other's liquidity (Bhattacharya and Gale (1987)), or on the central bank's liquidity (Repullo (2005)), market power and strategic behavior (Acharya, Gromb and Yorulmazer (2007)), and regulatory solvency constraints and marking to market of the assets (Cifuentes, Ferrucci and Shin (2005)).

Our paper, in general, is also related to the papers on bankruns (Diamond and Dybvig (1983)), runs in wholesale markets (Huang and Ratnovski (2011), Gorton and Metrick (2009), and He and Xiong (2009))[11], shortening of maturities during stress periods (Brunnermeier and Oehmke (2009)), drying up of liquidity and market freezes (Acharya, Gale and Yorulmazer (2011)), the interaction between market and funding liquidity (Brunnermeier and Pedersen (2009)), and the determinants of illiquidity risk (Morris and Shin (2009)).

[10] Also, see Freixas et al. (1999) for an excellent survey on interbank markets.

[11] Shin (2009) and Goldsmith-Pinkham and Yorulmazer (2010) provide analyses of the Northern Rock episode in the UK in 2007 and the role of excessive reliance on wholesale markets in creating financial fragility and rollover risk.

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6 Appendix A: Model description

In this Appendix, we provide a detailed, yet self-contained, description of the model used to analyze the laisser-faire economy and the market clearing conditions at $t = 2$ that has been discussed in Section 3.

Date 0 Recall that bankers are initially endowed with one unit of the asset and one unit of cash. At date 0, bankers choose whether to consume their cash immediately or retain one unit in their portfolios for future use. We call the bankers who retain the cash *liquid* and those who do not *illiquid*. Let $0 \leq \alpha \leq 1$ denote the measure of illiquid bankers. The α illiquid bankers end the period with a portfolio $(1, 0)$ and the $1 - \alpha$ liquid bankers end the period with a portfolio $(1, 1)$.

Date 1 At the beginning of date 1, a fraction θ_1 of bankers receive the liquidity shock. The $(1 - \alpha)\,\theta_1$ liquid bankers who receive the shock can discharge their debt using their cash holdings and end the period with a portfolio $(1, 0)$. The alternative is to default and lose everything. The $\alpha\theta_1$ illiquid bankers who receive a liquidity shock sell part of their asset holdings in exchange for cash to discharge their debt and end the period with a portfolio $(1 - p_1, 0)$, where $0 \leq p_1 \leq 1$ denotes the price of one unit of cash. If some of these bankers cannot obtain cash to discharge their debt, they must be indifferent between obtaining cash and default. This will be the case if $p_1 = 1$.

The $\alpha\,(1 - \theta_1)$ illiquid bankers who do not receive a shock do not trade and end the period with a portfolio of $(1, 0)$. We will see later that this is the optimal strategy for them.[12] The $(1 - \alpha)\,(1 - \theta_1)$ liquid bankers who do not receive a liquidity shock have the option of acquiring p_1 units of the asset using their one of cash. Liquid bankers who use their cash to purchase the asset are called *buyers*; those who do not are called *hoarders*. We

[12] We will show that, in equilibrium, the price of cash at date 1 is equal to the expected price of cash at date 2. This is sufficient to prove that an illiquid banker cannot improve his payoff by purchasing cash at date 1.

assume that a measure $(1-\alpha)(1-\theta_1)\lambda$ of these bankers become buyers and end the period with a portfolio $(1+p_1, 0)$. The remaining $(1-\alpha)(1-\theta_1)(1-\lambda)$ become hoarders and end the period with a portfolio $(1, 1)$.

Date 2 At the beginning of date 2, a fraction θ_2 of the bankers who did not receive a liquidity shock at date 1 receive a liquidity shock. Bankers who received a liquidity shock at date 1 have no cash, so there is nothing for them to do at date 2. Without loss of generality we assume they remain inactive.

The $\alpha(1-\theta_1)\theta_2$ illiquid bankers who receive a shock at date 2 can purchase one unit of cash for a price $p_2 \geq 0$. It will be optimal for them to do so as long as $p_2 \leq 1$, but since the buyers have $1+p_1$ units of the asset, the price may rise above one unit of the asset. In any case, these bankers will end the period with a portfolio of $(\max\{1-p_2, 0\}, 0)$. The $\alpha(1-\theta_1)(1-\theta_2)$ illiquid bankers who do not receive a shock at either date have no gains from trade. They are assumed not to trade and end the period with a portfolio $(1, 0)$.

The $(1-\alpha)(1-\theta_1)\lambda\theta_2$ buyers who receive a liquidity shock at date 2 can purchase one unit of cash for a price $p_2 \geq 0$. It will be optimal for them to do so as long as $p_2 \leq 1+p_1$. In any case, they will end the period with a portfolio $(1+p_1-p_2, 0)$. The remaining $(1-\alpha)(1-\theta_1)\lambda(1-\theta_2)$ buyers who do not receive a shock at either date have no gains from trade and are assumed not to trade. They will end the period with a portfolio $(1+p_1, 0)$.

Finally, consider the hoarders. The $(1-\alpha)(1-\theta_1)(1-\lambda)\theta_2$ hoarders who receive a liquidity shock at date 2 use their unit of cash to discharge their debt and end the period with a portfolio $(1, 0)$. The alternative is to default and lose all their wealth. The $(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)$ hoarders who do not receive a liquidity shock can supply cash to the illiquid bankers and buyers who did receive a liquidity shock. It is optimal to supply cash as long as $p_2 \geq R^{-1}$ and it is strictly optimal to supply all their cash if $p_2 > R^{-1}$. These bankers end the period with a portfolio equal to $(1, 1)$ or $(1+p_2, 0)$, depending on the price p_2.

The allocation of assets in the first two dates is illustrated in Figure 2 and the allocation of assets at the end of date 2 is illustrated in Figure 3.

Date 3 At the last date, bankers receive the payoffs from the portfolios of cash and the asset carried forward from date 2. Bankers who have not already discharged their debts must pay their creditors one unit of cash. The terminal payoffs, which are easily calculated from the terminal allocation, are illustrated in Figure 4.

Market clearing at date 2 Suppose that the state of the economy at date 2 is (θ_1, θ_2). We can ignore the bankers who received a shock at date 1 and are inactive at date 2. We can also ignore the hoarders who receive a shock at date 2; they will use their own cash to discharge their debts and will have no gains from trade.[13] And we can ignore the buyers and the illiquid bankers who do not receive a shock. Since they have assets but no cash and no need for cash, they will have no incentive to trade either.

Thus, there are three groups of bankers who might engage in trade at date 2. First, there are the hoarders who do not receive a shock. These are the potential suppliers of liquidity. Then there are the buyers and the illiquid bankers who receive a shock. They are the potential demanders of liquidity.

The available supply of cash at date 2 is equal to the number of hoarders (a fraction $(1-\alpha)(1-\theta_1)(1-\lambda)$), who did not receive a liquidity shock at date 2 (a fraction $1-\theta_2$). Thus, the available supply is

$$(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2).$$

It is optimal to supply no cash if $p_2 < R^{-1}$, optimal to supply some cash if $p_2 = R^{-1}$ and optimal to supply all the cash if $p_2 > R^{-1}$. The supply of cash is illustrated in Figure A1a.

—Figure A1 about here—

[13] We are assuming that the agents in this class must discharge their own debt or default and lose the value of any assets they hold. This implies that they cannot trade cash for assets with agents who hold a large number of assets but need cash.

We can construct the demand curve similarly. The demand for cash from buyers comes from the buyers (a fraction $(1-\alpha)(1-\theta_1)\lambda$), who received a liquidity shock at date 2 (a fraction θ_2). Thus, the maximum demand for cash from buyers is

$$(1-\alpha)(1-\theta_1)\lambda\theta_2.$$

Each of the buyers has $1+p_1$ units of the asset. It is optimal for them to sell all of these assets for cash if $p_2 < 1+p_1$ and to sell some of these assets for cash if $p_2 = 1+p_1$.

The number of illiquid bankers demanding cash is equal to the number of illiquid bankers at date 0 (a fraction α), who did not receive a liquidity shock at date 1 (a fraction $1-\theta_1$), and who received a liquidity shock at date 2 (a fraction θ_2). Thus, the maximum demand for cash from illiquid bankers is

$$\alpha(1-\theta_1)\theta_2.$$

Each of these bankers has one unit of the asset. It is optimal for them to sell all of their assets for cash if $p_2 < 1$ and optimal for them to sell some of their assets if $p_2 = 1$. The demand function is illustrated in Figure A1b.

In Panel c of Figure A1 we illustrate the different configurations of the demand and supply curves that may arise for different values of the liquidity shock θ_2. It is clear from Panel c that, except for a set of states of probability zero, the intersection of the supply and demand curves will correspond to one of three regimes. The regime in Panel c(i) occurs when the supply of cash is greater than the maximum demand for cash from illiquid bankers and buyers. In this regime, some hoarders will not be able to exchange cash for the asset, so they must be indifferent between holding and selling cash. This will occur only if the market clearing price is $p_2 = R^{-1}$. The regime in Panel c(ii) occurs when the supply of cash is sufficient to meet the needs of the buyers and some, but not all, illiquid bankers. Then the market will clear if and only if the price is $p_2 = 1$. Finally, the regime in Panel c(iii) occurs when the supply of cash is insufficient to meet even the needs of all the buyers. The market will clear if and only if the price is $p_2 = 1+p_1$.

We can characterize the three different regimes at date 2 in terms of the critical values of θ_2 that divide them. Consider first the regime in Panel c(iii), which occurs if and only if

$$(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2) < (1-\alpha)(1-\theta_1)\lambda\theta_2.$$

This inequality is equivalent to $\theta_2 > \theta_2^{**}$, where θ_2^{**} is implicitly defined by the condition that

$$(1-\lambda)(1-\theta_2^{**}) = \lambda\theta_2^{**}$$

or $\theta_2^{**} = 1-\lambda$.

Next consider the regime in Panel c(ii), which corresponds to

$$(1-\alpha)(1-\theta_1)\lambda\theta_2 < (1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2) < (1-\alpha)(1-\theta_1)\lambda\theta_2 + \alpha(1-\theta_1)\theta_2.$$

These inequalities are equivalent to $\theta_2^* < \theta_2 < \theta_2^{**}$, where θ_2^* is defined by

$$(1-\alpha)(1-\lambda)(1-\theta_2^*) = (1-\alpha)\lambda\theta_2^* + \alpha\theta_2^*$$

or $\theta_2^* = (1-\alpha)(1-\lambda)$.

Then it is easy to see that the regime in Panel c(i) occurs if and only if $\theta_2 < \theta_2^*$.

This gives us the market-clearing price at date 2 as in Proposition 2

$$p_2(\theta_1,\theta_2) = \begin{cases} R^{-1} & \text{for } 0 \leq \theta_2 < \theta_2^*; \\ 1 & \text{for } \theta_2^* < \theta_2 < \theta_2^{**}; \\ 1 + p_1(\theta_1) & \text{for } \theta_2^{**} < \theta_2 \leq 1; \end{cases}$$

where $\theta_2^* = (1-\alpha)(1-\lambda(\theta_1))$ and $\theta_2^{**} = 1-\lambda(\theta_1)$.

7 Appendix B: Proofs

Proof of Proposition 1 Let $m_0 \geq 0$ denote the quantity of cash held at the end of date 0, let $m_1(\theta_1) \geq 0$ denote the amount of cash held at the end of date 1 in state θ_1, and let $m_2(\theta_1, \theta_2) \geq 0$ denote the amount of cash held at the end of date 2 in state (θ_1, θ_2). Feasibility requires

$$m_0 \geq m_1(\theta_1) \geq m_2(\theta_1, \theta_2), \tag{6}$$

for every value of (θ_1, θ_2). The amount of cash distributed at date 1 in state θ_1 is denoted by $x_1(\theta_1)$ and defined by putting

$$x_1(\theta_1) = m_0 - m_1(\theta_1) \geq 0,$$

for every value of θ_1. The amount distributed at date 2 in state (θ_1, θ_2) is denoted by $x_2(\theta_1, \theta_2)$ and defined by putting

$$x_2(\theta_1, \theta_1) = m_1(\theta_1) - m_2(\theta_1, \theta_2) \geq 0,$$

for every value of (θ_1, θ_2).

The expected output from the planner's policy in state (θ_1, θ_2) is

$$R\{x_1(\theta_1) + x_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\} + x_1(\theta_1) + x_2(\theta_1, \theta_2) + m_2(\theta_1, \theta_2) \tag{7}$$

The total amount of the asset at date 3 will be equal to the amount of cash distributed to bankers who receive a liquidity shock at dates 1 and 2, that is, $x_1(\theta) + x_2(\theta_1, \theta_2)$, plus the number of bankers who do not receive a liquidity shock at either date, that is, $(1 - \theta_1)(1 - \theta_2)$. The total amount of cash at date 3 is equal to the amount held by the planner, $m_2(\theta_1, \theta_2)$, plus the amount distributed to the creditors, $x_1(\theta_1) + x_2(\theta_1, \theta_2)$. Multiplying the amounts of cash and the asset by their respective returns and summing them gives the expression in (7). The total surplus is equal to the expected output minus the cost of obtaining liquidity, that is,

$$\begin{aligned} &R\{x_1(\theta_1) + x_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\} + x_1(\theta_1) + x_2(\theta_1, \theta_2) + m_2(\theta_1, \theta_2) - \rho m_0 \quad (8) \\ &= R\{m_0 - m_2(\theta_1, \theta_2) + (1 - \theta_1)(1 - \theta_2)\} + (1 - \rho) m_0, \end{aligned}$$

where we eliminate the constant term $R(1-\theta_1)(1-\theta_2)$ for simplicity. The planner chooses $(x_0, x_1(\cdot))$ to maximize the expected value of (8) subject to the constraints in (6).

We start the analysis at $t = 2$ and go backwards. Suppose that the planner has m_1 units of cash at the beginning of date 2 and the state is (θ_1, θ_2). There are $(1-\theta_1)\theta_2$ bankers in need of cash and the optimal distribution strategy is to supply

$$x_2(\theta_1, \theta_2) = \min\{(1-\theta_1)\theta_2, m_1\}.$$

Thus, the value of m_1 units of cash in state (θ_1, θ_2) is

$$\begin{aligned} V_2(m_1, \theta_1, \theta_2) &= R\min\{(1-\theta_1)\theta_2, m_1\} + m_1 - \min\{(1-\theta_1)\theta_2, m_1\} + \min\{(1-\theta_1)\theta_2, m_1\} \\ &= R\min\{(1-\theta_1)\theta_2, m_1\} + m_1. \end{aligned}$$

For a fixed value of θ_1, the value of m_1 units of cash at the end of date 1 (before θ_2 has been realized) is

$$\begin{aligned} V_2(m_1, \theta_1) &= E[V_2(m_1, \theta_1, \theta_2)|\theta_1] \\ &= R\int_0^{\frac{m_1}{1-\theta_1}} (1-\theta_1)\theta_2 f_2(\theta_2)\, d\theta_2 + m_1 R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + m_1. \end{aligned}$$

The derivative of V_2 with respect to m_1 is calculated to be

$$\begin{aligned} V_2'(m_1, \theta_1) &= R(1-\theta_1)\frac{m_1}{1-\theta_1} f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} - Rm_1 f_2\left(\frac{m_1}{1-\theta_1}\right)\frac{1}{1-\theta_1} + \\ &\quad R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + 1 \\ &= R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + 1. \end{aligned}$$

The expression for $V_2'(m_1, \theta_1)$, the marginal value of cash carried forward to date 2, is quite intuitive. One unit of cash can be converted into one unit of consumption at any date, whether it is held by a creditor or a banker or the planner, but in some cases it has an additional value because it can be used to "save" one unit of the asset that would otherwise be lost in default. This happens if the total supply of cash at date 2, m_1, is less than the demand $(1-\theta_1)\theta_2$ and the probability of this happening is $1 - F_2\left(\frac{m_1}{1-\theta_1}\right)$. So the value of an extra unit of cash is one plus the probability that m_1 is less than $(1-\theta_1)\theta_2$ times R.

Now consider the planner's problem at date 1. He has m_0 units of cash in state θ_1 and must choose the amount x_1 to distribute to bankers. Feasibility requires $0 \leq x_1 \leq m_0$ and, without loss of generality we can assume $x_1 \leq \theta_1$ since there is no point giving cash to a banker who has not received a liquidity shock. Thus, the planner will choose x_1 to maximize

$$(R+1)\,x_1 + V_2\left(m_0 - x_1, \theta_1\right)$$

subject to

$$0 \leq x_1 \leq \min\left\{m_0, \theta_1\right\}. \tag{9}$$

If the constraint (9) is non-binding, the first-order condition

$$\begin{aligned} R+1 &= V_2'\left(m_0 - x_1, \theta_1\right) \\ &= R\left(1 - F_2\left(\frac{m_1}{1-\theta_1}\right)\right) + 1 \end{aligned}$$

must be satisfied. This is possible only if $F_2\left(\frac{m_1}{1-\theta_1}\right) = 0$ or $m_1 = m_0 - x_1 = 0$, a contradiction. Thus, the constraint (9) must bind and this implies that the optimal policy is $x_1 = \min\left\{m_0, \theta_1\right\}$ or

$$m_1\left(\theta_1\right) = \max\left\{m_0 - \theta_1, 0\right\}.$$

Substituting this decision rule into the objective above, we obtain the value function

$$V_1\left(m_0, \theta_1\right) = (R+1)\min\left\{\theta_1, m_0\right\} + V_2\left(\max\left\{m_0 - \theta_1, 0\right\}, \theta_1\right)$$

At the end of date 0, before θ_1 is realized, the value of m_0 units of cash is given by

$$\begin{aligned} E\left[V_1\left(m_0, \theta_1\right)\right] &= \int_0^1 \left[(R+1)\min\left\{\theta_1, m_0\right\} + V_2\left(\max\left\{m_0 - \theta_1, 0\right\}, \theta_1\right)\right] f_1\left(\theta_1\right) d\theta_1 \\ &= \int_0^{m_0} \left[(R+1)\,\theta_1 + V_2\left(m_0 - \theta_1, \theta_1\right)\right] f_1\left(\theta_1\right) d\theta_1 + (R+1)\,m_0\left(1 - F_1\left(m_0\right)\right). \end{aligned}$$

The derivative is easily calculated to be

$$\begin{aligned}
& [(R+1)\,m_0 + V_2\,(0,m_0)]\,f_1\,(m_0) - (R+1)\,m_0 f_1\,(m_0) + \int_0^{m_0} V_2'\,(m_0-\theta_1,\theta_1)\,f_1\,(\theta_1)\,d\theta_1 + \\
& (R+1)\,(1-F_1\,(m_0)) \\
= & \int_0^{m_0} \left[R\left(1-F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)\right)+1\right] f_1\,(\theta_1)\,d\theta_1 + (R+1)\,(1-F_1\,(m_0)) \\
= & (R+1)\,F_1\,(m_0) - \int_0^{m_0} RF_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1\,(\theta_1)\,d\theta_1 + (R+1)\,(1-F_1\,(m_0)) \\
= & R\left(1-\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1\,(\theta_1)\,d\theta_1\right)+1.
\end{aligned}$$

This expression has an intuitive interpretation. The value of an extra unit of cash at date 0 is at least one because cash can be converted into one unit of consumption at any date. In some states, an extra unit of cash is worth an additional R units because it allows the planner to "save" one unit of the asset. This event occurs if and only if m_0 is less than $\theta_1 + (1-\theta_1)\,\theta_2$. The expression in parentheses is simply the probability that m_0 is less than $\theta_1 + (1-\theta_1)\,\theta_2$.

At date 0, the choice of how much liquidity to hold is determined by equating the marginal cost of cash, ρ, to the marginal value of cash. That is, m_0 will be chosen to satisfy the first-order condition

$$R\left(1-\int_0^{m_0} F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right) f_1\,(\theta_1)\,d\theta_1\right)+1=\rho.$$

Proof of Proposition 3 The buyers end date 1 with $1+p_1\,(\theta_1)$ units of the asset and no cash; the hoarders end the period with one unit of the asset and one unit of cash. Consider the buyers first. A fraction θ_2 of the buyers receive a liquidity shock and have a payoff $(1+p_1\,(\theta_1)-p_2\,(\theta_1,\theta_2))\,R$; a fraction $(1-\theta_2)$ do not receive a shock and have a payoff $(1+p_1\,(\theta_1))\,R-1$. Thus, the buyers' expected payoff at date 1 is

$$\begin{aligned}
& \int_0^1 \{\theta_2\,(1+p_1\,(\theta_1)-p_2\,(\theta_1,\theta_2))\,R + (1-\theta_2)\,((1+p_1\,(\theta_1))\,R-1)\}\,f_2\,(\theta_2)\,d\theta_2 \\
= & \int_0^1 \{(1+p_1\,(\theta_1)-\theta_2 p_2\,(\theta_1,\theta_2))\,R-(1-\theta_2)\}\,f_2\,(\theta_2)\,d\theta_2,
\end{aligned}$$

where $p_2\,(\theta_1,\theta_2)$ is a function of θ_2 (given θ_1). Now consider the hoarders. A fraction θ_2 of the hoarders receive a liquidity shock and have a payoff R and a fraction $(1-\theta_2)$ do not receive a shock and have a payoff $(1+p_2(\theta_1,\theta_2))R-1$. Thus, the hoarders' payoff at date 1 is

$$\begin{aligned}
& \int_0^1 \{\theta_2 R + (1-\theta_2)((1+p_2(\theta_1,\theta_2))R-1)\} f_2(\theta_2)\,d\theta_2 \\
= & \int_0^1 \{(1+(1-\theta_2)p_2(\theta_1,\theta_2))R-(1-\theta_2)\} f_2(\theta_2)\,d\theta_2,
\end{aligned}$$

where $p_2(\theta_1,\theta_2)$ is, again, a function of θ_2. It is optimal to buy if and only if the buyers' payoff is at least as great as the hoarders, that is,

$$\int_0^1 \{(1+p_1(\theta_1)(\theta_1)-\theta_2 p_2(\theta_1,\theta_2))R\} f_2(\theta_2)\,d\theta_2 \geq \int_0^1 \{(1+(1-\theta_2)p_2(\theta_1,\theta_2))R\} f_2(\theta_2)\,d\theta_2,$$

or

$$p_1(\theta_1) \geq \int_0^1 p_2(\theta_1,\theta_2) f_2(\theta_2)\,d\theta_2.$$

Similarly, it will be optimal to hoard if and only if

$$p_1(\theta_1) \leq \int_0^1 p_2(\theta_1,\theta_2) f_2(\theta_2)\,d\theta_2.$$

Now we can prove that equilibrium requires $0<\lambda(\theta_1)<1$. From Proposition 2, we know that the distribution of the random variable $p_2(\theta_1,\theta_2)$ is

$$p_2(\theta_1,\theta_2) = \begin{cases} R^{-1} & \text{w. pr. } F_2((1-\alpha)(1-\lambda(\theta_1))) \\ 1 & \text{w. pr. } F_2(1-\lambda(\theta_1)) - F_2((1-\alpha)(1-\lambda(\theta_1))) \\ 1+p_1(\theta_1) & \text{w. pr. } 1-F_2(1-\lambda(\theta_1)) \end{cases}$$

and the expected value of $p_2(\theta_1,\theta_2)$ is

$$\begin{aligned}
E[p_2(\theta_1,\theta_2)|\theta_1] &= F_2((1-\alpha)(1-\lambda(\theta_1)))R^{-1} + (F_2(1-\lambda(\theta_1)) - F_2((1-\alpha)(1-\lambda(\theta_1)))) + \\
& \quad (1-F_2(1-\lambda(\theta_1)))(1+p_1(\theta_1)) \\
&= F_2((1-\alpha)(1-\lambda(\theta_1)))\left(R^{-1}-1\right) - F_2(1-\lambda(\theta_1))p_1(\theta_1) + 1 + p_1(\theta_1).
\end{aligned}$$

Suppose that $\lambda(\theta_1)=0$. Then market clearing at date 1 requires $p_1(\theta_1)=1$ and

$$E[p_2(\theta_1,\theta_2)|\theta_1] = F_2(1-\alpha)\left(R^{-1}-1\right)+1<1.$$

But optimality of hoarding at date 1 requires $p_1(\theta_1) \leq E[p_2(\theta_1, \theta_2)|\theta_1]$. This contradiction establishes that $\lambda(\theta_1) > 0$.

Next, suppose that $\lambda(\theta_1) = 1$. Then market clearing at date 2 requires that

$$E[p_2(\theta_1, \theta_2)] = 1 + p_1(\theta_1).$$

But the optimality of buying at date 1 requires that $p_1(\theta_1) \geq E[p_2(\theta_1, \theta_2)|\theta_1]$, which is clearly impossible. This contradiction establishes that $\lambda(\theta_1) < 1$.

Since $0 < \lambda(\theta_1) < 1$, the liquid bankers must be indifferent between hoarding and buying. From the optimality conditions derived earlier, it is obvious that $p_1(\theta_1) = E[p_2(\theta_1, \theta_2)|\theta_1]$.

Proof of Proposition 4 From Proposition 3, we know what

$$\begin{aligned} p_1(\theta_1) &= E[p_2(\theta_1, \theta_2)|\theta_1] \\ &= F_2((1-\alpha)(1-\lambda(\theta_1)))\left(R^{-1}-1\right) - F_2(1-\lambda(\theta_1))\,p_1(\theta_1) + 1 + p_1(\theta_1) \end{aligned}$$

which implies that

$$p_1(\theta_1) = \frac{1 - F_2((1-\alpha)(1-\lambda(\theta_1)))(1-R^{-1})}{F_2(1-\lambda(\theta_1))}.$$

Using this equation, we can define a function $\tilde{p}(\lambda)$ by putting

$$\tilde{p}(\lambda) = \frac{1 - F_2((1-\alpha)(1-\lambda))(1-R^{-1})}{F_2(1-\lambda)}$$

for any $\lambda \in (0,1)$. The function $\tilde{p}(\lambda)$ is increasing in λ and varies from $1 - F_2((1-\alpha))(1-R^{-1})$ to ∞ as λ varies from 0 to 1. Then there exists a unique value $\bar{\lambda}$ such that $\tilde{p}\left(\bar{\lambda}\right) = 1$ and $\tilde{p}(\lambda) < 1$ if and only if $\lambda < \bar{\lambda}$.

If $\tilde{p}(\lambda(\theta_1)) < 1$ then market clearing requires

$$(1-\alpha)(1-\theta_1)\lambda(\theta_1) = \alpha\theta_1$$

or

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

Let $\bar{\theta}_1$ be the unique value of θ_1 that satisfies

$$\bar{\lambda} = \frac{\alpha \bar{\theta}_1}{(1-\alpha)\left(1-\bar{\theta}_1\right)}.$$

Since the right hand side is increasing in θ_1 and varies from 0 to ∞ as θ_1 varies from 0 to 1 there is a unique solution to this equation and it satisfies $0 < \bar{\theta}_1 < 1$.

We claim that the equilibrium value of λ, call it $\lambda(\theta_1)$, satisfies

$$\lambda(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \bar{\lambda}\right\}$$

for any θ_1. If $\theta_1 < \bar{\theta}_1$ then

$$(1-\alpha)(1-\theta_1)\bar{\lambda} > \alpha\theta_1$$

and market clearing requires $\lambda(\theta_1) < \bar{\lambda}$. Then $p_1(\theta_1) = \tilde{p}(\lambda(\theta_1)) < 1$ implies that all illiquid bankers who receive a liquidity shock must obtain liquidity, that is,

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} < \bar{\lambda}.$$

If $\theta_1 \geq \bar{\theta}_1$, then

$$(1-\alpha)(1-\theta_1)\bar{\lambda} \leq \alpha\theta_1$$

and equilibrium requires $\lambda(\theta_1) = \bar{\lambda}$. To see this, recall that $\lambda(\theta_1) > \bar{\lambda}$ implies that $\tilde{p}(\lambda(\theta_1)) > 1$, which is impossible, and that $\lambda(\theta_1) < \bar{\lambda}$ implies that $(1-\alpha)(1-\theta_1)\lambda(\theta_1) < \alpha\theta_1$ and $\tilde{p}(\lambda(\theta_1)) < 1$, a contradiction. This completes the proof of our claim. Hence,

$$\begin{aligned} p_1(\theta_1) &= \tilde{p}\left(\min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \lambda(\theta_1)\right\}\right) \\ &= \min\left\{\tilde{p}\left(\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right), 1\right\}. \end{aligned}$$

Proof of Proposition 6 We can calculate the expected return of a banker who chooses to hold cash at $t=0$ and chooses to become a hoarder at $t=1$ as follows. With probability θ_1 he is hit by the liquidity shock at $t=1$ and uses his cash for his own investment so that his return is R. With probability $(1-\theta_1)\theta_2$ he is not hit by the liquidity shock at $t=1$ but gets hit at $t=2$, in which case, his return is again R. And with probability $(1-\theta_1)(1-\theta_2)$

he is not hit by the liquidity shock and can use his spare liquidity to acquire $p_2(\theta_1, \theta_2)$ units of the asset at $t = 2$ and his return is $(1 + p_2(\theta_1, \theta_2))R - 1$. Hence, the expected return of a liquid banker that chooses to become a hoarder at $t = 1$ can be written as:

$$\begin{aligned}
& \int_0^1 \int_0^1 \{\theta_1 R + (1-\theta_1)\theta_2 R + (1-\theta_1)(1-\theta_2)((1+p_2(\theta_1,\theta_2))R - 1)\} \; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - \rho \\
= \;\; & R + \int_0^1 \int_0^1 \{(1-\theta_1)(1-\theta_2)(p_2(\theta_1,\theta_2)R - 1)\} \; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - \rho \\
= \;\; & R + \int_0^1 (1-\theta_1)R \underbrace{\left[\int_0^1 p_2(\theta_1,\theta_2) f_2(\theta_2)d\theta_2\right]}_{=p_1(\theta_1)} f_1(\theta_1)d\theta_1 - \\
& \int_0^1 \int_0^1 (1-\theta_1)(\theta_2 p_2(\theta_1,\theta_2)R + (1-\theta_2)) \; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - \rho \\
= \;\; & R + \int_0^1 (1-\theta_1)p_1(\theta_1)R \; f_1(\theta_1)d\theta_1 - \\
& \int_0^1 \int_0^1 (1-\theta_1)(\theta_2 p_2(\theta_1,\theta_2)R + (1-\theta_2)) \; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 - \rho.
\end{aligned}$$

In other words, a hoarder is always guaranteed to have a return of R from his own investment but in case he is not hit by a liquidity shock, he can make an additional return from acquiring assets at $t = 2$.

We can calculate the expected return of illiquid bankers as follows. With probability $(1-\theta_1)(1-\theta_2)$ he is not hit by the liquidity shock and his return is $R-1$. With probability θ_1 he is hit by the liquidity shock at $t = 1$, and sells a fraction of his assets for cash so that his return is $(1 - p_1(\theta_1))R$. With probability $(1-\theta_1)\theta_2$ he is not hit by the liquidity shock at $t = 1$ but gets hit at $t = 2$, in which case his return is $\max\{0, (1 - p_2(\theta_1, \theta_2))R\}$. Hence, the expected return of an illiquid banker can be written as:

$$\begin{aligned}
& \int_0^1 \int_0^1 \{\theta_1(1-p_1(\theta_1))R + (1-\theta_1)(1-\theta_2)(R-1) + (1-\theta_1)\theta_2 \max\{0,(1-p_2)R\}\} \, f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 \\
= \; & R - \int_0^1 \theta_1 p_1(\theta_1) R f_1(\theta_1)d\theta_1 - \int_0^1 \int_0^1 (1-\theta_1)(\theta_2 p_2(\theta_1,\theta_2)R + (1-\theta_2)) \; f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2 + \\
& \int_0^1 \int_{\theta_2 > \theta_2^{**}}^1 (1-\theta_1)\theta_2 p_1(\theta_1) R \; f_2(\theta_2)f_1(\theta_1)d\theta_2 d\theta_1,
\end{aligned}$$

since $1 - p_2(\theta_1, \theta_2) = -p_1(\theta_1)$ for $\theta_2 > \theta_2^{**}$.

In equilibrium, illiquid bankers and hoarders (therefore buyers) should have the same expected return. Note that the first and the third terms in the expected returns for a hoarder and an illiquid banker is common. Hence, in equilibrium, we obtain

$$\int_0^1 p_1(\theta_1)\, f_1(\theta_1) d\theta_1 - \frac{\rho}{R} = \int_0^1 (1-\theta_1) p_1(\theta_1) \underbrace{\left[\int_{\theta_2 > \theta_2^{**}}^1 \theta_2 f_2(\theta_2) d\theta_2\right]}_{=(1-F_2(\theta_2^{**}))E[\theta_2|\theta_2>\theta_2^{**}]} f_1(\theta_1) d\theta_1,$$

which can be written as

$$\int_0^1 p_1(\theta_1) \left\{1 - (1-\theta_1)(1-F_2(\theta_2^{**}))E\left[\theta_2 \left|\theta_2 > \theta_2^{**}\right.\right]\right\} \ f_1(\theta_1) d\theta_1 = \frac{\rho}{R}.$$

Proof of Proposition 7 If a banker chooses to remain illiquid at date 0, his payoff in state (θ_1, θ_2) is

$$\theta_1 R\left(1 - p_1(\theta_1)\right) + (1-\theta_1)\,\theta_2 R\left(1 - p_2(\theta_1, \theta_2)\right) + (1-\theta_1)(1-\theta_2)(R-1), \qquad (10)$$

since with probability θ_1 he receives a liquidity shock at date 1 and gives up $p_1(\theta_1)$ units of the asset for cash (or defaults in the case $p_1(\theta_1) = 1$), with probability $(1-\theta_1)\,\theta_2$ he receives a liquidity shock at date 2 and gives up $p_2(\theta_1, \theta_2)$ units of the asset for cash (or defaults in the case $p_2(\theta_1, \theta_2) = 1$), and with probability $(1-\theta_1)(1-\theta_2)$ he receives no liquidity shock and retains one unit of the asset. By comparison, if he decides to become liquid at date 0, his payoff in state (θ_1, θ_2) is

$$R + (1-\theta_1)(1-\theta_2)\left(p_2(\theta_1, \theta_2) R - 1\right) - \rho, \qquad (11)$$

since the banker can keep his asset for certainty and in the event that he does not receive a liquidity shock, his one unit of cash is worth $p_2(\theta_1, \theta_2)(\theta_1, \theta_2) R$ at date 2. Note that we are here using the fact that hoarding is optimal at date 1. The expected value of (10) is

$$\begin{aligned}
&E\left[\theta_1 R\left(1 - p_1(\theta_1)\right) + (1-\theta_1)\,\theta_2 R\left(1 - p_2(\theta_1, \theta_2)\right) + (1-\theta_1)(1-\theta_2)(R-1)\right] \\
&= E\left[\theta_1 R\left(1 - p_2(\theta_1, \theta_2)(\theta_1, \theta_1)\right) + (1-\theta_1)\,\theta_2 R\left(1 - p_2(\theta_1, \theta_2)\right) + (1-\theta_1)(1-\theta_2)(R-1)\right] \\
&= E\left[R - (\theta_1 + (1-\theta_1)\,\theta_2)\, p_2(\theta_1, \theta_2)\, R - (1-\theta_1)(1-\theta_2)\right].
\end{aligned}$$

Comparing this with the expected value of the payoff (11),

$$E\left[R+\left(1-\theta_1\right)\left(1-\theta_2\right)\left(p_2\left(\theta_1,\theta_2\right)R-1\right)\right]-\rho,$$

we see that not holding liquidity is optimal if and only if

$$E\left[\left(1-\theta_1\right)\left(1-\theta_2\right)p_2\left(\theta_1,\theta_2\right)R\right]-\rho\leq E\left[-\left(\theta_1+\left(1-\theta_1\right)\theta_2\right)p_2\left(\theta_1,\theta_2\right)R\right]$$

or

$$E\left[p_2\left(\theta_1,\theta_2\right)R\right]\leq\rho.$$

From the planner's problem, we have the first-order condition

$$R+1-R\int_0^{m_0}F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)f_1\left(\theta_1\right)d\theta_1=\rho.$$

Since

$$\begin{aligned}E\left[p_2\left(\theta_1,\theta_2\right)|\theta_1\right] &= R^{-1}F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)+\left(1-F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)\right)\\ &= 1-\left(1-R^{-1}\right)F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right),\end{aligned}$$

for $\theta_1<m_0$ and 1 otherwise,

$$\begin{aligned}E\left[p_2\left(\theta_1,\theta_2\right)\right] &= \int_0^{m_0}\left\{1-\left(1-R^{-1}\right)F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)\right\}f_1\left(\theta_1\right)d\theta_1+1-F_1\left(m_0\right)\\ &= 1-\left(1-R^{-1}\right)\int_0^{m_0}F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)f_1\left(\theta_1\right)d\theta_1.\end{aligned}$$

Then

$$\begin{aligned}E\left[p_2\left(\theta_1,\theta_2\right)R\right] &= R-\left(R-1\right)\int_0^{m_0}F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)f_1\left(\theta_1\right)d\theta_1\\ &\leq R+1-R\int_0^{m_0}F_2\left(\frac{m_0-\theta_1}{1-\theta_1}\right)f_1\left(\theta_1\right)d\theta_1\\ &= \rho,\end{aligned}$$

as required.

Proof of Proposition 8 Let $(1-\alpha)(1-\theta_1)\lambda(\theta_1)$ be the measure of buyers at $t=1$, which in equilibrium equals the number of illiquid bankers that manage to borrow. There are three cases to consider at $t=2$.

i) For $\theta_2 < \theta_2^*$, there is enough liquidity at $t=2$ for all bankers that got hit by the liquidity shock at $t=2$. In that case, there are $1-\alpha\theta_1+(1-\alpha)(1-\theta_1)\lambda(\theta_1)$ units of the asset since all the assets except for the ones held by illiquid bankers hit by the shock at $t=1$ who could not get the needed liquidity (a measure of $\alpha\theta_1-(1-\alpha)(1-\theta_1)\lambda(\theta_1))$ are pursued until $t=3$. In that case, the assets have a return of $(1-\alpha\theta_1+(1-\alpha)(1-\theta_1)\lambda(\theta_1))R$ at $t=3$. Furthermore, the creditors received $(1-\alpha)(1-\theta_1)\lambda(\theta_1)+(1-\alpha)\theta_1$ and $\theta_2(1-\theta_1)$ at $t=1$ and $t=2$, respectively. And, there are $(1-\alpha)-[(1-\alpha)(1-\theta_1)\lambda(\theta_1)+(1-\alpha)\theta_1+\theta_2(1-\theta_1)]$ units of cash left with the hoarders. Hence, the total output at $t=3$ is

$$(1-\alpha\theta_1+(1-\alpha)(1-\theta_1)\lambda(\theta_1))R+(1-\alpha).$$

ii) For $\theta_2^* < \theta_2 < \theta_2^{**}$, there is enough liquidity for all buyers that get hit by the liquidity shock at $t=2$ but not enough for all illiquid bankers that get hit at $t=2$. Hence, some of the long assets held by illiquid bankers that got hit at $t=2$ need to be liquidated prematurely, in addition to the $\alpha\theta_1-(1-\alpha)(1-\theta_1)\lambda(\theta_1)$ units that got liquidated at $t=1$.

At $t=2$, the supply of cash comes from the hoarders that did not get hit by the liquidity shock at $t=2$, which has a measure of $(1-\alpha)(1-\theta_1)(1-\lambda(\theta_1))(1-\theta_2)$. The buyers who got hit by the liquidity shock at $t=2$ are the ones to receive cash first so that only $(1-\alpha)(1-\theta_1)(1-\lambda(\theta_1))(1-\theta_2)-(1-\alpha)(1-\theta_1)\lambda(\theta_1)\theta_2$ units of cash is left for illiquid bankers hit by the shock at $t=2$. Hence, the measure of assets that get liquidated prematurely at $t=2$ can be calculated as:

$$\begin{aligned} & \alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)(1-\lambda(\theta_1))(1-\theta_2)+(1-\alpha)(1-\theta_1)\lambda(\theta_1)\theta_2 \\ = \ & \alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)\left[(1-\lambda(\theta_1))(1-\theta_2)-\lambda(\theta_1)\theta_2\right] \\ = \ & \alpha(1-\theta_1)\theta_2-(1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1)) \end{aligned}$$

Hence, the number of assets that got liquidated prematurely (both at $t = 1$ and $t = 2$) is:

$$\begin{aligned} & \alpha\theta_1 - (1-\alpha)(1-\theta_1)\lambda(\theta_1) + \alpha(1-\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1)) \\ = & \alpha\theta_1 + \alpha(1-\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\theta_2) \\ = & \alpha - (1-\theta_1)(1-\theta_2). \end{aligned}$$

Hence, the total output at $t = 3$ is

$$(1 - \alpha + (1-\theta_1)(1-\theta_2))R + (1-\alpha).$$

iii) For $\theta_2 > \theta_2^{**}$, there is not enough liquidity even for all buyers that got hit by the liquidity shock at $t = 2$. Hence, some of the long assets held by illiquid bankers that got hit at $t = 2$ and all the assets held by illiquid bankers that got hit at $t = 2$ need to be liquidated prematurely, in addition to the $\alpha\theta_1 - (1-\alpha)(1-\theta_1)\lambda(\theta_1)$ units that got liquidated at $t = 1$.

At $t = 2$, the supply of cash comes from the hoarders that did not get hit by the liquidity shock at $t = 2$, which has a measure of $(1-\alpha)(1-\theta_1)(1-\lambda(\theta_1))(1-\theta_2)$. Hence, only a measure $(1-\alpha)(1-\theta_1)(1-\lambda(\theta_1))(1-\theta_2)$ of the buyers who got hit by the liquidity shock at $t = 2$ can get liquidity at $t = 2$, whereas the rest, which has a measure $(1-\alpha)(1-\theta_1)\lambda(\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\lambda(\theta_1))(1-\theta_2)$, gets liquidated. Hence, the measure of assets that gets liquidated prematurely at $t = 2$ can be calculated as:

$$\begin{aligned} & \alpha(1-\theta_1)\theta_2 + \{(1-\alpha)(1-\theta_1)\lambda(\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\lambda(\theta_1))(1-\theta_2)\}(1+p_1(\theta_1)) \\ = & \alpha(1-\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1))(1+p_1(\theta_1)) \end{aligned}$$

Hence, the number of assets that got liquidated prematurely (both at $t = 1$ and $t = 2$) is:

$$\begin{aligned} & \alpha\theta_1 - (1-\alpha)(1-\theta_1)\lambda(\theta_1) + \alpha(1-\theta_1)\theta_2 - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1))(1+p_1(\theta_1)) \\ = & \alpha_1 - (1-\theta_1)(1-\theta_2) - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1))p_1(\theta_1). \end{aligned}$$

Hence, the total output at $t = 3$ is

$$(1 - [\alpha_1 - (1-\theta_1)(1-\theta_2) - (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1))p_1(\theta_1)])R + (1-\alpha).$$

Using the output for the three different regions of θ_2 given above, we can calculate the total expected output as

$$\begin{aligned} E(\Pi) &= R + (1-\alpha) - \int_0^{\theta_2^*} \left[\alpha\theta_1 - (1-\alpha)(1-\theta_1)\lambda(\theta_1)\right] R f_2(\theta_2) d\theta_2 \\ & -\int_{\theta_2^*}^{\theta_2^{**}} \left[\alpha - (1-\theta_1)(1-\theta_2)\right] R f_2(\theta_2) d\theta_2 - \int_{\theta_2^{**}}^{1} \left[\alpha - (1-\theta_1)(1-\theta_2)\right] R f_2(\theta_2) d\theta_2 \\ & + \int_{\theta_2^{**}}^{1} (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1)) p_1(\theta_1) R f_2(\theta_2) d\theta_2, \end{aligned}$$

which can be written as

$$\begin{aligned} E(\Pi) &= (1-\alpha)(R+1) + (1-\theta_1)(1-E[\theta_2])R \\ & +R\int_0^{\theta_2^*} \left[\alpha(1-\theta_1) + (1-\alpha)(1-\theta_1)\lambda(\theta_1) - (1-\theta_1)(1-\theta_2)\right] f_2(\theta_2) d\theta_2 + \\ & +R\int_{\theta_2^{**}}^{1} (1-\alpha)(1-\theta_1)(1-\theta_2-\lambda(\theta_1)) p_1(\theta_1) f_2(\theta_2) d\theta_2. \end{aligned}$$

In what follows, we restrict attention to the case where $p_1(\theta_1) \equiv 1$, for reasons we explain later. Using Leibniz's rule, we can obtain the effect on total output of a small change in $\lambda(\theta_1)$ as

$$\begin{aligned} & R\left[\alpha(1-\theta_1) + (1-\alpha)(1-\theta_1)\lambda(\theta_1) - (1-\theta_1)(1-\theta_2^*)\right] f_2(\theta_2^*) \frac{d\theta_2^*}{d\lambda(\theta_1)} + R(1-\alpha)(1-\theta_1) F_2(\theta_2^*) - \\ & R(1-\alpha)(1-\theta_1)(1-\theta_2^{**}-\lambda(\theta_1)) p_1(\theta_1) f_2(\theta_2^{**}) \frac{d\theta_2^{**}}{d\lambda(\theta_1)} - R(1-\alpha)(1-\theta_1)(1-F_2(\theta_2^{**})). \end{aligned}$$

Using $\theta_2^{**} = 1-\lambda(\theta_1)$ and $\theta_2^* = (1-\alpha)(1-\lambda(\theta_1))$, we can show that

$$\begin{aligned} & \left[\alpha(1-\theta_1) + (1-\alpha)(1-\theta_1)\lambda(\theta_1) - (1-\theta_1)(1-\theta_2^*)\right] = \\ & \qquad (1-\theta_1)\left[\alpha + (1-\alpha)\lambda(\theta_1) - (1-(1-\alpha)(1-\lambda(\theta_1)))\right] = \\ & \qquad\qquad (1-\theta_1)\left[\alpha - 1 + (1-\alpha)\right] = 0, \end{aligned}$$

and

$$(1-\alpha)(1-\theta_1)(1-\theta_2^{**}-\lambda(\theta_1)) = (1-\alpha)(1-\theta_1)(1-(1-\lambda(\theta_1))-\lambda(\theta_1)) = 0.$$

Hence, the derivative reduces to

$$\begin{aligned} & R(1-\alpha)(1-\theta_1)F_2(\theta_2^*) - R(1-\alpha)(1-\theta_1)(1-F_2(\theta_2^{**})) \\ = \; & R(1-\alpha)(1-\theta_1)\{F_2(\theta_2^*) - (1-F_2(\theta_2^{**}))\} \end{aligned}$$

and the sign of the derivative is determined by the sign of

$$F_2(\theta_2^*) - (1-F_2(\theta_2^{**})) = F_2((1-\alpha)(1-\lambda(\theta_1))) - (1-F_2(1-\lambda(\theta_1))).$$

Now we have to consider two cases, depending on whether θ_1 is greater or less than $\bar{\theta}_1$.

Case 1: Suppose that $\theta_1 > \bar{\theta}_1$. Then $p_1(\theta_1) \equiv 1$ and the equilibrium condition is

$$F_2(1-\lambda(\theta_1)) = 1 - F_2((1-\alpha)(1-\lambda(\theta_1)))\left(1-R^{-1}\right).$$

But $1 - R^{-1} < 1$ implies that

$$F_2(1-\lambda(\theta_1)) > 1 - F_2((1-\alpha)(1-\lambda(\theta_1))),$$

so

$$F_2((1-\alpha)(1-\lambda(\theta_1))) - (1-F_2(1-\lambda(\theta_1))) > 0$$

and an increase in $\lambda(\theta_1)$ increases total output.

Case 2: Now suppose that $\theta_1 < \bar{\theta}_1$ so that all liquidity needs are met at date 1. Then $\lambda(\theta_1)$ cannot be increased. If $\lambda(\theta_1)$ is decreased a small amount, there will be excess demand for liquidity and the price will jump to $p_1(\theta_1) = 1$. The effect of a small change in $\lambda(\theta_1)$ will correspond to our earlier calculation with $p_1(\theta_1) = 1$. Also, for $\theta_1 < \bar{\theta}_1$,

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} < \lambda(\theta_1),$$

so

$$\begin{aligned} & \frac{d}{d\lambda(\theta_1)}\{F_2((1-\alpha)(1-\lambda(\theta_1))) - (1-F_2(1-\lambda(\theta_1)))\} \\ = \; & -(1-\alpha)f_2((1-\alpha)(1-\lambda(\theta_1))) - f_2((1-\lambda(\theta_1))) < 0. \end{aligned}$$

implies that

$$F_2\left(\left(1-\alpha\right)\left(1-\lambda\left(\theta_1\right)\right)\right)-\left(1-F_2\left(1-\lambda\left(\theta_1\right)\right)\right) > F_2\left(\left(1-\alpha\right)\left(1-\lambda\left(\theta_1\right)\right)\right)-\left(1-F_2\left(1-\lambda\left(\theta_1\right)\right)\right) > 0.$$

So if we increase hoarding a little bit at date 1, this result tells us that it is better to reduce hoarding, i.e., increase $\lambda\left(\theta_1\right)$. In the limit, when $\lambda\left(\theta_1\right)$ reaches its equilibrium value, there will be a jump in the allocation, as the drop in $p_1\left(\theta_1\right)$ triggers a non-negligible transfer of assets back to the illiquid bankers. This will have a further positive impact on output, since the illiquid bankers cannot receive another liquidity shock and so it is better for them to hold more assets. Thus, it is not optimal to reduce $\lambda\left(\theta_1\right)$ and it is not feasible to increase $\lambda\left(\theta_1\right)$.

From the analysis of the two cases above, we can characterize the socially optimal level of $\lambda\left(\theta_1\right)$ as follows:

$$\lambda^{soc}(\theta_1) = \min\left\{\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}, \tilde{\lambda}\left(\theta_1\right)\right\},$$

where $\tilde{\lambda}\left(\theta_1\right)$ is determined implicitly by the FOC

$$F_2((1-\alpha)\left(1-\tilde{\lambda}\left(\theta_1\right)\right)) + F_2(1-\tilde{\lambda}\left(\theta_1\right)) = 1.$$

Proof of Proposition 9 We have the expected output as a function of θ_1 as follows:

$$\begin{aligned} E(\Pi\left(\theta_1\right)) &= (1-\alpha)(R+1-\rho) + (1-\theta_1)(1-E(\theta_2)) + \\ &\quad (1-\theta_1)R\int_0^{\theta_2^*}\left(\alpha + (1-\alpha)\lambda\left(\theta_1\right) - (1-\theta_2)\right) f_2(\theta_2)d\theta_2 - \\ &\quad (1-\theta_1)R\int_{\theta_2^{**}}^1 \left\{(1-\alpha)\left(\lambda\left(\theta_1\right) - (1-\theta_2)\right) p_1\left(\theta_1\right)\right\} f_2(\theta_2)d\theta_2. \end{aligned}$$

Using the Leibniz's rule, we obtain:

$$
\begin{aligned}
\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-\rho) \\
&+(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\
&+(1-\theta_1)R\left(\alpha+(1-\alpha)\lambda(\theta_1)-(1-\theta_2^*)\right)f_2(\theta_2^*)\left[\frac{d\theta_2^*}{d\alpha}\right] \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^1\left((1-\theta_2)-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)p_1(\theta_1)\;\; f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(\frac{dp_1(\theta_1)}{d\alpha}\right)\{(1-\alpha)\left(\lambda(\theta_1)-(1-\theta_2)\right)\}f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R(1-\alpha)\left(\lambda(\theta_1)-(1-\theta_2^{**})\right)p_1(\theta_1)\left[\frac{d\theta_2^{**}}{d\alpha}\right].
\end{aligned}
$$

Using $\theta_2^* = (1-\alpha)(1-\lambda(\theta_1))$ and $\theta_2^{**} = 1-\lambda(\theta_1)$, we obtain $\lambda(\theta_1)-(1-\theta_2^{**}) = 0$, and

$$
\alpha+(1-\alpha)\lambda(\theta_1)-(1-\theta_2^*) = \alpha+(1-\alpha)\lambda(\theta_1)-(1-(1-\alpha)(1-\lambda(\theta_1))) = 0,
$$

so that the 3rd and the 6th expressions disappear, which gives us

$$
\begin{aligned}
\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-\rho) \\
&+(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^1\left((1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)p_1(\theta_1)\;\; f_2(\theta_2)d\theta_2 \\
&+(1-\theta_1)R\int_{\theta_2^{**}}^1\theta_2 p_1(\theta_1)\;\; f_2(\theta_2)d\theta_2 \\
&-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(\frac{dp_1(\theta_1)}{d\alpha}\right)\{(1-\alpha)\left(\lambda(\theta_1)-(1-\theta_2)\right)\}f_2(\theta_2)d\theta_2.
\end{aligned}
$$

From the equilibrium condition at $t=0$, we have

$$
R\int_0^1 p_1(\theta_1)\;\; f_1(\theta_1)d\theta_1-\rho = R\int_0^1(1-\theta_1)p_1(\theta_1)\left[\int_{\theta_2^{**}}^1\theta_2\;\; f_2(\theta_2)d\theta_2\right]f_1(\theta_1)d\theta_1.
$$

Even though the condition holds on average over θ_1, we can still plug this in the above derivative to get:

$$\begin{aligned}\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-\rho) \\ &+(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\ &-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)p_1(\theta_1)\;\; f_2(\theta_2)d\theta_2+(p_1(\theta_1)R-\rho) \\ &-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(\frac{dp_1(\theta_1)}{d\alpha}\right)\{(1-\alpha)(\lambda(\theta_1)-(1-\theta_2))\}f_2(\theta_2)d\theta_2.\end{aligned}$$

Case 1: $\theta_1 < \overline{\theta}_1$

For $\theta_1 < \overline{\theta}_1$, we have $\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}$ so that $\frac{\partial\lambda(\theta_1)}{\partial\alpha} = \frac{\theta_1}{(1-\alpha)^2(1-\theta_1)}$. Hence,

$$1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha} = 1+\frac{\theta_1}{1-\theta_1} = \frac{1}{1-\theta_1}.$$

Using this, we can obtain:

$$\begin{aligned}\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-\rho) \\ &+RF_2(\theta_2^*)-Rp_1(\theta_1)(1-F_2(\theta_2^{**}))+p_1(\theta_1)R-\rho \\ &-(1-\theta_1)R\int_{\theta_2^{**}}^1\left(\frac{dp_1(\theta_1)}{d\alpha}\right)\{(1-\alpha)(\lambda(\theta_1)-(1-\theta_2))\}f_2(\theta_2)d\theta_2.\end{aligned}$$

Note that, in this region,

$$p_1(\theta_1) = \frac{1-F_2(\theta_2^*)\left(1-\frac{1}{R}\right)}{F_2(\theta_2^{**})},$$

so that $Rp_1(\theta_1)F_2(\theta_2^{**}) = R\left[1-F_2(\theta_2^*)\left(1-\frac{1}{R}\right)\right]$. Using this, we obtain

$$\frac{dE(\Pi(\theta_1))}{d\alpha} = -(1-F_2(\theta_2^*))-R(1-\theta_1)\int_{\theta_2^{**}}^1\left(\frac{\partial p_1(\theta_1)}{\partial\alpha}\right)\{(1-\alpha)(\lambda(\theta_1)-(1-\theta_2))\}f_2(\theta_2)d\theta_2.$$

In this case, we know that

$$p_1(\theta_1) = \frac{1-F_2\left(\frac{1-\alpha-\theta_1}{1-\theta_1}\right)\left(1-\frac{1}{R}\right)}{F_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)}.$$

Hence, we obtain:

$$\frac{dp_1(\theta_1)}{d\alpha} = \frac{f_2\left(\frac{1-\alpha-\theta_1}{1-\theta_1}\right)\left(\frac{1}{\theta_1}\right)\left(1-\frac{1}{R}\right)}{F_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)} + \frac{\left[1-F_2\left(\frac{1-\alpha-\theta_1}{1-\theta_1}\right)\left(1-\frac{1}{R}\right)\right] f_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)\left(\frac{\theta_1}{(1-\alpha)^2(1-\theta_1)}\right)}{\left[F_2\left(\frac{1-\alpha-\theta_1}{(1-\alpha)(1-\theta_1)}\right)\right]^2}.$$

Note that $\frac{dp_1(\theta_1)}{d\alpha} > 0$. Hence, we obtain $\frac{dE(\Pi(\theta_1))}{d\alpha} < 0$.

Case 2: $\theta_1 > \overline{\theta}_1$

For $\theta_1 > \overline{\theta}_1$, we have $p_1(\theta_1) = 1$. Using this, we obtain:

$$\begin{aligned}\frac{dE(\Pi(\theta_1))}{d\alpha} &= -(R+1-\rho) \\ &\quad +(1-\theta_1)R\int_0^{\theta_2^*}\left(1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\ &\quad -(1-\theta_1)R\int_{\theta_2^{**}}^1\left((1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)f_2(\theta_2)d\theta_2 \\ &\quad \underbrace{+(1-\theta_1)R\int_{\theta_2^{**}}^1 \theta_2 p_1(\theta_1)\ \ f_2(\theta_2)d\theta_2}_{=p_1(\theta_1)R-\rho \text{ in equilibrium at } t=1}.\end{aligned}$$

We can write the above expression as:

$$\frac{dE(\Pi(\theta_1))}{d\alpha} = -1 + R\left[F_2(\theta_2^*) - 1 + F_2(\theta_2^{**})\right]\left(1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right).$$

Furthermore, from the equilibrium at $t=1$, we have

$$F_2(\theta_2^*) - 1 + F_2(\theta_2^{**}) = F_2(\theta_2^*)\left(\frac{1}{R}\right).$$

Hence, we get

$$\frac{\partial E(\Pi)}{\partial\alpha} = -1 + \left(1-\lambda(\theta_1)+(1-\alpha)\frac{d\lambda(\theta_1)}{d\alpha}\right)\left[F_2(\theta_2^*)\right].$$

Using the implicit function theorem, we get

$$-\frac{d\bar{\lambda}(\theta_1)}{d\alpha}f_2(1-\bar{\lambda}(\theta_1)) = f_2((1-\alpha)(1-\bar{\lambda}(\theta_1)))\left(1-\frac{1}{R}\right)\left[(1-\alpha)\frac{d\bar{\lambda}(\theta_1)}{d\alpha}+1-\bar{\lambda}(\theta_1)\right],$$

so that

$$\frac{d\bar{\lambda}\left(\theta_1\right)}{d\alpha} = -\frac{f_2((1-\alpha)(1-\bar{\lambda}\left(\theta_1\right)))\left(1-\frac{1}{R}\right)(1-\bar{\lambda}\left(\theta_1\right))}{f_2(1-\bar{\lambda}\left(\theta_1\right)) + f_2((1-\alpha)(1-\bar{\lambda}\left(\theta_1\right)))\left(1-\frac{1}{R}\right)(1-\alpha)} < 0.$$

This gives us

$$\frac{\partial E(\Pi)}{\partial \alpha} = \underbrace{-1 + (1-\lambda\left(\theta_1\right))F_2(\theta_2^*)}_{<0} + \underbrace{\left((1-\alpha)\frac{d\lambda\left(\theta_1\right)}{d\alpha}\right)}_{<0}\left[F_2(\theta_2^*)\right] < 0.$$

Figure 1: Timeline

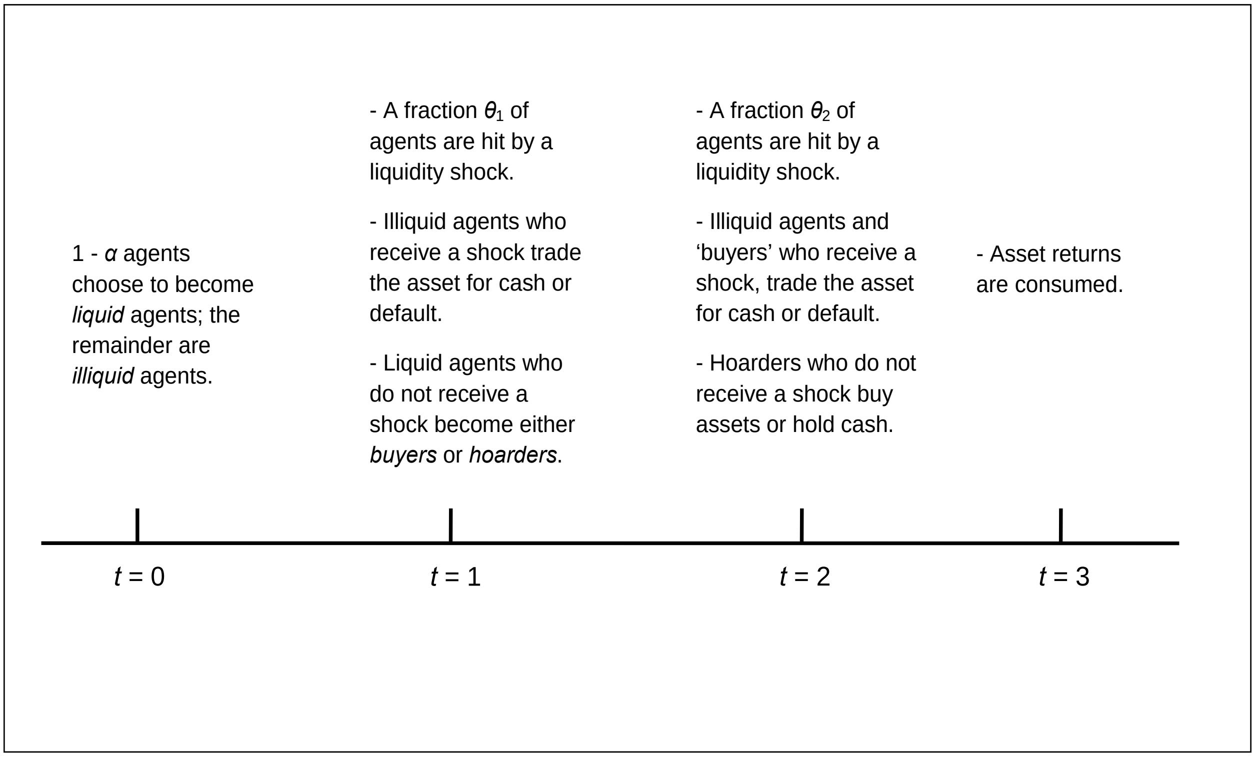


Figure 2: Allocations at dates 0 and 1

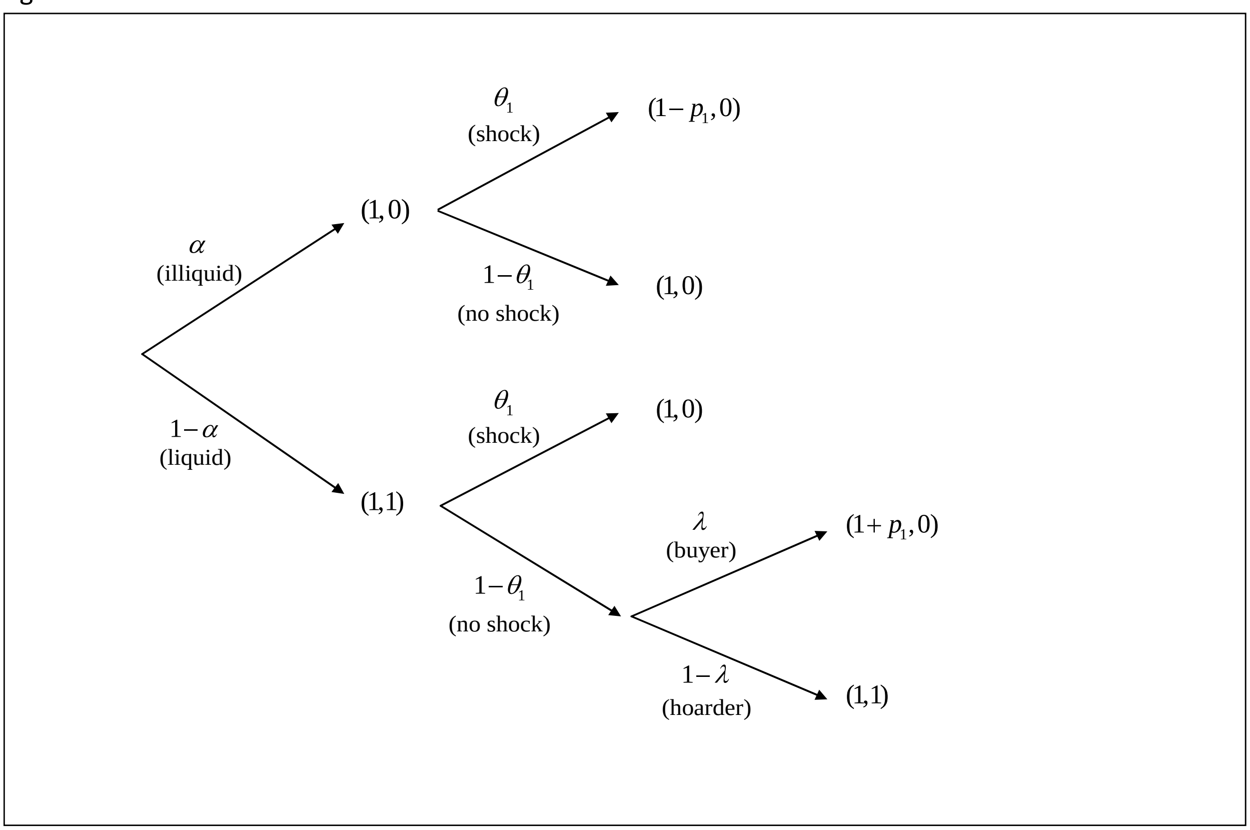


Figure 3a: Allocations at date 2

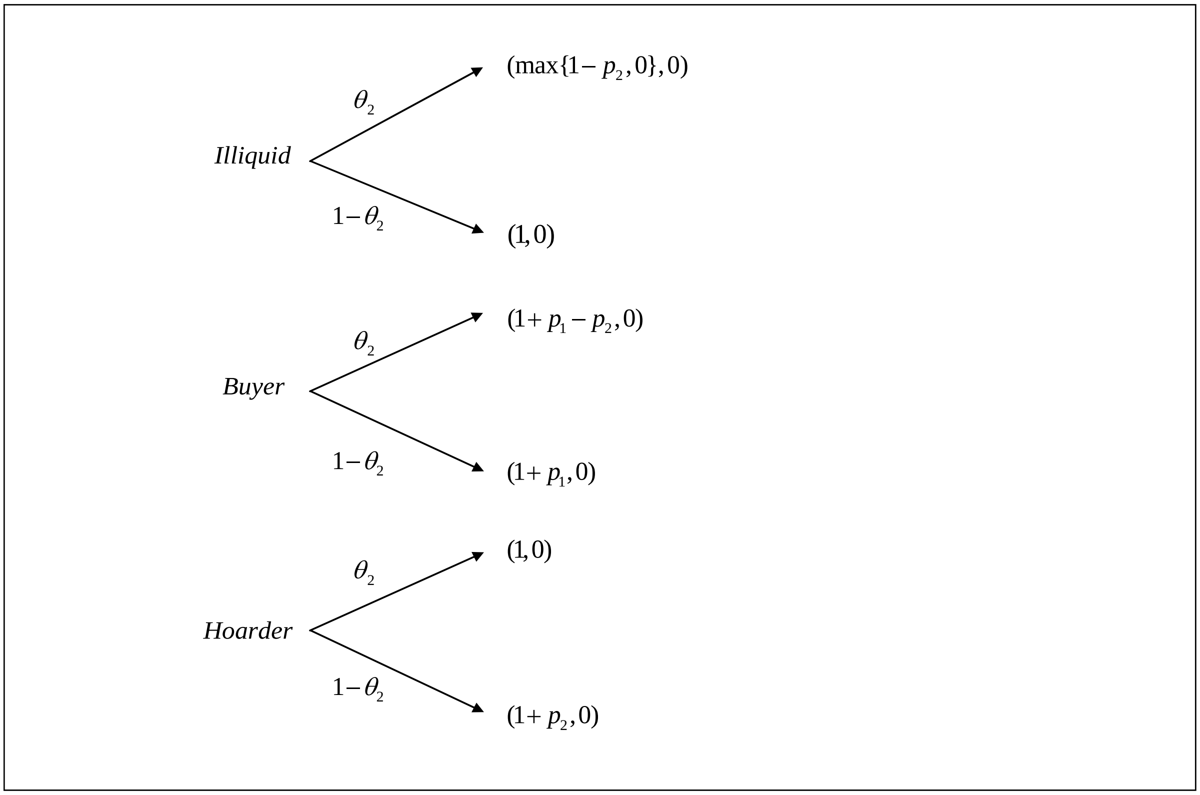


Figure 3b: Allocations at date 0, 1 and 2.

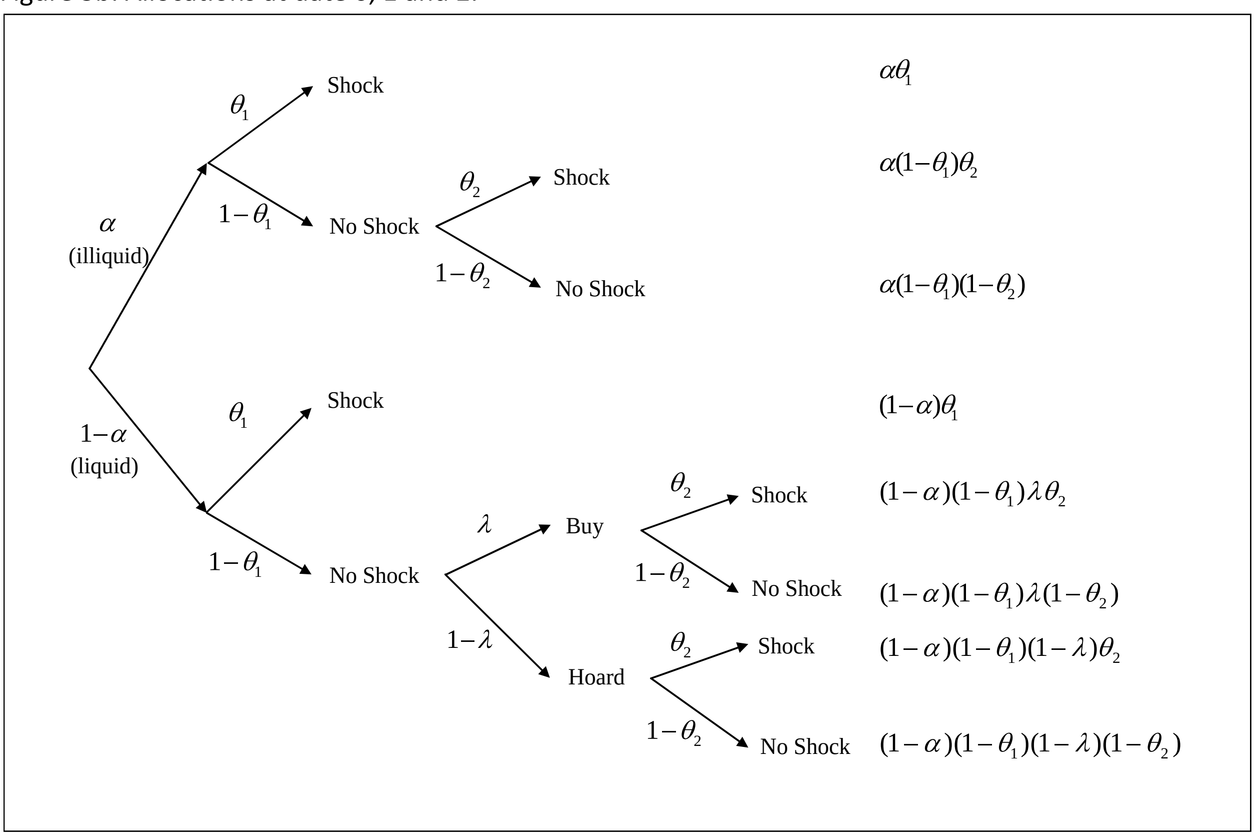


Figure 4: Terminal payoffs

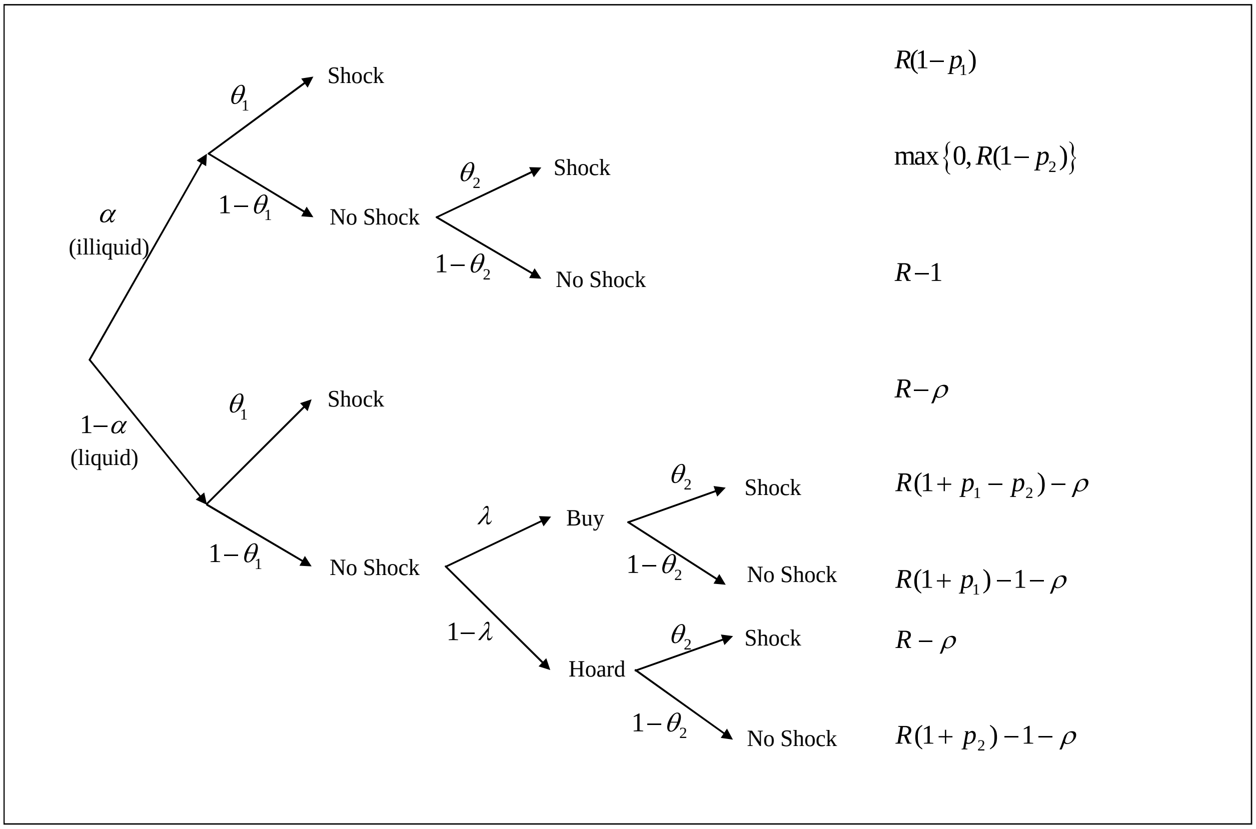


Figure 5: Market-clearing price p_2 (Proposition2)

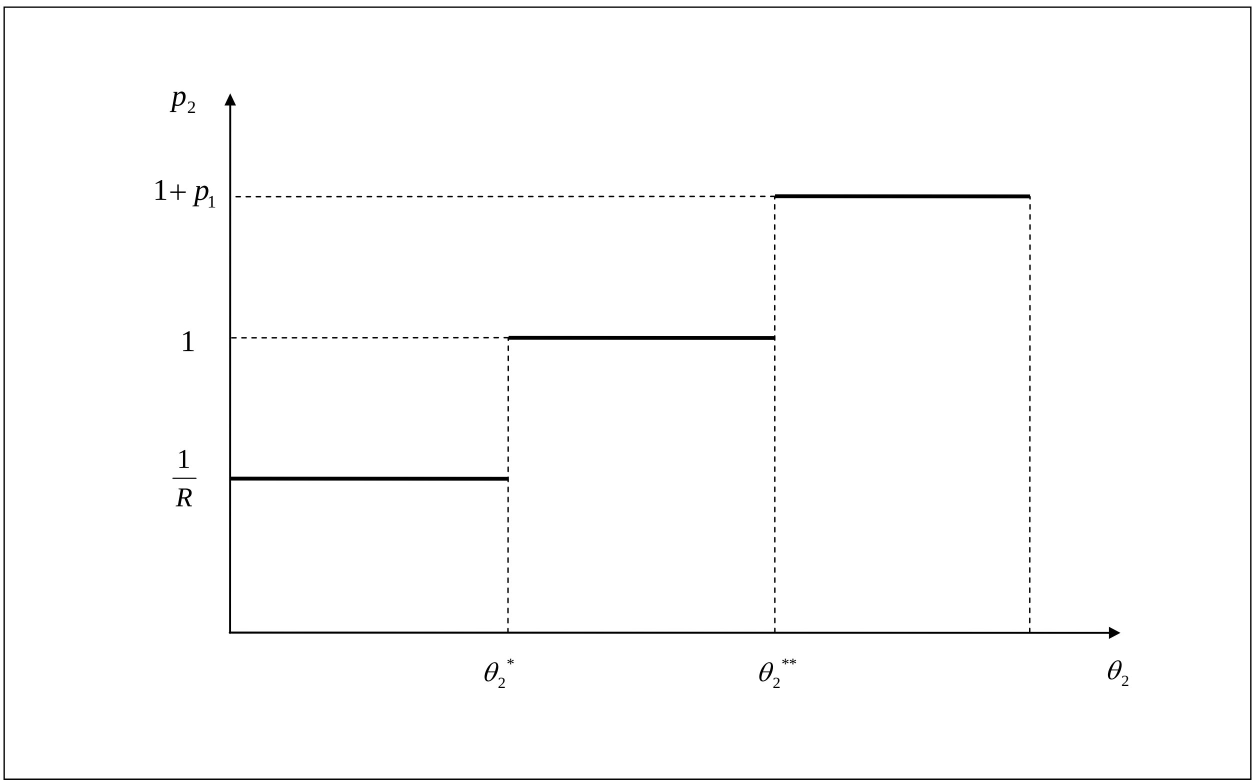


Figure6a: Equilibrium $\lambda(\theta_1)$

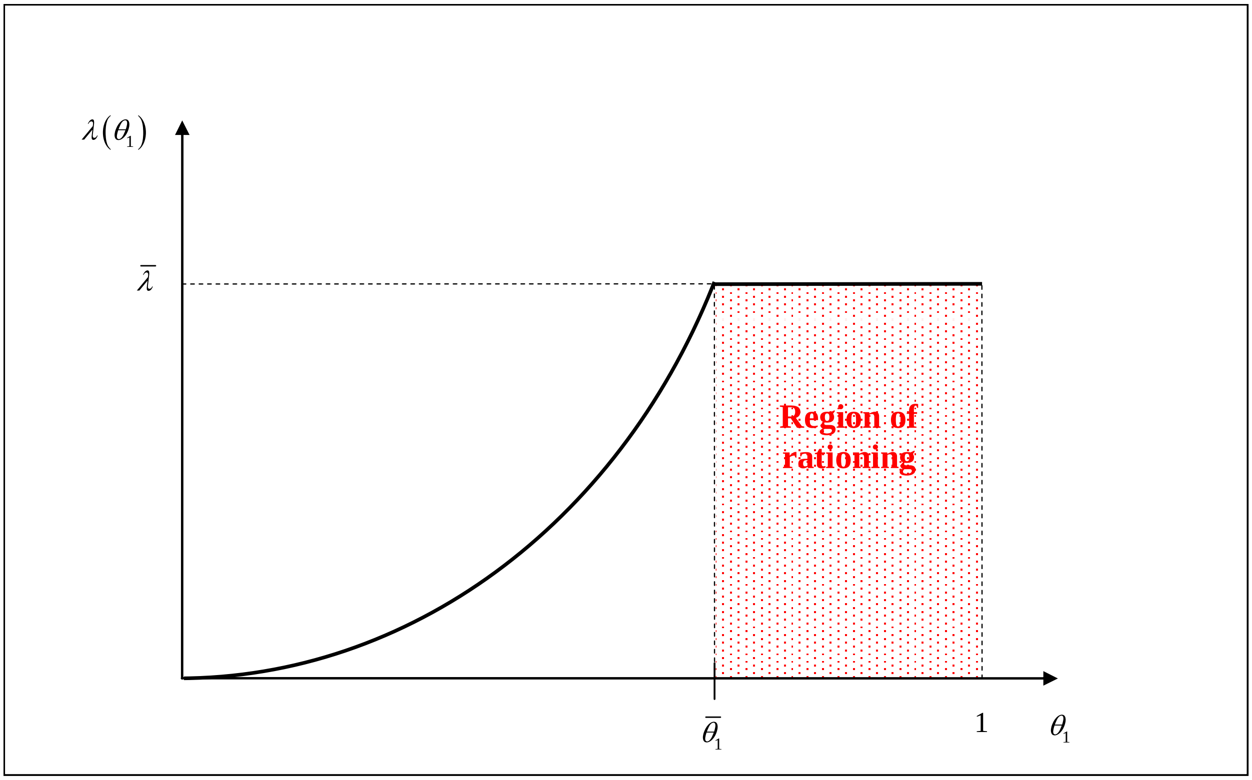


Figure6b: Equilibrium $p_1(\theta_1)$

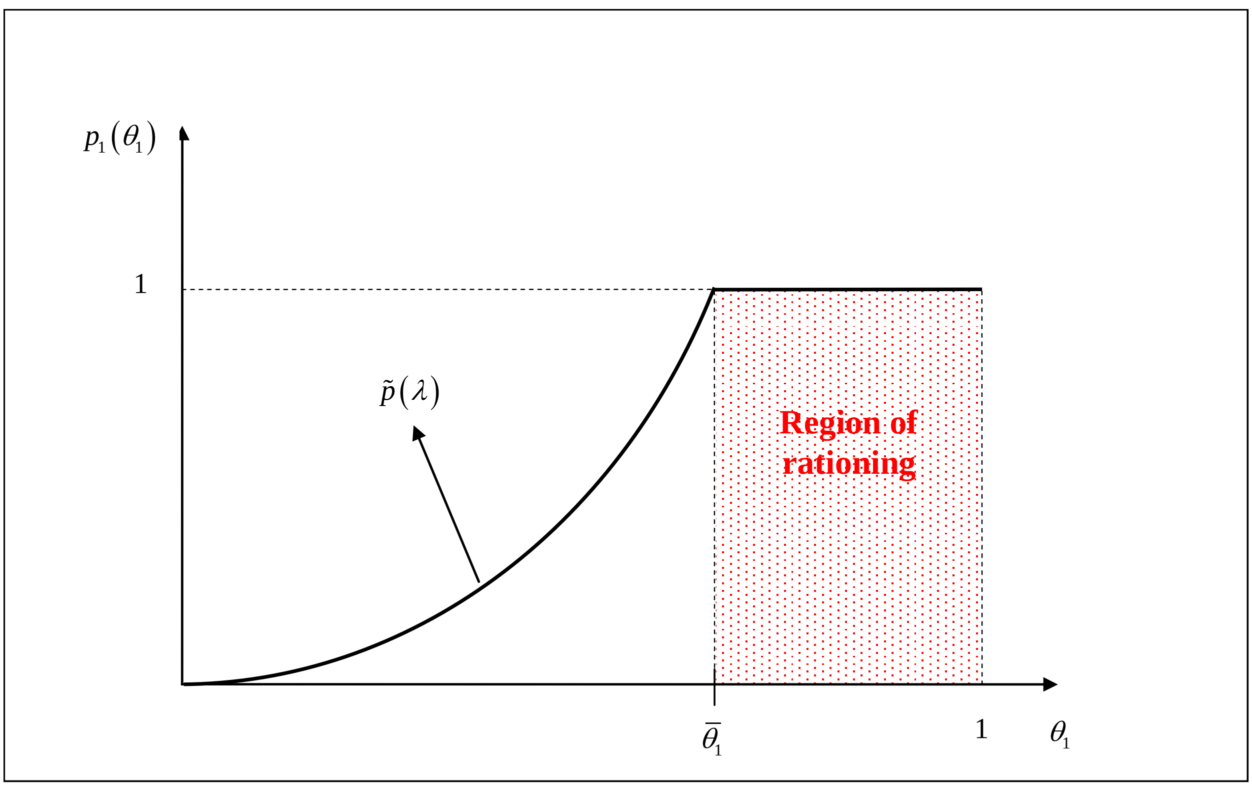


Figure 7a: Equilibrium and socially optimal levels of λ as a function of θ_1 for R=3 and ρ=2

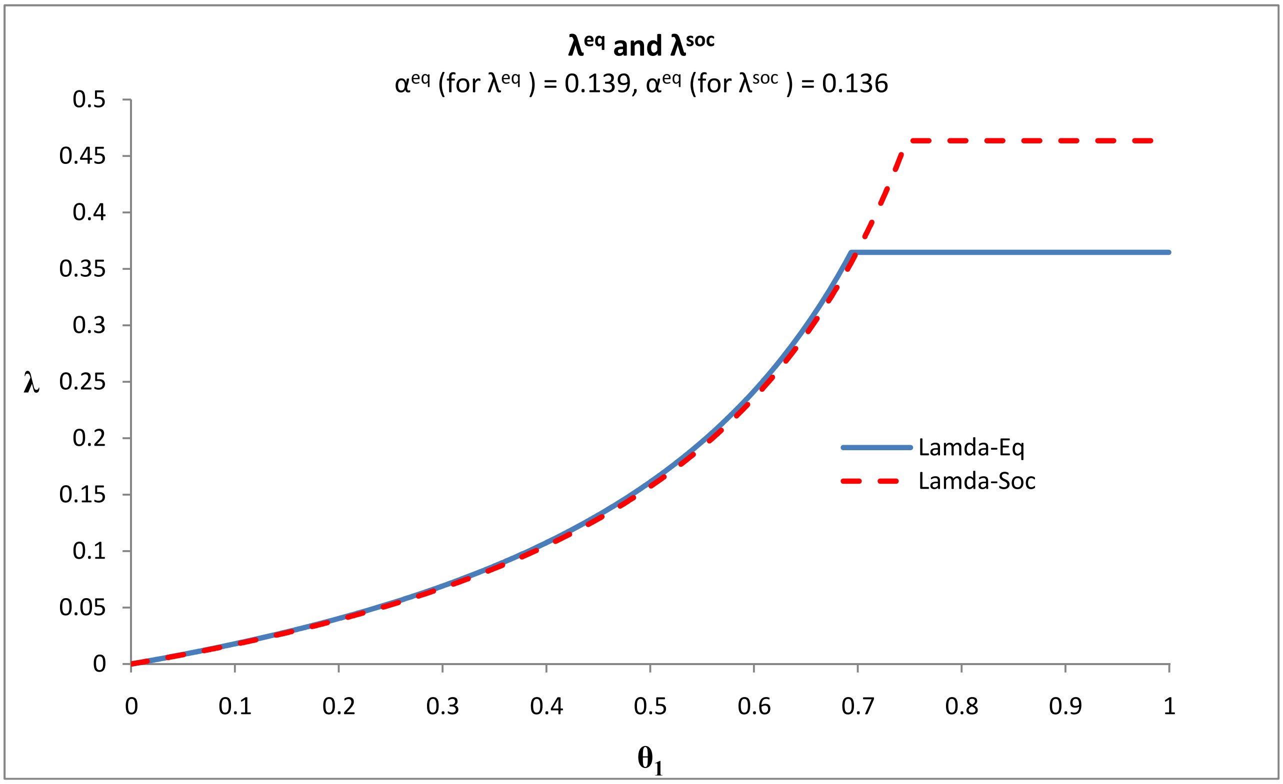
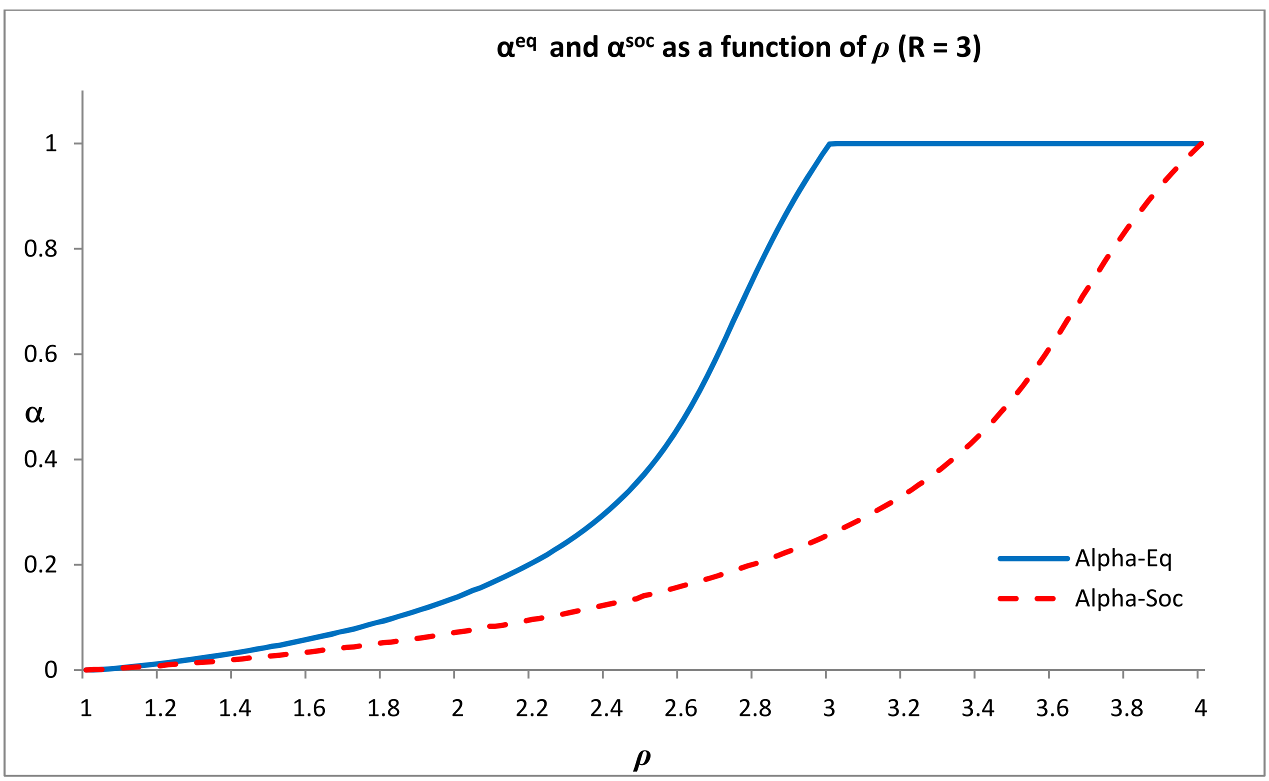


Figure 7b: Equilibrium and socially optimal levels of α as a function of ρ for R=3



Figures for Appendix A

Figure A1a: Supply of cash at date 2

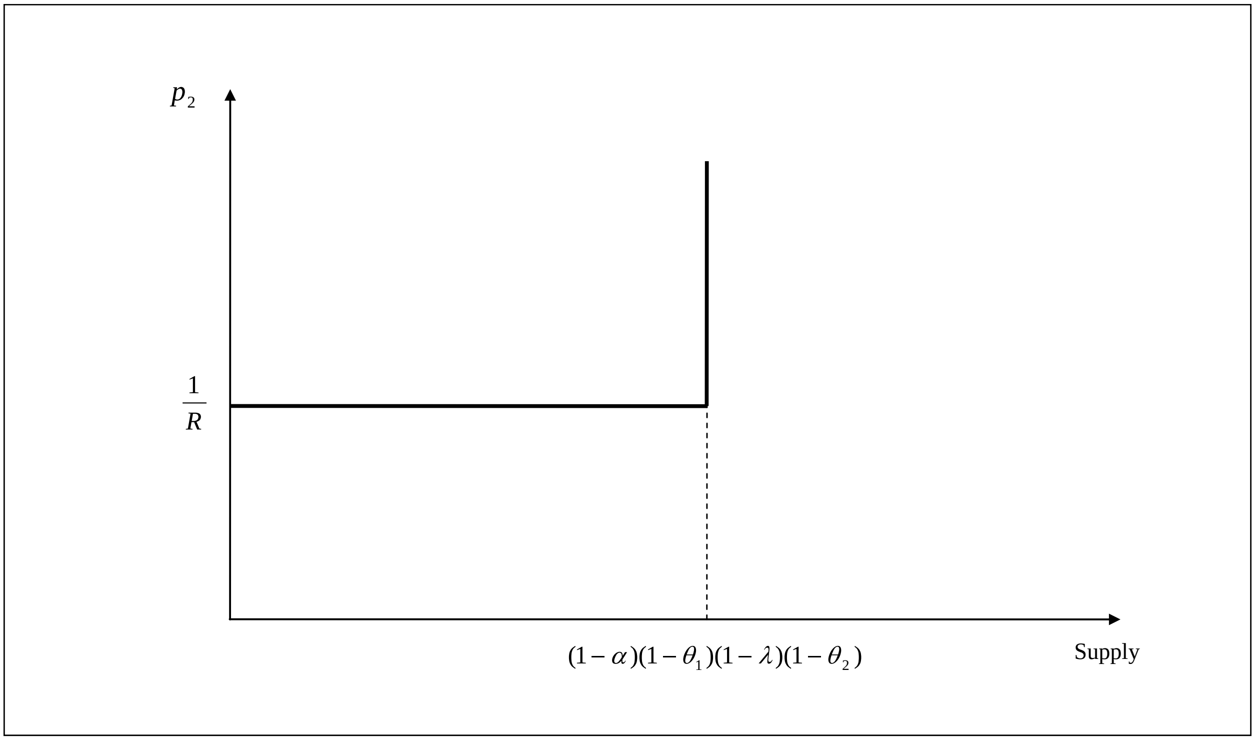


Figure A1b: Demand for cash at date 2

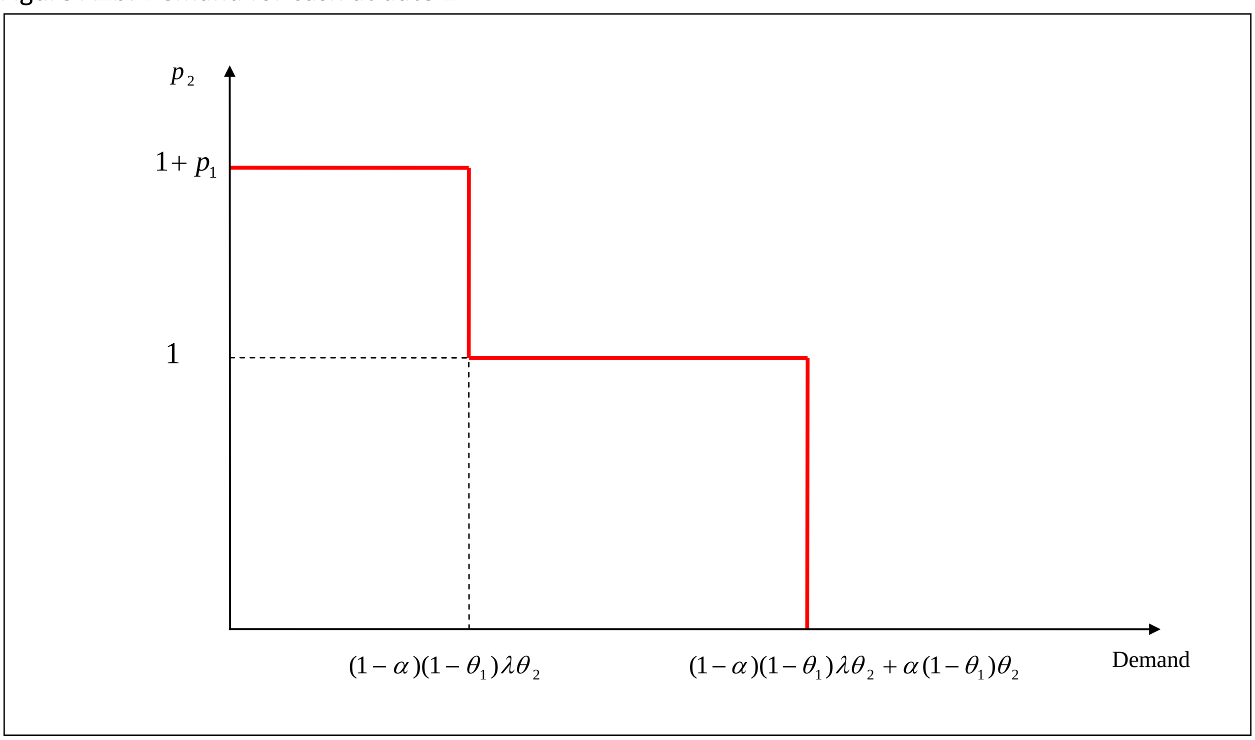


Figure A1c: Different demand and supply regimes