

Fractal states in quantum information processing

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Abstract

The fractal character of some quantum properties has been shown for systems described by continuous variables. Here, a definition of quantum fractal states is given that suits the discrete systems used in quantum information processing, including quantum coding and quantum computing. Several important examples are provided.

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1. Introduction

The fractal character of some properties of quantum systems has often been noted, usually for continuous properties and mainly in connection with specific spatial, temporal or spectral distributions—for example, see [1–4]. Here, the fractal character of discrete quantum systems is considered in the context of quantum information processing, where quantum parallelism naturally gives rise to states with very large numbers of computational basis components. Recently, a specific such class of self-similar entangled quantum states known as Bell gem states was defined and shown to include states useful for quantum error correction and quantum computation [5]. Here, a more general class of self-similar quantum states, which we will refer to as “fractal states” that includes Bell gem states, is introduced. Like geometric fractals, these states are characterized by self-similarity and potentially fractional values of topological dimensionality. In addition to showing how quantum concatenation coding can produce fractal states, other states known to be useful for quantum information processing are shown to be elements of sequences of states that are fractal in character when extended to the limiting scale.

Let us begin by recalling the properties of fractals in general, in order motivate the definition of quantum fractal states. A fractal is, broadly speaking, a self-similar entity (e.g. a set) having some property that becomes infinite (say, the number of subentities, e.g. subsets) and another property that remains finite (e.g., a volume in which it lies) in the limiting scale [6]. The common feature of fractals is that some property (e.g. a symmetry it possesses) is retained with change of scale. Fractals may be exactly self-similar, approximately self-similar, or stochastically self-similar. One can also view fractals as limiting elements of sequences of entities.

The fractals most often considered are geometric sets, exemplified by the Cantor set—see Fig. 1. This set can be viewed as derived from the interval $[0, 1) \in \mathbb{R}$ by successive subdivisions into pairs of sub-intervals of equal length, one located to the left and one to the right, with the central interval being excluded. In the first subdivision of the unit interval, $[0, 1)$, the successor element in the sequence of sets leading to the Cantor set is thus the set formed by set-theoretic union of the remaining intervals, $[0, 1/3) \cup [2/3, 1)$; the successor to this set is one in which both these intervals are similarly divided, and so on. The properties of fractal geometric sets allow one to attribute them a well-defined topological (“fractal”) dimension d , via the relation $c = s^d$ where c is the number of subdivisions of every component with each change of scale by a factor s , so that $d = \ln c / \ln s$. For the Cantor set, for example, $c = 2$ and

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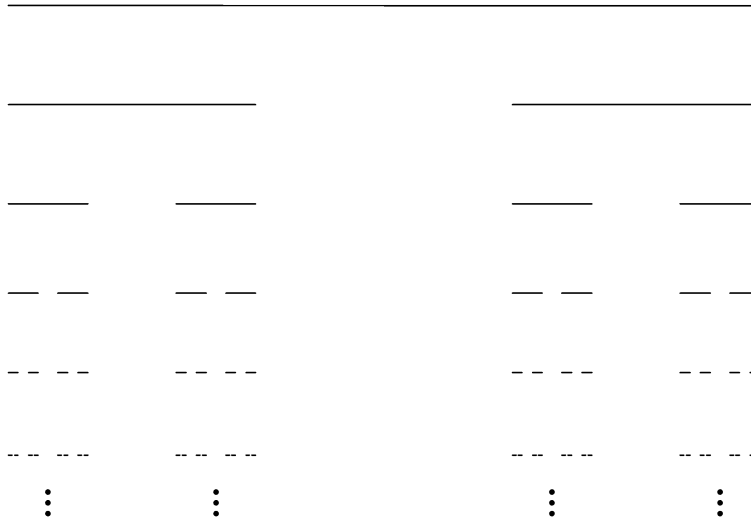


Fig. 1. The Cantor set, lying in the interval $[0, 1] \subset \mathbb{R}$. Dots indicate that the subdivision and scaling of line segments continues indefinitely.

$s = 3$, so that $d = \ln 2 / \ln 3$, a non-integral real number. Fractals are thus capable of possessing non-integral topological dimensionality. It is important to note, however, d may take integral as well as fractional values.

Now let us define quantum fractal states for quantum systems having discrete states such as might be used in quantum information applications. This can be done by making correspondences between quantum state-vector properties and those of geometric sets, motivating our definition. In particular, we take vector multiplication in Hilbert space to play a role analogous to that played by set-theoretic union in the case of geometric sets. The ratios of probabilities of outcomes of precise quantum measurements at successive scales can then be taken to correspond to ratios of lengths of geometric sets, providing a scaling factor, s , for the fractal. These probabilities are given by squared magnitudes of (complex) quantum amplitudes of the components of the quantum state under consideration in the eigenbases determined by precise quantum measurements. The number of quantum subsystems arising with each change of scale is taken to correspond to the number of subintervals in the case of geometric sets, providing the parameter c . With these two quantities in hand, one can determine the fractal dimension of the corresponding quantum fractal state, specifying its fractal character. One can thus formalize the notion of a quantum fractal state as in the following definition.

Definition. A quantum fractal state (QFS) is a quantum state that is self-similar, having topological dimensionality d related to the number $c > 1$ of subsystems arising with each discrete change of scale indexed by $n \in \mathbb{N}$, and to the scaling factor $s = p^{(n)} / p^{(n+1)} \geq 1$ of probabilities of measurement outcomes in a given eigenbasis relative to those at the successor scale:

$$d = \ln c / \ln s \tag{1}$$

with $s, c \in \mathbb{N}$, where d is defined to be $d = \log_2 c$ at the bounding scale $s = 1$.

In the above, the number of successor systems per change of scale is then $c = s^d$ where, again, the probabilities $p^{(m)}$ are

those obtained from quantum state amplitudes by squaring as per the Born rule. The following expression provides a specific class of pure quantum fractal state characterized by the parameters c and s , by relating a term in the sequence of states leading to the fractal state to its predecessor:

$$|\Phi(n + 1, c, s)\rangle = \sum_{i_1, i_2, \dots, i_c=0}^{s-1} \alpha_{i_1, i_2, \dots, i_c}^{(n+1)} |\Psi_{i_1}\rangle |\Psi_{i_2}\rangle \cdots |\Psi_{i_c}\rangle, \tag{2}$$

where $|\Psi_{i_j}\rangle$ are normalized basis vectors for $\mathcal{H}^{(n)} = \mathcal{H}^{\otimes c^n}$, the c^n -dimensional Hilbert space corresponding to a system at scale n , with at least one of the $|\Psi_{i_j}\rangle$ being the predecessor state $|\Phi(n, c, s)\rangle$ at the previous scale, under the constraints $|\alpha_{i_1, i_2, \dots, i_c}^{(n+1)}|^2 \in \{0, \frac{1}{s}\}$ and $\sum_{i_1, i_2, \dots, i_c} |\alpha_{i_1, i_2, \dots, i_c}^{(n+1)}|^2 = 1$, ensuring similarity between scales; a state $|\Phi(0, c, s)\rangle$ at the initial scale and a basis for Hilbert space \mathcal{H} of which it is an element must also be identified. Simple, but by no means necessary, choices in the quantum information processing setting are the computational basis state $|\Phi(0, c, s)\rangle = |0\rangle$ and the qu- N -it space $\mathcal{H} = \mathbb{C}^N$, $N \in \mathbb{N}$. To find a representative fractal state, $|\mathcal{F}(c, s)\rangle$, for a given pair of parameters (c, s) , it suffices to take $|\Psi_{i_1}\rangle = |\Phi(0, c, s)\rangle$, $|\Psi_{i_j}\rangle = |j\rangle$ for $j > 0$, $\alpha_{i_1, i_2, \dots, i_c}^{(n+1)} = 1/\sqrt{s}$ for $i_1 = i_2 = \cdots = i_c$ and $\alpha_{i_1, i_2, \dots, i_c}^{(n+1)} = 0$ otherwise, at all scales. The states given by Eq. (2) are elements of a sequence leading to a quantum fractal state in the limit $n \rightarrow \infty$. This particular state is a quantum product state.

As a specific example in which the above convenient choices are made that illustrates the fractal character of such states, consider a quantum state having the characteristic parameters of the Cantor set, namely $c = 2$ and $s = 3$, which we will call a ‘‘Cantor state’’, arrived at through the following sequence of states:

$$|\Phi(0, 2, 3)\rangle = |0\rangle, \tag{3}$$

$$|\Phi(1, 2, 3)\rangle = \frac{1}{\sqrt{3}}(|0\rangle)(|0\rangle + |1\rangle + |2\rangle)$$

$$= \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |02\rangle), \tag{4}$$

$$\begin{aligned}
 |\Phi(2, 2, 3)\rangle &= \frac{1}{\sqrt{3}} \left[\frac{1}{\sqrt{3}} (|0\rangle)(|0\rangle + |1\rangle + |2\rangle) \right] \\
 &\quad \times [|00\rangle + |11\rangle + |22\rangle] \\
 &= \frac{1}{\sqrt{9}} (|0000\rangle + |0011\rangle + |0022\rangle \\
 &\quad + |0100\rangle + |0111\rangle + |0122\rangle \\
 &\quad + |0200\rangle + |0211\rangle + |0222\rangle), \\
 &\vdots \tag{5}
 \end{aligned}$$

where, in particular, the computational basis for \mathbb{C}^3 was chosen. In the limit $n \rightarrow \infty$ this sequence approaches the representative fractal state $|\mathcal{F}(2, 3)\rangle$ having topological dimension $d = \ln 2 / \ln 3$.

As a second example *without* the above convenient choices but still of the form of Eq. (2), consider Bell gem states of the form $(1/\sqrt{2})(|i\rangle|j\rangle \pm |j\rangle|i\rangle)$, lying in a $d = 2^{2^m}$ -dimensional Hilbert space, where $|i\rangle \neq |j\rangle$ are elements of the same form but dimensionality $d' = 2^{2^{(m-1)}}$, $m \in \mathbb{N}$, $m \geq 2$ [5]; the simplest such states are

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{6}$$

Take as the state at scale $n = 0$, $|\Psi^- \rangle$; at the successor scale $n = 1$, one can then have the states

$$\frac{1}{\sqrt{2}}(|\Psi^+\rangle|\Psi^- \rangle + |\Psi^- \rangle|\Psi^+\rangle) = \frac{1}{\sqrt{2}}(|0101\rangle - |1010\rangle), \tag{7}$$

$$\frac{1}{\sqrt{2}}(|\Psi^+\rangle|\Psi^- \rangle - |\Psi^- \rangle|\Psi^+\rangle) = \frac{1}{\sqrt{2}}(|1001\rangle - |0110\rangle), \tag{8}$$

having the form of the right-hand side of Eq. (2) with $c = s = 2$, where in the first case $\alpha_{01}^{(1)} = \alpha_{10}^{(1)} = 1/\sqrt{2}$, all other coefficients being zero, and in the second case $\alpha_{01}^{(1)} = 1/\sqrt{2}$, $\alpha_{10}^{(1)} = -1/\sqrt{2}$, all other coefficients being zero. In the limit $n \rightarrow \infty$, given similarly chosen $\alpha_{ij}^{(m)}$ according with these symmetries at all scales, one obtains quantum states having topological dimensionality $d = 1$, which are fractals.¹

2. Relation to computational states

One can further relate states used in various quantum information processing applications to quantum fractal states, as we will now see. Consider quantum code states formed by the continued concatenation of coding maps. Such states can be viewed as increasingly well approximating fractal states, in the sense of being self-similar at an increasing number of scales. The simplest case of code concatenation involves a single concatenation step: one uses an M -qubit code $C^{\text{out}} = (\mathcal{E}^{\text{out}}, \mathcal{D}^{\text{out}})$ referred to as the outer code, and a second M' -qubit code $C^{\text{in}} = (\mathcal{E}^{\text{in}}, \mathcal{D}^{\text{in}})$, referred to as the inner code, where \mathcal{E} and \mathcal{D} indicate encoding

and decoding operations, which are CPTP maps of the statistical operators ρ corresponding to computational states—in our context, $\rho = |\Phi(m, c, s)\rangle\langle\Phi(m, c, s)|$. The logical qubits of the inner code form M' -qubit blocks used by the outer code, comprising a fully encoded quantum computational register, in effect using the concatenated encoding map $\tilde{\mathcal{E}} = \mathcal{E}^{\text{out}} \circ (\mathcal{E}^{\text{in}})^{\otimes M}$. An error correction scheme can coherently correct each code block using the inner code and then the entire MM' -qubit register using the outer code [9]. To operate at increasingly larger scales, a larger concatenation code repeatedly using the same code for *both* inner and outer codes at successive scales can be used to produce code states. Such states are by definition quantum fractal states, being of the form given in Eq. (2).

As a specific example, consider the states provided by the bit-flip code, where the computational basis elements $|0\rangle, |1\rangle$ are mapped onto states of several qubits, forming logical qubits used for quantum error detection and correction—see, for example [8]. This three-qubit repetition code is implemented by the following mapping

$$|0\rangle \mapsto |0\rangle_L = |000\rangle, \tag{9}$$

$$|1\rangle \mapsto |1\rangle_L = |111\rangle, \tag{10}$$

where “ L ” indicates a logical qubit state. One can then see that employing a 3-qubit bit-flip code for both inner and outer codes, allowing for the correction of both physical bit-flip errors and first-level logical-bit-flip errors, requires a 9-qubit physical register of three 3-qubit blocks. This quantum bit-flip code, when continually applied, produces fractal quantum states with parameters values $c = 3$ and $s = 1$; the states of the left-hand side of Eqs. (9)–(10) being those of the scale 0 and those of the right-hand side being those of the scale 1; the single coefficient $\alpha^{(m)}$ at each scale being simply 1. Such an error-correction code can be used when physical and logical bit-flip errors are possible simultaneously at different scales. In particular, if a quantum computational system is found susceptible to logical-bit value errors at n different scales, the following sequence of concatenation code states can be used:

$$|\mathcal{F}(0, 3, 1)\rangle = |i\rangle, \tag{11}$$

$$|\mathcal{F}(1, 3, 1)\rangle = |i\rangle^{\otimes 3}, \tag{12}$$

$$|\mathcal{F}(2, 3, 1)\rangle = |i\rangle^{\otimes 9}, \tag{13}$$

$$\vdots \tag{14}$$

$$|\mathcal{F}(n, 3, 1)\rangle = |i\rangle^{\otimes 3^n}, \tag{15}$$

with $i = 0, 1$, which, if continued indefinitely, leads to quantum fractal states of topological dimensionality $d = \log_2 3$. A series of n quantum cloning machines each set to output 3 copies, input serially, will produce these code states. This is an extremely simple example. A less simple example of a concatenated quantum code, based on the encoding map from logic bits to the Bell states $|\Psi^\pm\rangle$, is that provided as the second example of the previous section, known to be useful for carrying out quantum error correction and quantum computation—see [5,7] and references therein.

¹ A common misconception regarding fractals is that d must be fractional for them, which is not necessarily the case, as it is not in this example [6].

Yet another place where sequences of self-similar states might be expected to arise in quantum information processing is in spin clusters—that is, arrays of qubits. One class of states that has been proposed for use in cluster quantum computing on a linear qubit array is given by the expression

$$|\phi_N\rangle = \frac{1}{2^{N/2}} \bigotimes_{a=1}^N (|0\rangle\sigma_z^{a+1} + |1\rangle), \quad (16)$$

with the convention that $\sigma_z^{N+1} \equiv \mathbb{I}$, which states are generally considered as given by a representative states reachable by a local unitary transformation and so are considered here [8]. For example, when $N = 4$, one has

$$|\phi_4\rangle = \text{l.u.t.} \frac{1}{2} (|0\rangle|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle|1\rangle + |1\rangle|1\rangle|0\rangle|0\rangle - |1\rangle|1\rangle|1\rangle|1\rangle), \quad (17)$$

which can be viewed as of the form $|\Phi(1, 2, 2)\rangle$, with $|\Phi(0, 2, 2)\rangle = |\phi_2\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ and $\alpha_{(00)(00)}^{(1)} = \alpha_{(00)(11)}^{(1)} = \alpha_{(11)(00)}^{(1)} = (1/\sqrt{2}) = -\alpha_{(11)(11)}^{(1)}$. The prescription of Eq. (16) gives rise in the limit of large N to a quantum fractal state much like the very first example above but with topological dimensionality $d = 1$ since $c = s = 2$; the effect of the σ_z^{a+1} is to provide varying phases in the $\alpha^{(N)}$.

Finally, note that the various examples provided above demonstrate that quantum fractal states need not be specifically only factorable or only entangled.

3. Conclusion

A new class of quantum states was introduced, the quantum fractal states, characterized by self-similarity and the capacity for fractional dimensionality. A specific example class of quantum fractal states was given and the self-similarity of a number of quantum states known to be useful for quantum information processing was pointed out. A stimulus for the broader recognition of self-similarity in states used for quantum information processing, where multiple-component systems and quantum error-correction coding are necessary, has thereby been provided.

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