"*Das Überraschende*: Wittgenstein on the Surprising in Mathematics", in Jon Ellis and Daniel Guevara, eds., *Wittgenstein and the Philosophy of Mind* (Oxford University Press, ~~2010).~~ 201

Das Überraschende: Wittgenstein on the Surprising in Mathematics[1]

Juliet Floyd, Boston University

It is an uncontroversial fact that mathematics partly depends upon the surprising, the unexpected, the puzzling, the beautiful, and the illuminating to retain our interest in practicing it. Mathematics, as Wittgenstein writes, appears to us a "many-colored" mixture in its multifarious applications and uses.[2] Is this a merely psychological or an epistemically relevant fact? Should consideration of how we experience it play a role in philosophical discussions of the foundations of mathematics?

It is remarkable how relatively little has been written in recent decades about the decisive role played in mathematics by our capacity to be struck, preoccupied, satisfied, and turned by pictures, diagrams, symbolic generalizations, and new forms of conceptual

[1] The following is a revised version of Floyd 2008, a companion piece to Floyd forthcoming. In making my revisions, I have benefited from conversations with Avner Baz, Robert Briscoe, Robert Bowditch, Laurence Goldstein, Daniel Guevara, Jonathan Ellis, Akihiro Kanamori, Matthias Kross, Montgomery Link, Jean-Philippe Narboux, Norbert Schappacher, Peter Simons, Hartley Slater, Alan Thomas, Anja Weiberg and especially Felix Mühlhölzer (the entire essay is indebted to Mühlhölzer 2002, which initiated discussion in print of Wittgenstein's RFM I Appendix III on the surprising). Audiences at the Einstein Forum, Potsdam, the University of Kent, Canterbury and the University of Chicago, as well as the 2006 conference at the University of California, Santa Cruz on Wittgenstein's philosophy of mind offered helpful criticisms. And in the final stages of editing I was favored with very generous support from the Lichtenberg-Kolleg, Georg August Universität Göttingen. Thanks to Kyle Robertson for correcting a Latin Square error at proof stage.

[2] RFM III 46: "I'm inclined to say that mathematics is a <u>colorful</u> <u>mix</u> of techniques of proof [ein <u>buntes</u> <u>Gemisch</u> von Beweistechniken] --And upon this is based its manifold applicability and its importance". Following Mülhölzer (2006) I reject Anscombe's (influential) translation of "bundes Gemisch" by "motley": the latter has unnecessarily negative, chaotic connotations (as does the French translation, "bigarrée" (Wittgenstein 1983)). It is perhaps worth noting that the remark (MS 122, p. 86r, 8 January 1940) appears to be tentative, with an added question mark in the margin beside the underlining: Wittgenstein is struggling to formulate a line of thought about the "unsurveyability" of proofs in *Principia Mathematica*, without being fully satisfied. Mülhölzer 2006, forthcoming a, forthcoming b insightfully document the ways in which Wittgenstein was reacting to Hilbert in RFM III in emphasizing that mathematical proofs are "überblickbar", "übersehbar", or "übersichtlich". Proper treatment of this substantial commentary and the issue of "surveyability" of proof lie outside the scope of this essay. Mühlholzer argues, interestingly, that Wittgenstein does not use "surveyability" to mean "understanding", but instead something "purely formal". Here, perhaps in contrast, I connect the issue of seeing aspects with understanding. But it should be said that I argue here for two ideas that are in agreement with Mülhölzer's interpretation of the Hilbert-Wittgenstein relation. 1. Wittgenstein is *not* demanding that proofs be "taken in" at a glance. 2. Aspect perception and "surveyability" are connected with extra- and intra-theoretical *applications* of mathematics (on which compare Floyd forthcoming).

and representational structure[3], and even less on the topic of (what might arguably be called) aesthetic criteria, important as these are in determining what is to be taken as a formulation of an interesting problem or a canonical set of axioms for a branch of mathematics.[4] These lacunae are especially striking when one notices, as even a brief glance at the history of mathematics shows, that among mathematicians terms of evaluation such as "surprising" and "beautiful" are crucial, quite ordinary, and ubiquitous terms of art.

What I want to suggest, with some help from certain passages in Wittgenstein, is that these markers of mathematical experience cannot be deemed irrelevant or "non-cognitive" without loss, even if talk about our mathematical experiences presents, as Frege warned, the psychologistic danger of confusing the grounds for a judgment with its context of discovery. In what follows I shall primarily be arguing by illustration, attempting to illuminate some of Wittgenstein's remarks on mathematics with easily accessible and vivid examples.

My ulterior interpretive motive is to shed light on a thematic strand within Wittgenstein's writing on philosophy and mathematics from which even his most sympathetic interpreters have too often shied away: his recurrent interest in exploring talk of aspect perception, of showing and saying, of picturing and seeing or "taking in". Having written for more than a decade on Wittgenstein's remarks on mathematics, I have

[3] Recent exceptions to this include Brown 1999, D. Barker-Plummer et.al 2002, Mancosu, Jørgensen and S. Pedersen eds 2005, Grialou, Longo, Okada eds. 2005, Norman 2006. Wilson 2006 and Grosholz 2007 emphasize the conceptual drift and productive ambiguities involved in our our applications of mathematical diagrams and representations.

[4] Putnam 1994, Rota 1997 and Kennedy (manuscript).

myself only belatedly[5] come to recognize that there is no possibility of doing interpretive justice to these remarks as a whole without somehow accounting for his obsession with the "patter" [*Geschwätz*] surrounding mathematical activity (RFM IV §27), that is, the "raw material"[6] or "prose"[7] that mathematicians throw off--with what one might call, informally, the intuitive or experiential, including, e.g., expressions concerning heuristics, evaluations, diagrams, proof-pictures, notations, symbolisms, and models, geometrical and otherwise. To be sure, Wittgenstein forwards many criticisms of mentalistic appeals to self-evidence or intuition or interior states of direct insight, and he rejects the idea that configurations of signs or diagrams can interpret or apply themselves, referring to an abstract reality on their own. In his writings on mathematics he articulates these criticisms partly by appealing to the well-known, broadly constructivist thought that the algorithmic, explicitly rule-governed, humanly calculable and controllable elements of mathematics (e.g. in calculations and proofs) lie at its core as an objective practice, and not its referential features or propositional content alone. However, for Wittgenstein (even in the *Tractatus*) mathematics involves more than simply rules or calculations: it involves a variety of methods of proof and argumentation and articulation, a variety of

---

[5] In Floyd 2005 and Floyd forthcoming I stress the importance of pictorial metaphors and aspect perception for Wittgenstein's remarks on mathematics and logic.

[6] "Raw material": PI §254. MS 124 p. 35 (its original version) is directed at Hardy 1940 (discussed below in part III). Gerrard 1991 draws a useful analogy between Wittgenstein's discussion of the quotation from Augustine in PI and his remarks about Hardy 1940. I discuss Wittgenstein's remarks on Hardy 1941 in this way below, in part IV.

[7] "Prose": PR pp. 324,330,335; PG 286,269,375-6; RFM V §46, VII §41; cf. Wittgenstein to Schlick 13.7.1935, quoted and discussed below. Floyd 2001 and Floyd forthcoming discuss the issue of "prose" at greater length, taking Wittgenstein to refer to potentially misleading ways of understanding glosses on, and presentations of, mathematical proofs and results (cf. Mühlhölzer 2002). Mühlhölzer forthcoming b argues that Wittgenstein's main idea in investigating "prose" in his manuscripts tends not to be so general, but usually involves a more specific rejection of the idea that mathematics involves descriptive uses of language, or consists of propositions. This restricted reading of "prose" certainly does illuminate the manuscript remarks cited here, with the exception, I think, of Wittgenstein's letter to Schlick, which I interpret it below in the more general way—a way which I take to lie behind the more specific point emphasized by Mühlhölzer.

"machinery" and ways of thinking that we develop and apply, for a variety of purposes.[8] Wittgenstein is concerned, therefore, to criticize unreflective views of what rules-in-themselves or proof-patterns can explain, apart from consideration of our abilities and activities in applying them. Thus he includes in the "prose" surrounding mathematics the "unsurveyable" formalized proof structures from *Principia Mathematica* (RFM V §46, VII §41). And in the same vein he revisits the idea throughout his life that in mathematics (and in logic and in philosophy) there are no genuinely surprising, "deep" truths or facts. In these areas of human cognition--unlike in empirical or perceptual or experimental cognition--"process and result are equivalent"[9], i.e., the *processes* of calculating or proving or clarifying or defining or arguing or seeing or communicating are what lend significance to results (tell us what the results *are*). To know a result in mathematics is to be able to see it (portray it, communicate it) *as* necessary, inevitable, unsurprising, in the context of proof (or calculation, or definition, or method of argumentation, or representation, or explanation).[10] Wittgenstein's ways of stressing this, his remarks to the effect that it follows from the equation of process and result that there are no surprising results in mathematics (or in logic and philosophy), that in a certain sense the aim of proof (or deductive pattern or conceptual investigation) is to make surprise *disappear*, are, of course, starkly counterintuitive given the prevalence of experiences and discussions of surprise in the practice and the applications of these subjects, given the fact that we are by no means logically omniscient, and given the

[8] On "proof machinery" see MS 117 p. 170, and MS 126, p. 127 (discussed below). On applications of mathematics and aspect perception in the pre-*Tractatus* period, see Floyd forthcoming.

[9] "process and result are equivalent": MN (NB p. 114), NB 24.4.15, TLP 6.1261, PG p. 457, RFM I §§82,154, III §24, IV §50.

[10] Mühlhölzer 2006 emphasizes that by the "surveyability" of proof Wittgenstein means that a proof can be reproduced "with certainty and in the manner in which we reproduce pictures". The metaphor of proof as a picture or model (*Bild*) is, I believe, central for Wittgenstein in RFM, forcing talk of aspect perception upon him.

apparently cognitively ampliative character of at least some deductive reasoning.[11] Wittgenstein intends, however, to unmask misleading accounts of surprise as "deep", to place their depth into proper philosophical perspective. He is not denying that certain interesting matters may be hidden, but recasting our ideas about what kinds of hiddenness there might be, and what kinds of openness and revelation mathematics, logic and philosophy might, on a realistic view[12], provide.

And thus it is significant that visual and musical comparisons, metaphors, and investigations of untutored expressions of puzzlement, surprise, and experiences of changes in our concepts and ways of seeing are strewn with increasing frequency over time throughout Wittgenstein's writing, and not always in ways that are unmasked or criticized.[13] But what are we to say about these?

It is tempting to conclude from certain remarks that Wittgenstein's allusions to what and how we see, as well as his investigations of proof-pictures and aspect perception, are primarily directed at views (Platonistic and/or quasi-causal) that try to lodge explanations of the meaning and objectivity of mathematics in the sheer existence of abstract entities, grasped by us in mere visualizations of meaning, rules, or *a priori* intuitions. If this is all there is to say, however, then Wittgenstein's talk of what we see

[11] Two relevant discussions of tensions facing debates about the ampliative or non-ampliative character of deduction are Dummett 1978 and Dreben and Floyd 1991.
[12] Here I allude to the sense of "realism" discussed in the title essay of Diamond 1991, i.e., an open-eyed, sober attitude that is realistic, rather than fantastical. See also Diamond 1997 for a discussion of "realism" and a parallel between Wittgenstein's treatment of mathematics and ethics that has influenced me greatly in this essay.
[13] It is true that Wittgenstein is often reflexively suspicious of his own appeals to aspect talk (on the significance of this general point, see Baz forthcoming). However, he keeps returning to talk of aspects throughout his life, as if he can never quite rid himself of the focus such talk requires. A good example of the ambivalence is RFM III §§46-50 , where he first rejects and then reapplies aspect language and he first accepts, then rejects, the notion of "discovery" of an aspect. The distinction between "discovery" and "invention" is sometimes important to Wittgenstein, and sometimes not, as the same series of passages illustrates. For discussion of Wittgenstein's talk of aspects and paradigms, see Floyd forthcoming, Narboux forthcoming.

and how we see and experience it is really just transitional talk, to be criticized and then jettisoned as in the end irrelevant to a proper characterization of mathematics. Anything else might seem to risk falling into psychologistic mentalism or metaphysical phenomenology.[14] If , however, Wittgenstein's remarks about seeing and experiencing and surveying images and pictures are taken to have been intended to further an empiricistic finitism or anthropologism, it is most unclear how they could manage to do so in any convincing way, for they are amorphous, often metaphorical, and unprincipled, lacking any clear epistemological basis or content.[15] Moreover, in the context of Wittgenstein's later philosophy as a whole, the remarks mentioning aspect-perception seem to be precisely directed *against* traditional empiricist views of how we acquire and justify our knowledge, as has been emphasized by those interpreting Part II xi of *Philosophical Investigations* (Budd 1989 ch. III, Mulhall (1990), Glock (1996): 40).[16] Logic and mathematics provided Wittgenstein with useful sounding boards – perhaps even the key ones – for the development of his talk of aspect perception.[17] Part I of *Remarks on the Foundations of Mathematics*, for example-- where first-person remarks about seeing aspects and puzzlement frequently occur--forwards a series of criticisms of the myth of an immediately given experience or representation of meaning, number and/or logical necessity. The mode of argumentation here differs strikingly from Frege's,

[14] Wright 2007 p. 490 emphasizes that phenomenology has little hope of answering the concerns inspired by, e.g., Wittgenstein's exploration of rule-following.
[15] This may explain why interpreters have repeatedly tried to provide such a basis inspired by Wittgenstein. Wright 1980, 1993, Marion 1998, 2009 are correct, I believe, in taking Wittgenstein to be focusing on a loose, indeterminate form of finitism as a *style* of mathematical practice, rather than strict finitism of the principled kind broached by Bernays, Wang and Kreisel and discussed in Dummett 1959.
[16] There are nevertheless some who read Wittgenstein's view as broadly empiricistic and naturalistic, e.g., Steiner (1996), who emphasizes the ways in which we "harden" empirical regularities into rules of description as central to Wittgenstein's view of mathematics.
[17] This is my argument in Floyd forthcoming, where I highlight Wittgenstein's early treatment of probability. The discussion of this should be compared with Raïd 2009.

even if it is indebted to Frege's criticisms of empiricism and psychologism about our concept of number (his criticisms of the idea that there must be an idea attached to each mathematical word as a unit considered on its own, his rejection of number words as referring to ideas, and so on).[18]

Without denying that Wittgenstein's criticisms of uncritical Platonism, mentalism and empiricism are central and powerful to his philosophy, I should like to broach an alternative, more constructive interpretation in what follows, if only to ward off the idea that the whole force of Wittgenstein's considerations may be summarized as an essentially negative line of thought. I suggest that we take Wittgenstein's focus on our immediate expressive responses to proof pictures--on our puzzlement, surprise, frustration, pleasure, and on immediate visualizations of experiences and descriptions of these-- as neither primarily epistemological or verificationist in intent, nor wholly or primarily aimed at a rejection of Platonism or 'private', Cartesian views of sensation. Instead, I propose that we (also) regard it as a logical (grammatical) focus, designed to sophisticate and complicate our conception(s) of the expression of cognitive content or thought in language. This suggestion allows us to see these remarks on aspects offering one way (among others) to reinterpret and revise, both Frege's conception of what anti-psychologism and anti-empiricism require, and Russell's conception of what empiricism and logicism may and may not explain. Wittgenstein's programmatic, constructive idea was to investigate, refine, and clarify philosophically untutored, naïve talk of interests, expectations, and experiences in mathematics (his philosophy's "raw material"[6]) with an eye toward deriving general philosophical lessons from the exercise. The idea is partly

[18] *How* precisely RFM I is indeed to Frege is an interesting question, which it would be useful to explore.

programmatic, and was applied directly to, e.g., remarks of the mathematicians K. Gödel and G.H. Hardy, as well as to Frege and Russell (for more on Hardy and Gödel, see III-IV below). This was a significant theme in the larger evolution of Wittgenstein's philosophy precisely because Frege stood for Wittgenstein as the most powerful philosopher to have tried to portray such voicings of mathematical experience as largely irrelevant to our understanding of logic and/or arithmetic, as merely tonal coloring distinct from sense or content, and Russell stood as the most powerful philosopher to have tried to explain these experiences in causal or psychological terms, and their grammar as "purely verbal" (Russell 1936: 140ff). It is the idea that what we say about how and why certain things strike and interest us can *a priori* have no cognitive significance, but *merely* sensory or aesthetic or psychological or historical significance, that forms the target of a good deal of Wittgenstein's writing on mathematics. And this anti-non-cognitivism works side by side with his rejection of the idea (from at least 1929 on) that human experience (say, the field of visual space) may be held to have a given, necessary, determinate, intrinsic mathematical structure—that is, his rejection of a certain view of the "intuitive" content of geometry and/or our intuition of space. There is to be found in these writings, I believe, a quite distinctive effort at "undoing the psychologizing of psychology" (as Cavell 1976: 91 puts it), that is, at deempiricizing our talk about mathematical experience in such a way as to avoid falling into either Frege's or Russell's respective psychologicizations of mathematical experience, or a conventionalist's view of mathematics as merely stipulative. Wittgenstein's remarks turn on an image of meaning as alive, when it is alive, for each one of us, one by one--just as the empiricist supposes--yet alive also in what we do and say to and with one another, in the flavor of how we make demands upon one another, in how we react, discriminate, make comparisons, and view situations in light of these experiences. Our interests are revealed in the ways we express ourselves with concepts, and Wittgenstein is interested in what interests us (cf. PI §570, LW II p. 46).

I am not going to argue that Wittgenstein forwarded a worked-out, systematic, philosophy of mathematics, for I believe he did not. What he did do was to point, suggestively, toward a range of specific ways in which talk of mathematical experience and practice, even talk of intuition, might make sense. At least he was right to raise a series of questions about the logicist idea that Frege and Russell refuted, as *a priori* irrelevant to philosophy of arithmetic, all possible talk of mathematical experience or intuition. His remarks on mathematics turn on folding features of our descriptions and vocalizations of our experiences into the description of what mathematical practice is, on grammaticalizing the intuitive. There was a need for this because talk of the "intuitively given" was prevalent in work of mathematicians he read in the 1920s who took themselves to be rejecting "formalism", though in a way different from Frege's. Already in the *Tractatus* Wittgenstein had written that "to the question whether intuition is required for the solving of mathematical problems, the answer is that language itself provides the necessary intuition. The process of *calculation* brings about this intuition. Calculation is no experiment." (TLP 6.233-6.2331). This is a logicist-inspired criticism of Kant that would later on express both a criticism of Brouwer's transcendental psychology and an expression of some sympathy for Brouwer's constructivist preferences; it evinces a suggestive turn of thought--from philosophical appeals to intuition to a focus on the activity of calculation in the context of our "everyday" language--that remained attractive to Wittgenstein throughout his life (cf. TLP 6.211; Kremer 2002). By investigating particular examples of proofs and diagrams and notations in logic and mathematics Wittgenstein is, at bottom, proposing a new kind of criticism directed at our paths of interest, appreciation, and preoccupation when we discuss mathematics in philosophy. He is proceeding on the assumption that there are qualitative ways of exploring our talk, experiences and activities within logic and mathematics as part of the ordinary, the everyday, the familiar, and, hence, as talk that may be repudiated, miscast, misapplied, and misunderstood when we philosophize. This is not to be read as an obsession with freezing language as it is, or an assumption that logical and mathematical necessity are "purely linguistic" (whatever that would mean), but instead as a call to refine our understanding of our "raw" material, i.e., our talk.

I

Surprise is neither simply an experience, nor simply an attitude or point of view on the world--although it involves and reflects elements of each of these. Adam Smith was right to distinguish between *wonder*, *surprise* and *admiration* in his history of philosophy. Although he allied himself, like Smith, with the ancient Greek idea that philosophy originates in wonder, rather than in surprise (cf. PI §§522ff), since Wittgenstein was not always as clear as Smith, I shall quote Smith here:

> Wonder, Surprise and Admiration, are words which, though often confounded, denote, in our language, sentiments that are indeed allied, but that are in some respects different also, and distinct from one another. What is new and singular, excites that

> sentiment which, in strict propriety, is called Wonder; what is unexpected, Surprise; and what is great or beautiful, Admiration.
>
> We wonder at all extraordinary and uncommon objects, at all the rarer phaenomena of nature, at meteors, comets, eclipses, at singular plants and animals, and at every thing, in short, with which we have before been either little or not at all acquainted; and we still wonder, though forewarned of what we are to see.
>
> We are surprised at those things which we have seen often, but which we least of all expected to meet with in the place where we find them; we are surprised at the sudden appearance of a friend, whom we have seen a thousand times, but whom we did not imagine we were to see then.
>
> We admire the beauty of a plain or the greatness of a mountain, though we have seen both often before, and though nothing appears to us in either, but what we had expected with certainty to see.
>
> …Surprise, therefore, is not to be regarded as an original emotion of a species distinct from all others. The violent and sudden change produced upon the mind, when an emotion of any kind is brought suddenly upon it, constitutes the whole nature of Surprise…(Smith 1795, 1)

One of Smith's (grammatical) points is that, like skepticism, surprise is not so much factive (responding to the existence of something) as evaluative: it brings into play our discriminative cognitive capacities, our concepts and our sense of what is natural and familiar and known. In mathematics, to call a particular result "surprising" may express doubt, astonishment, difficulty in comprehension, pleasure or praise, but it must denote the appearance of the result in a context which is *worthy* of our interest.[19] Thus surprise ought not to be taken to issue only from ignorance or turned expectation, but also from our interests and needs, from what it is we find noteworthy and what we do not.

In mathematics "surprising" is a working term of art, and it may or may not be expressed impersonally. Its function is, however, not to be reduced to that of a description of a particular psychological state. A quick glance as the role of the term's ordinary usage is enough to convince one of this. A famous mathematician works on a result, cannot prove it, and someone else does. The mathematician then writes in a major treatise that the solution is "surprisingly simple" (or "important", or "remarkable", or "decisive" or perhaps even "amazing"), and readers repeat this, on authority. Now the readers are not themselves surprised at the simplicity, for they had no expectation in the first place, they had not grappled with a problem made difficult by an earlier point of view—unless they can work themselves back into the original problem context, and allow themselves to be struck by what was once new, but is now jejune or trivial or obvious. A mathematician has recently written that "undoubtedly, many occurrences of mathematical beauty eventually fade or fall into triviality as mathematics progresses" and there is truth in the idea--a favored one of Wittgenstein's--that part of the aim of mathematics is to make what is puzzling or astonishing vanish into triviality (cf. Rota 1997: 175, Mühlholzer 2002). What is conveyed by such terms of criticism are not merely records of one person's experience, or even the perception of an historical moment in mathematics when a community turned in a new direction. They are also possible experiences, and matters of normative judgment. In this sense the notion of

[19] There are many examples that could be collected from working correspondence among mathematicians, and I commend to the reader their collection and examination. Some instructive expressions of surprise may be found in Gödel's correspondence with Bernays; see, in particular, Gödel 2003: 91-2.

surprise forms part of an account of the interest and significance of what we do, and why: of what, in short, mathematics *is*.

The following passage about Hobbes, which may be found in Aubrey's *Lives*, gives us a flavor of these points.

> [Hobbes] was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's Elements lay open, and 'twas the 47 El.libri I. He read the Proposition. *By* G----sayd he (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible*! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read…that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.

This remark is repeated by the historian of mathematics (e.g., by Stillwell 1989: 13) not so much for its biographical or philosophical interest (interesting though it is *as* that), but for its portrayal of a paradigmatic experience that is familiar. The pleasures of surprise in the realm of the necessary are distinctive, and widely known. Those who work mathematical puzzles in the daily newspaper are not focused on generating hitherto unknown truths, but on the distinctive pleasure of their morning *divertissement*.

Such capturing of our attention is in a certain sense ephemeral: surprise must be able to completely disappear if it is (grammatically speaking) to be able to appear (cf. PI §§133, 524), and it is localized in the direction of its attention. If every step in the writing out of an elementary arithmetical series is "new" and "different" and "surprising" in the same way, then something is not, as we say, mastered. But it is not right to say that that if the generality of the sequence is to be mastered, every single step must come to be viewed as equally mundane and a matter of course. It is also that such mastery entails that we are able to draw contrasts between the interesting and the jejune, the informative and the redundant, the noteworthy and the obvious. Thus as Wittgenstein remarks, "if you are surprised [at a mathematical result] then you have not understood it yet…when the surprise comes to you at the end of a chain of inference it is only a sign that unclarity or some misunderstanding remains" (RFM I App. II §2). Yet the proper order of sentiments in relation to philosophy (and mathematics) is sequential, in Smith's (and the classical tradition's) view, a view I am using to clarify Wittgenstein's remarks. The proper sequence is from surprise, to wonder, to admiration, there being no route to the proper vanishing of surprise without surprise. *Surprise* is an initial rush[20] of puzzlement or confusion that vanishes; *Wonder* asks, How is this possible?, and only after some kind of connection is drawn that makes sense of the specific reasons for the discord or puzzlement is proper *Admiration* or true *Appreciation* of the result to be had.

In its natural habitat, surprise inflicts a peculiar stamp upon human intentionality that is saturated with our values, preoccupations, concepts, preferences, choices, and interests as we see them now. Like the boring, the trivial, and the beautiful, the

[20] This may explain the attraction of psychologists and philosophers to surprise as something that may be investigated causally, in terms of bodily reactions; like laughter or the startle reflex, it has long been associated by philosophers (including Wittgenstein) with what is in at least some contexts merely psychological or bodily, quasi-intentional at best, brutely reactive, part of our animal nature (cf. RFM I App. II §2, RPP I §568). The psychological literature I have in mind aims to portray the tapestry of human emotion as if it is woven from a finite palette of universal emotional substrata (at least seven basic emotions, of which surprise is one), substrata mirrored in the muscular physiognomy of the human face. These substrata are seen as cognitive and evolved, largely automatic and unconscious, rather than conventional or culturally relative. See Ekman 2003. For a discussion of the quasi-intentional, psychological state of the startle reflex connecting to the concept of surprise, see Robinson (1995), and, more critically, Shanker 1998, Shanker and Greenspan 2004.

surprising is an evaluative phenomenon. If this is right, then the phenomenon of surprise is not something that can underwrite any explanation of the mind-independent existence or ontological reality of eternal, unchanging, abstract entities, or any theory of the cognitively ampliative character of deduction, if only because nothing is, *qua* the thing that it is, *intrinsically* surprising. I say this because the phenomenon of the surprising in mathematics--the very *hardness* of finding or understanding a proof of what one might not have expected--has sometimes been adduced as evidence of the mind- transcendent reality of mathematical objects. But a moment's reflection should show that this appeal to phenomenology or experience can in no way establish or refute realism, much less underwrite the idea that the mathematician is a discoverer *rather than* an inventor. A fictionalist about mathematics can certainly point toward the fact that fictional narratives and characters may inform us of certain discoveries about the human, precisely by necessitating one or another way of looking at things, and not just things properly belonging to the fictional world.[21]

Here I am suggesting a change or supplementation of emphasis in the *prima facie* way one might read Wittgenstein's Appendix on the Surprising in *Remarks on the Foundations of Mathematics*--a swatch of remarks that he himself evidently considered to be rough, because he excised it from the early version of *Philosophical Investigations*. There would be an understandable tendency here to focus on the relation of these remarks

[21] Radical (step by step, stipulative) conventionalism is another story. As Dummett (who interpreted Wittgenstein as such a conventionalist) said (1959) a proof does not drive us along "willy nilly", but directs and necessitates us, and this phenomenological point does seem relevant to the radical conventionalist position. If we look, as did Hume, we do not "see" necessity given to us in a certain sense. As Diamond 1991 argues, however, Wittgenstein relies on the fact that we do not simply *choose* or *stipulate* what will compel or strike us: he is no *radical* conventionalist. Attention to Wittgenstein's investigations of extra-mathematical applications of mathematics helps, I believe, to make this point, as well as a stress on the fact that Wittgenstein's aim is to reject the idea of a single step or result or fact that is intrinsically surprising *per se*, rather than rejecting the idea that there can be surprising proofs, methods, arguments, and so on.

to anti-Platonism, to Wittgenstein's longstanding commitment to the non-descriptive character of logic, mathematics and philosophy in contrast to physics, a distinction with which he tended throughout his life to associate the phrase that "there are no surprises" (cf. Mühlhölzer 2002). In these later remarks he has not wholly worked himself free from this line of thought, which he associated explicitly with the idea that the mathematician is not (merely) a discoverer [*Entdecker*], but also (praise from Wittgenstein) an inventor [*Erfinder*].[22] But in his later writings this line of thought is taking place within a wider context in which he is coming to explore its coloring and tonality, its connections with what he explicitly calls "aesthetic" matters and with his idea that there can be the discovery or finding of a new aspect of things in mathematics and in language games (RFM I §167, App. II §2, II §38, IV §47, PI §23). To this extent he is interested as much in how we talk about and what we do with surprise as he is in denying the concept any place in our discussions of mathematics (cf. RFM I App. II §§1-3,13). He therefore contrasts the image of mathematical practice as discovery of amazing or astonishing facts or mysteries with an image of mathematics as a practice allowing us "to see the value of a train of thought in its bringing something surprising to us" (RFM I Appendix II §1; cf. RFM II §40). My claim is that Wittgenstein is interested in what interests and occupies us, in our abilities and how we describe them, and thus his remarks on mathematics, like

[22] Thus there is much truth in the reading of these remarks offered by Mühlhölzer 2002, which takes Wittgenstein's "conjecture" or "suspicion" to be that in mathematics there are only representation-facts, and no intrinsically surprising facts--that is to say, there are only "facts" whose surprising character vanishes to "triviality" once they are understood (the trivial here is seen to be the antithesis of the surprising). I would add to this, however, that the antithesis of mystery-mongering need not be intrinsic triviality and that the mathematician's (frequent) ordinary use of the word "trivial" does not usually mean, as it does in everyday life, "silly", "jejune", "uninteresting" or "off-point", but rather something like "producible by means everyone with requisite knowledge of mathematics will be able easily to apply" (such uses appear in Rota 1997 and in Hardy 1940 and are quoted in this essay). Mühlhölzer says (in conversation, 2009) that he would emphasize more now Wittgenstein's therapeutic intent: one might expect that every sentence that surprises us will reveal something, but one may not be right. With this suggestion I agree.

his remarks on aspect perception generally, are devoted to bringing discussion of our interests and values into a discussion of what our experiences of necessity are. That there can be misplaced articulations of our interests, needs, and abilities is clear enough; what I want to consider in what follows is the manner in which Wittgenstein aims to criticize such misarticulations.

II

Allow me first to discuss an example, drawn from finite combinatorics.[23] It will be important to what follows that the epistemic and ontological themes ordinarily involved with finitism do not appear in my description of the interest of this example: I shall not be concerned with the proof-theoretic information value of a finitistic or "surveyable" proof in relation to a non-finitistic or less compact one, for example, nor with the socially embedded context of mathematics, nor with the deductive rules of a particular logical system--relevant though these issues are to Wittgenstein's writings--but solely with what is striking, beautiful, and interesting about the ways in which mathematics can (and does) allow us to change our ways of viewing things.

Most readers will by now have heard of "Sudoku", the "mania" that has swept European and North American newspapers in the last few years (cf. http://en.wikipedia.org/wiki/Sudoku). Sudoku may be played with no knowledge of mathematics; these puzzles are solved by logic alone, although there are, interestingly, no

[23]There is every reason to assume that Wittgenstein had studied finite modular arithmetic and its applications; an allusion to this may be found at RFM VII §§18-20. I won't examine here the question of how principled and/or restrictive Wittgenstein's preference for finitistic and/or calculational elements of mathematics was or should or could have been. Because of his later philosophy's strong commitment to the complexity and variety of elements of linguistic practice (including mathematics), it seems to me perfectly imaginable that, inspired by large swatches of Wittgenstein's writing, one might develop a Wittgenstein-inspired, grammaticalized philosophy of the infinite that is not strongly revisionist of set theory (compare Moore 1990 and Kanamori 2005). But for this one would have to go beyond Wittgenstein's own writings.

words in them.[24] Nevertheless, teachers of elementary discrete mathematics still know how to see Sudoku as an instance of a much older part of mathematics, the study of Latin Squares.

Here is a Sudoku. The reader might try to solve it by filling in with the digits 1 through 9 in such a way that each digit occurs exactly once in every row and every column of the large array, and also exactly once within each of the nine 3 x 3 square arrays of which the larger square is composed:

[24] Felix Mühlhölzer has objected in conversation that logic, for Wittgenstein, has to do primarily with the forms of sentences, and so *must* have to do with words. A suitable response lies beyond the scope of what I can write here. Perhaps Sudoku problems are logical in an extended sense of the term. But I take the idea of exhausting all possibilities as central to their solution, and to Wittgenstein's own idea of the logical. Note that his analysis of tautologies in the *Tractatus* dispenses altogether with words or even particular variables and syntactic forms such as quantifiers, just as Sudoku problems do.

				8	3	4		
3					4	8	2	1
7								
		9	4		1		8	3
4	6		5		7	1		
1	2	5	3					9
		7	2	4				

Let us now consider the wider mathematical context that leads to an analysis of this game.

For any positive integer *n,* a *Latin Square L* (*i, j*) can be defined as an *n* x *n* array in which each of one of symbols 1,2,…*n* occurs exactly once in each row *i* and once in each column *j*. There are only two Latin Squares of order 2:

1	2
2	1

2	1
1	2

A Sudoku problem is simply to complete a partially presented Latin Square of order 9 subject to the further condition that the nine 3 x 3 arrays of which the 9 x 9 array is composed contain each one of the signs 1 through 9.[25]

To stick with Wittgenstein's Tractarian terminology, we might say "sign" and not "symbol" when we define a Latin Square. There is of course no necessity involved in the use of the signs "1" through "9" for this game: we might have played with the signs "*", "&", "^" or words or colors or letters. That we do not presumably reflects the contingent but significant fact that most of us have been trained to quickly take in, distinguish

[25] "Killer Sudoku" is a variant combination game, in which filled in series of squares must sum to a particular amount; see kakuro.com.

between and order the numeric signs with ease: it is part of the (very widely shared) pleasure of this game not to force us to appeal to knowledge of words or mathematics as we play it. Thus we are here viewing the numerals as (at best) indices, not as arguments (cf. *Tractatus* 5.02), i.e. as arbitrary distinct signs within a larger square array, not as numerals denoting the natural numbers or forming any part of a picture of mathematical or physical reality. These Sudoku signs contribute nothing to any proposition concerning cardinalities of sets or properties of the natural numbers.

The distinction between "sign" and "symbol" is a matter of how we distinguish and do things, not one of direct or literal sensory (empirical) perception or appearance on the page, nor one explained through the notion of an interpreted or projected

representation (much less a mental one).[26] This is witnessed by Dürer's famous etching *Melancholia*. In this etching, the square Dürer includes in the upper right hand corner consists of an array of numbers which add up to the same sum no matter in which direction one adds the rows and columns--a "magic square".[27] The array is pictured from within Dürer's representation, and so is a picture-array, rather than just a mathematical object or array *tout court*. It belongs, so to speak, to a fictional world. Yet all the same, it loses nothing mathematical for this, for

[26] The idea that the sign/symbol distinction can be understood by noting that it involves the way in which we "read" a sign is underdescribed, inviting mentalistic or transcendental psychological accounts of how meaning emerges. Cf. Potter 2000: 165 for the kind of underdescription that concerns me.

[27] According to wikipedia.com--where the etching may be viewed--Dürer's order-4 magic square "is believed to be the first seen in European art. It is very similar to Yang Hui's square, which was created in China about 250 years before Dürer's time. The sum 34 can be found in the rows, columns, diagonals, each of the quadrants, the center four squares, the corner squares, the four outer numbers clockwise from the corners (3+8+14+9) and likewise the four counter-clockwise … the two sets of four symmetrical numbers (2+8+9+15 and 3+5+12+14) and the sum of the middle two entries of the two outer columns and rows (e.g. 5+9+8+12), as well as several kite-shaped quartets, e.g. 3+5+11+15; the two numbers in the middle of the bottom row give the date of the engraving: 1514".

we can "see what is mathematically essential in the picture as well" (RFM I § 36ff). The traditional empiricist's conception of experience as a given sensory domain which causes us to construct the reality we do and impose necessity through convention seems not to apply here, where we meet what Wittgenstein called, in critically alluding to a paper of Russell's of the same name, "the limits of empiricism".

Now we can always ask, given any particular Latin Square, Are there other Latin Squares of the same order? It is simple to generate an *n* x *n* Latin Square for every *n*. (I say "generate" advisedly, for this verb cuts across the apparent divide between "discovering" and "inventing": with it, process and result are equivalent, i.e., neither way of regarding what we do is privileged.) The rule for generation is this: begin with the top row of 1…*n* and let successive rows cycle forward, to form *cyclic* Latin Squares. Our first example above is cyclic, as is this one, for *n*=4:

1	**2**	**3**	**4**
2	**3**	**4**	**1**
3	**4**	**1**	**2**
4	**1**	**2**	**3**

What should we say if someone failed to be able to see the pattern *in* these examples? What if someone did not see that they suffice to fix, not only a canonical form of cyclic Latin Square for *n* = 4, but also the *concept* of "*n* cyclic Latin Square" as well? Compare Wittgenstein's remark that "generality in mathematics does not stand to particularity in mathematics in the same way as the general to the particular elsewhere" (RFM V §25): the generality and the necessity here may be informal, but they are not *less* epistemologically forceful for this. (In fact, as we shall see, once we step to a more general setting, we lose our focus on this concept.)

Another quick way to generate Latin Squares of a given order is to take a permutation of the symbols 1,…*n* and relabel the square accordingly throughout.[28] For example, a permutation taking 1 to 4, 2 to 1, 3 to 3 and 4 to 2, respectively, when applied to the previous example, induces the following Latin Square:

4	**1**	**3**	**2**
1	**3**	**2**	**4**
3	**2**	**4**	**1**
2	**4**	**1**	**3**

However, this Latin Square is in a sense "essentially the same" as the previous one, for these squares are, with regard to their structural rearrangement, *isomorphic.* As is common in mathematics, and as we have already emphasized, the signs themselves do not matter, but rather how we draw and apply their relationships to one other.

Now we can ask, Are there *essentially* different (i.e., non-isomorphic in the relevant sense) Latin Squares for a given order *n*?   To answer this question it is not enough to use brute force, i.e., to write down every different Latin Square of order 4, for we have first to make sense of the concept of being "non-isomorphic in the relevant sense" (this is because every Latin Square of a given order is, in another purely cardinal sense, strictly isomorphic to every other of that order).

To reflect on this question mathematically, we codify the notion of two Latin Squares of the same order being as different (as unlike or various in internal structure) as possible. A way to visualize this variety is to imagine superimposing one square on top

[28] I say "another" way, but the reader will be able to see that cyclic Latin Squares are also produced by permutations of a systematic sort.

of the other. If we never see more than one array place at which each of the superimposed digits 1 through n match up, then the squares are as different as they can be. For example, for any pair chosen from the following family of Latin Squares, the two squares differ from each other in this way:

3	4	1	2
2	1	4	3
4	3	2	1
1	2	3	4

4	3	2	1
2	1	4	3
1	2	3	4
3	4	1	2

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Call two Latin Squares of order n, L_1 and L_2, *orthogonal* if for each ordered pair of numbers $\langle k, k' \rangle$ there is just one position (i, j) for which $L_1 (i, j) = k$ and $L_2 (i, j) = k'$. The above group of Latin Squares is an orthogonal family, in that each pair of two is orthogonal, no matter which two are chosen.

In 1781 Euler, the most important mathematician of the eighteenth century, posed the following problem.[29] Given thirty-six officers of six different ranks from six different regiments, can they be arranged in a square in such a way that each row and each column contains one officer of each rank and one officer from each regiment?

This question is easy to state: no fancy new concepts to be determined, no infinitely large classes of objects, no muss, no fuss. Yet despite the problem's apparent simplicity, Euler was unable to solve it. ("When it looks as if..., we should look out" (*Remarks on the Foundations of Mathematics* II §241).) And in fact he conjectured, within about a year, that there *was* no such pair of squares for all squares of order n where n is of the form $4m + 2$ for some number m (i.e., is 6, 10, 14, 18…).

[29] Wittgenstein alludes to an Euler diagram in his discussion of Gödel at RFM VII §19 (see n. 34 below).

We now know that there is no solution to the problem of how to arrange the thirty-six officers. Well over a century after Euler considered the problem, G. Tarry (in 1900) verified Euler's conjecture for n = 6 by conducting an exhaustive enumeration of all possible Latin squares of order 6. With this precedent, for many years it was widely believed that there was, similarly, no pair of orthogonal Latin squares of order 10, the intractability of exhaustive enumeration for this higher order being regarded as a daunting obstacle.

Yet quite unexpectedly, in 1960, Bose, Parker, and Shrikhande showed by a very involved construction that for $n = 10$, and in fact for any $n = 4m + 2$ for $m > 1$, there *is* a pair of orthogonal Latin squares of order n. Thus is Euler's conjecture refuted for all but $n = 6$, the original thirty-six officers problem. Now we can ask: How many such mutually orthogonal squares exist for these n? Even today it is not known whether or not there are three mutually orthogonal Latin squares of order 10.

The daunting calculational difficulties in making progress on orthogonal Latin squares of these orders stands in stark contrast to great progress on other orders that depended directly on central mathematical developments in algebraic number theory in the 19th Century. Here is a beautiful application of field theory, allowing us to see Latin Squares in a new way, and to see one system of mathematics *in* another. The key to seeing the connection between Gauss's work in algebra, Euler's problem of the 36 officers, and the theory of Latin Squares is to understand that we may view a Latin Square as a table for the arithmetic of, for example, the field of natural numbers mod 3:

$\underline{Z_3\text{, Arithmetic mod 3}}$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

After the structure of a *field* was isolated, relying on purely algebraic features of the real numbers, Galois constructed finite fields of cardinality p^k for every prime p and positive integer k. In general it is a theorem that if p is prime and i, j and t are non-zero elements of Z_p, then the rule

$$L_t(i, j) = ti + j$$

defines a Latin Square L_t, and the Latin Squares L_t, L_u are orthogonal when $t \neq u$. From Gauss's study of finite fields, if we let n be a positive integer > 2, let p be prime, and let $n = p^t$ for some positive integer t, then there are n - 1 Latin Squares of order n that are pairwise orthogonal.

Now we can ask, What are the first numbers n that are not > 2 and which are not powers of a prime in this way? The answer is: 6 (= 2 · 3), and 10 (= 2 · 5), just those orders of Latin Squares that gave Euler and others even today such trouble. By contrast, for 9 (= 3^2) it is simple to use the Galois field with 9 elements to generate 8 mutually orthogonal Latin squares of order 9. (These serve, I suppose, as a partial basis for generating many different Sudoku problems.[30])

When one first encounters Sudoku puzzles it may be quite surprising that the order of the square should matter so much. But, having encountered the process of generation of Sudoku through finite fields, one sees the situation with more refinement,

[30] This is an understatement. Since I first wrote this paper, the mathematics of Sudoku is now a burgeoning field of research. Cf. http://en.wikipedia.org/wiki/Mathematics_of_Sudoku for some of the indications.

as no longer governed by exhaustive enumeration of possibilities but by the way of thinking and calculating afforded by field theory (see, e.g., Grimaldi 2004). Once we bring in the algebraic interpretation of the problem in field theory, the difficulties Euler encountered for the case of order 6 appear to be strikingly different from the simplicity of the case of the order of the prime 7 and of the orders of the powers of primes 8 and 9. This contrasts with the unknown terrain of order 10 Latin Squares. We have come to see differences between the orders that matter to us, and that would have surprised Euler himself.

If the structure of fields affords us a new way of looking at Sudoku, seeing Sudoku in the context of field theory affords us also a new way of seeing a wide variety of empirical problems--a Wittgensteinian point.[31] For there is a branch of mathematics known as *design of experiments*, developed initially in agricultural applications in the 1920s--and still taught today as part of a thriving and important branch of mathematics--in which one applies the theory of Latin Squares to real life problems like that of the 36 officers.[32]

Suppose, for example, that you have n different kinds of medication that you wish to test on n different subjects in twofold combination. One can see this problem as affording an opportunity for the application of what we have just seen about Latin Squares and orthogonality: what is needed is a “combined block design”, two orthogonal Latin Squares of order n. We know from our brief study of Euler’s problem that if $n = 6$ we cannot solve this

[31] Mathematics must appear *in mufti*, i.e., in civilian clothes, i.e., in applications (RFM V §2). This is not a generalized semantical indispensability point about mathematics in physics, but only an echo of the logicist’s philosophy of mathematics. For it occurs where Wittgenstein is offering a revision or reinterpretation of the logicist demand that the semantics of number words make general sense of the applicability of number. On pure mathematics see RFM I §167.

[32] The theory of design of experiments is treated in great detail, with many illustrative examples of actual uses of so-called “block designs” in Roberts 1984.

problem directly (it "makes no sense"[33] to resolve it this way) but that we can if n is less than 6 or prime. For $n = 4$, that is, for four drugs 1,2,3,4 and four subjects A,B,C,D we superimpose two of our order 4 orthogonal squares (these were listed above) to obtain the following schedule, in which each paired element of the array (i, j) stands for a specific dosage to a specific patient on a specific month.

	Month 1	Month 2	Month 3	Month 4
A	(1,4)	(2,1)	(3,2)	(4,3)
B	(2,3)	(1,2)	(4,1)	(3,4)
C	(3,1)	(4,4)	(1,3)	(2,2)
D	(4,2)	(3,3)	(2,4)	(1,1)

Idiosyncrasies of these four subjects and order of dosage are assumed to be irrelevant in this block design. Each particular combination of drugs will be administered twice, to two different subjects, thereby controlling for differing reactions of individuals. This does not mean we have controlled for all (possibly relevant) factors, or that that one could not design another experiment that would control for these same factors better, if need be (perhaps over hundreds of trials). This shows that the suitability of any block design is subject to a restricted way of viewing the problem it is intended to solve. It also shows that our ordinary knowledge, our values, and our interests shape engineering and experimental problems.

What is philosophically important is that we can and do shape the course of our experience here--the testing of the effects of the dosage itself--by choosing to arrange our experiences *in terms of* the theory of Latin Squares. This is not to hold that "Calculating, if it is to be practical, must be grounded in empirical facts", but it is to allow tables and calculations

[33] "Makes no sense" is not just philosopher's jargon; it, along with other terminology such as "appropriate" and "good" and "what we want" may be found throughout the usual presentation of design of experiments theory; cf. Roberts 1984: chapter 9.

to "determine what the empirical facts *are*", to use Wittgenstein's phrase (RFM VII §§18-20).[34] And in fact, the theory of Latin Squares has been used to test everything from fertilizers to prosthodontics to tire and brake brands.

Euler's problem of the thirty-six officers is a beautiful example of a problem that is easy to state, but surprisingly difficult to solve. It is also a beautiful example of a surprisingly simple *kind* of solution (by brute enumeration) to a problem that appeared to be intractably complicated to deal with precisely because it was embedded in the context of higher algebra. It is, finally, a beautiful example of a case in which the existence or non-existence of certain objects in a universally applicable sense of *object* is not what is of primary interest to us, nor are the generalized, acontextually specified semantics of our terms, nor is any general distinction between "pure" and "applied" mathematics. Instead, what is of interest is that our *way of looking at certain situations*—both empirical and purely mathematical—*is changed*, and is *changed for the more interesting*.

III

I want next to consider a passage from the great number theorist G. H. Hardy's *A Mathematician's Apology*, a book that Wittgenstein knew.[35] In his *Apology* Hardy went

---

[34] In this passage Wittgenstein discusses a quite analogous phenomenon of application in connection with a diagram used to pose the problem of finding the number of ways one can trace every join in a wall continuously, and without repetition. This is a version of the famous problem of the seven bridges of Königsberg which Euler also studied, namely, Is it possible to find a route that will cross all seven of the bridges of Königsberg without crossing any one of them twice? This problem stimulated Euler to study what are now called Euler graphs, objects that are still studied today as part of the mathematical theory of graphs, a vast subject that began with the seven bridges problem. (Wittgenstein's idea in RFM is to connect this example of change of aspect of a situation with the kind of "self-reference" at work in Gödel's undecidable sentence, thus treating Gödel's (notably finitistic) theorem as a piece of applied arithmetic which changes our way of *seeing* a given formula in the language of *Principia Mathematica*.)

Wright 1980 associates Wittgenstein's discussion of the joins in the wall with a kind of conventionalism about mathematical truth he associates with rule-following skepticism, the view that the only logical necessities there are are those that are recognized by humans explicitly. I hope to have said enough to question whether this is the only perspective capable of making sense of Wittgenstein's remark; for further discussion see Mühlhölzer 1997.

so far as to hold that the distinction between "real mathematics" and chess lies in the fact that in "real" mathematics "there is a very high degree of *unexpectedness*, combined with *inevitability* and *economy*" (Hardy 1940: 113). Hardy has been criticized for making this remark about the surprising (Rota 1997: 172), though not by Wittgenstein.[36] I quote the passage in full below, for it forms just the kind of "raw material" for philosophy which interested Wittgenstein (cf. PG 396, PI §254).

The questions governing in Hardy's *Apology* are, in his words, "Why is it really worthwhile to make a serious study of mathematics?" and "What is the proper justification of a mathematician's life?" (1940: 65). So in this work he is aiming to give an account of what gives mathematics its ultimate life and death significance, not merely as a scientific subject, but as a pursuit to which he and others have devoted the better parts of their lives. Hardy is aiming to "justify" his and others' "existence" (1940: 66). What is important is that he stresses, in this context, not the hard reality, but the beauty and aesthetic value of mathematics, its arresting character:

> There is still one point remaining over from §11, where I started the comparison between 'real mathematics' and chess. We may take it for granted now that in substance, seriousness, significance, the advantage of the real mathematical theorem is overwhelming. It is almost equally obvious, to a trained

---

[35] Hardy 1940 is mentioned at MS 124, p. 35--a draft remark for PI §254--where Wittgenstein writes that "the sentences that Hardy sets forth as expression of his philosophy of mathematics in his miserable book "Apology of a Mathematician" are in no way philosophy, but could—like all similar outpourings—be conceived as raw material of philosophizing".

Hardy had been, with Moore, the examiner for Wittgenstein's fellowship at Trinity (cf. Monk 1990: 304); compare King 1984: 73 for testimony that Wittgenstein would meet with Hardy.

[36] Wittgenstein did make critical remarks about Hardy's paper "Mathematical Proof" (see AWL and LFM). He also wrote remarks stemming from his reactions to Hardy's *Coursebook in Pure Mathematics* (cf. Wittgenstein 2000), some apparently drawn from his annotations his own copy of (the 1941 edition of) this textbook. (On the status and availability of these annotations, which I discuss in IV below, see n. 46 below).

intelligence, that it has a great advantage in beauty also; but this advantage is much harder to define or locate, since the *main* defect of the chess problem is plainly its 'triviality', and the contrast in this respect mingles with and disturbs any more purely aesthetic judgment. What 'purely aesthetic' qualities can we distinguish in such theorems as Euclid's and Pythagoras's? I will not risk more than a few disjointed remarks.

In both theorems (and in the theorems, of course, I include the proofs) there is a very high degree of *unexpectedness*, combined with *inevitability* and *economy*. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions. There are no complications of detail—one line of attack is enough in each case; and this is true too of the proofs of many much more difficult theorems, the full appreciation of which demands quite a high degree of technical proficiency. We do not want many 'variations' in the proof of a mathematical theorem: 'enumeration of cases', indeed, is one of the duller forms of mathematical argument. A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way.

A chess problem also has unexpectedness, and a certain economy; it is essential that the moves should be surprising, and that every piece on the board should play its part. But the aesthetic effect is cumulative. It is essential also (unless the problem is too simple to be really amusing) that the key-move should be followed by a good many variations, each requiring its own individual answer. 'If P-B5 then Kt-R6; if…then…; if….then….'—the effect would be spoilt if there

> were not a good many different replies. All this is quite genuine mathematics, and has its merits; but it is just that 'proof by enumeration of cases' (and of cases which do not, at bottom, differ at all profoundly [n. I believe that it is now regarded as a *merit* in a problem that there should be many variations of the same type] which a real mathematician tends to despise.
>
> I am inclined to think that I could reinforce my argument by appealing to the feelings of chess-players themselves. Surely a chess master, a player of great games and great matches, at bottom scorns a problemist's purely mathematical art. He has much of it in reserve himself, and can produce it in an emergency: 'if he had made such and such a move, then I had such and such a winning combination in mind.' But the 'great game' of chess is primarily psychological, a conflict between one trained intelligence and another, and not a mere collection of small mathematical theorems (1940: 112-113).

The correspondence of mathematics to a higher "reality" is not mentioned here, nor is its cognitive usefulness, even if Hardy presupposes these. He is, after all, trying to characterize, not the logic or the content or the utility of mathematics as a science, but its ultimate significance as a human pursuit, even as a way of life.

In this vein, Hardy speaks to the question of what distinguishes number theory ("real", "serious" mathematics, as he had pursued them in his life as a mathematician) from chess. By calling "the great game" of chess "primarily psychological" Hardy hives it off from something he takes to be of higher intellectual (and aesthetic) value.

But Hardy's attitude reflects the common culture of the professional mathematician, in which elementary computation and enumeration of cases by brute force are not regarded as the most significant parts of mathematics, but instead as, at best, relatively uninteresting busy work. He mentions the (quite ordinary) mathematical impulse to fashion surveyability, the "clear cut" presentation of a proof constellation, which usually involves something other than enumeration. Hardy does not, in fact, deny to chess its mathematical significance: he even appeals elsewhere in the *Apology* to the popularity of mathematical puzzles to defend the idea of mathematics as a practice with recognized intellectual interest and value (1940: 86ff). So he has not really defended mathematics as having anything like intrinsic value as such.

Indirectly Hardy is reacting to the influence of (Hilbert's) formalism upon the philosophy of mathematics of his day. The heart of Hilbert's program was often presented in terms of an analogy between chess and arithmetic, and Hilbert's "finitism" emphasized the concrete, finite, intuitive character of mathematical signs as they appear in mathematical proofs. (In his post-1929 writings Wittgenstein frequently investigates both Hilbert's remarks and the chess comparison, though without taking sides in the way Hardy does.[37])

Wittgenstein's remarks on mathematics have often been explicated, collectively, as first and foremost critical of Hardy, directed against remarks Hardy made elsewhere about "mathematical reality"—as if anti-realism or anti-Platonism give us the primary thrust of his remarks on mathematics (and his relation to Hardy). Exercised to reject such uncritical Platonist metaphysics, Wittgenstein is said to have proposed a "calculus"

[37] On Wittgenstein's reactions to Hilbert see Mühlhölzer 2006, forthcoming a and forthcoming b. Mühlhölzer (2006) notes that an early appearance of Wittgenstein's notion of "surveyability" in connection with proof in mathematics attaches to the distinction between finite and infinite, at BT p. 484.

conception, according to which mathematics consists of finitistic algorithms and "techniques", as opposed to truths (Gerrard 1990, Monk 1990: 339ff, Marion 1998, Rodych 2002), or perhaps a brand of anti-realism turning on skepticism about the very applicability or coherence of our intuitive notion of a correct step in following a rule (Fogelin 1987, Wright 1980, Kripke 1981).

I hope to have so far shown that there is another quite distinctive dimension, both to mathematics and to Wittgenstein's remarks on logic and mathematics that is equally worth stressing. This dimension is emphasized quite explicitly in this passage from Hardy's *Apology*. Hardy points toward the interest, the "aesthetic value" of mathematics, its ability to give us a distinctive kind of "intellectual 'kick'", to captivate, to surprise, and to interest us (1940: 88). *This*, Hardy says, is not *merely* psychological, but something more. And this something is not merely instrumental or practical (having to do, for example, with the usefulness of the applications of mathematics in physics[38]). For Hardy, it is instead "purely aesthetic". Like Moore in *Principia Ethica* Hardy is treating the ultimate good of mathematics as a good in itself, not subject to naturalistic or instrumental defense, but instead subject to something more like aesthetic appreciation.[39]

One does not have to agree with Hardy's use of the phrases "merely psychological" and "purely aesthetic" (or even with his comparative assessment of chess and "serious" mathematics) to find this dimension of interest. Nor does one have to reject as relatively unimportant the extra-mathematical applications of mathematics, as Hardy does (and Wittgenstein did not). We have only to compare and contrast Hardy's

[38] In fact, Hardy explicitly worries about the dangers of these applications in the sphere of technology.
[39] At MS 119 p. 88v when he is discussing a remark of Hardy's about the infinite Wittgenstein remarks on the "peculiar similarity between investigations in philosophy (perhaps especially in mathematics) and aesthetic ones, e.g., what is bad about this garment, how does it fit, etc.".

reliance on the phenomena of the unexpected in distinguishing mathematics from a game with, for example, Frege's, when he confronts the question of what distinguishes the significance of "real" arithmetic from that of chess.

In his *Grundgesetze der Arithmetik* §91 Frege had written,

> Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no thought. If it did so and every chess move conforming to the rules corresponded to a transition from one thought to another, applications of chess would also be conceivable. Why can arithmetical equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? Now, it is applicability alone which elevates arithmetic from a game to the rank of a science. So applicability necessarily belongs to it (Frege 1980: 167).

For Frege, until mathematicians get philosophically clear about what is wrong with the view that numbers are just counters in a game, like chess pieces, there can be no true science of arithmetic. What makes arithmetic "serious" and of value for Frege is its "elevation to the rank of a science", an activity whose primary aim is truth. This presupposes that arithmetic expresses *Gedanke*, thoughts, the kind of sense (*Sinn*) that for Frege is eligible to be recognized or denied in judgment and assertion. And the elevation of arithmetic to the rank of a science is secured, on Frege's view, by his understanding of the universal applicability of arithmetic, his logicist analysis that systematizes the role of number words in descriptions of situations, i.e., in so-called "mixed" contexts (contexts such as "there are five plums on the table") and in ordinary statements of pure arithmetic as well. In both contexts Frege sees the numerals functioning as names of objects. In fact, by means of his derivation of the Dedekind-Peano axioms of arithmetic from fundamental logical laws, Frege takes himself to have proved that the numbers are purely logical objects. And thus Frege takes himself to have refuted formalism--the kind of formalism which sees in arithmetic nothing but uninterpreted calculations, nothing but a movement of figures analogous to chess—by demonstrating how universal applicability "necessarily belongs" to arithmetic. What philosophers have long labelled the "inevitability" or "necessity" of mathematical truth is thus lodged, on Frege's account, in the maximal generality of logic. Like Moore and Russell, Frege holds that the sense of "necessity" or "inevitability" we experience in doing arithmetic (our failing, for example, to be able to imagine that two beans and two more beans makes five) is something merely psychological, or, if logical, reducible to generality.

From early on in his life, Wittgenstein understood that a key to understanding the nature of logic and mathematics would require reliance on a proper conception of necessity and possibility, which can neither be dismissed as psychological nor reduced to the terms of generality as understood by the quantifiers. He regarded Frege's (and Russell's) logicist dismissal of the concepts of possibility and necessity (i.e., their purported reduction of the logical content of these notions to the rubric of generality) as philosophically fatal. The problem he faced was how to revitalize the notions of necessity and possibility in the face of their psychologistic reduction, while still adhering to the (Kantian) idea that the value of the modalities lies in their relation to our human ("logical") forms and capacities of judgment alone, and not to their ultimate metaphysical reality in application to things in themselves. He sought to find a way of seeing and discussing the ultimate significance—the interest, objectivity and applicability--of logic, mathematics and philosophy that does not fall back on a conception of truth lodged in the generality of its application to objects and concepts.

Wittgenstein's solution was to appeal, over and over again, to a wide variety of phenomena of aspect perception in order to rethink the concepts of possibility and necessity. The idea was that we can reflect on "grammatical" possibilities of discrimination and representation and extract, through reflection, an understanding of what is involved in the applications and necessities of logic and mathematics (cf. Floyd forthcoming). Of course, Wittgenstein remained fully aware that Frege would have dismissed phenomena of aspect perception as "merely psychological", relegating them to the context of discovery, rather than that of justification. And he knew that after reading the *Tractatus*, Russell and Ramsey had each tried to psychologize and empiricize such phenomena, aiming to absorb such experiences into a causal theory of belief and knowledge. What was to become ever more central in Wittgenstein's thought, as time went on, was the need to scrutinize the philosophical requirements and presuppositions on which his extraction of necessities from aspect perception turned.

IV

Reverting once again to Adam Smith, it is important to note that there is an important difference between the surprising [*das Überraschende*] and the astonishing (or amazing) [*das Erstaunliche*]. The former, when it is positively evoked in Wittgenstein's writings, is an index of engagement with the local, with a rush (*"raschen"*) of absorption or preoccupation, with curiosity and its satisfaction, with finding solutions, with puzzlement and its vanishing, with openness to changing our point of view on a train of thought or a representation or a situation. A proof in mathematics allows us to see *how* a conclusion follows with necessity from its premises, *how* the line of thought, and the result, may be produced and seen as in this sense unsurprising. This does not mean that we may not be surprised *that* a proof can be proved here, or that a proof proceeds by particular methods we might not have foreseen, or that a proof may find applications in other parts of mathematics, or elsewhere in science. This is part of our very concept of proof: that it is communicable, that it is a process that may be made surveyable, that it may be used. Astonishment and amazement involve us, by contrast, in latching on to a particular result, world, or field of significance as essentially and intrinsically astounding. While amazement and astonishment may have their respectable places in human life (on which see LE), they do not qualify as true originators or satisfiers of philosophy, as does wonder. Severed from surprise and wonder, amazement may in fact too easily spill over into stupefaction, hyperbolic overtone, mysteriousness, wordlessness, fanaticism or astonishment ("*erstaunen*", to be turned to stone). The surprising disappears or vanishes, lending itself to transformation, *via* wonder, into proper admiration or appreciation. The astounding or amazing or mysterious or astonishing may not.

Like Smith, I am suggesting, Wittgenstein regarded the role of surprise, wonder and admiration to be important, their "influence…of far wider extent than we should be apt upon a careless view to imagine". For philosophy, as "an art of the imagination" spawned by puzzlement, surprise, and wonder

> …endeavours to introduce order into this chaos of jarring and discordant appearances, to allay this tumult of the imagination, and to restore it, when it surveys the great revolutions of the universe to that tone of tranquility and composure, which is both most agreeable in itself, and most suitable to its nature (Smith 1795 II.12)

The surprising is interested and engaged, pushing us forward into the details of the local and our reactions to it. The amazing or astonishing, by contrast, can sink us back into a mute wondering or, even more dangerous to Wittgenstein's mind, a *superstitious* celebration of the mysteriousness of the universe in itself.[40]

Astonishment and amazement are linked by Wittgenstein to wonder: wondering at the existence of the world, that there are objects or numbers at all, at the fact of a particular proof. Wonder is not always criticized by Wittgenstein; it may even be said that his writings are designed to cultivate and generate respect for it (cf. PI §524).[41] But he is sharply critical of the use of these notions in certain contexts--as if the legitimate forces of surprise, and the interests it reflects, might find themselves at any moment overwhelmed by philosophy.

Some of the most important examples of the contrast between the astonishing and the surprising in Wittgenstein's writings concern reactions mathematicians and

[40] Recall PI §§109ff. Compare these with Wittgenstein's remarks on the non-mysteriousness of $\aleph_0$ in *Wittgenstein's Lectures on the Foundations of Mathematics*, *Cambridge 1939* as well as his remarks criticizing James Jeans's title *The Mysterious Universe* in LA III §36.

[41] Gordon C. F. Bearn makes the importance of wonder central to his reading of Wittgenstein in *Waking to Wonder: Wittgenstein's Existential Investigations* (Albany, NY: State University of New York Press, 1997).

philosophers have had to the paradoxes of logic. We have only to investigate a few examples of how mathematicians talked about these to see the point.

When on June 22, 1902 Frege wrote to Russell to acknowledge receipt of Russell's letter informing him of the paradoxes, he wrote:

> Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic (Van Heijenoort 1967: 127-8).

Frege was "surprised" (*überrascht*, *bestürzt*) by Russell's paradoxes (and unpleasantly so) because he was in the process of finishing his two volume work, the *Grundegesetze der Arithmetik*, and had been shown his system of logic was inconsistent. But he makes it clear in his reply to Russell that he is determined to learn from the situation, and set arithmetic aright again. Frege was not professing stupefaction or amazement, but a surprise that would push him forward in his research.

This might be contrasted with what Wittgenstein called "the superstitious dread and veneration by mathematicians in the face of contradiction",[42] the manner of formulating the significance of the paradoxes that was expressed, for example, by Gödel in his (popular) article "Russell's Mathematical Logic". There Gödel wrote:

[42]This, Wittgenstein adds, is very funny [ist sehr *komisch*]" (*The Wittgenstein Nachlass*, item 118, p. 115v (MS Vol. XIV). The final italicized phrase of Wittgenstein's remark was dropped from later typescripts, and hence also from the published version of the remark that Gödel and others saw at *Remarks on the Foundations of Mathematics* I, Appendix III §). (For a discussion of Gödel's attitude toward this remark, see Floyd 2001). Wittgenstein's point seems to be, not that such superstition is to be condemned, but, rather, that it is to be investigated as a phenomenon in its own right. It may well be that he is unfair to Hilbert and other mathematicians here, for they seem not to have been gripped by a need for Cartesian certainty. For more on this point see Mühlhölzer 2006, forthcoming a.

> By analyzing the paradoxes to which Cantor's set theory had led, [Russell] freed them from all mathematical technicalities, thus bringing to light the amazing fact that our logical intuitions (i.e., intuitions concerning such notions as: truth, concept, being, class, etc.) are self-contradictory (Gödel 1944: 124).

This risks for Wittgenstein falling into cheap seduction, rather than proper stimulation of a reader toward thoughts and studies of his or her own. Not because it might not describe a possible experience of interest to the public, but because it risks, in its style of formulation, stimulating the wrong kind of head-spinning about the paradoxes, head-spinning disengaged from mathematical activity.

In 1935 Schlick wrote to Wittgenstein to ask him what he thought about a result that had proved very surprising to many people, the Gödel theorem concerning the incompleteness of first-order arithmetic. Schlick appears to have professed "astonishment" at the theorem, and seems to have been turning to Wittgenstein for advice on what to think (cf. Wittgenstein 2004; we have only Wittgenstein's reply to Schlick, the last letter he ever wrote to him). Wittgenstein's reply turns on the distinction between the surprising and the astonishing, as I--following Smith--have distinguished them:

> As to the application of what I've said to the case which you quote: I want now only to say this: If you hear someone has proved that there must be unprovable propositions in mathematics, there is *first of all nothing astonishing in this* [*vorerst gar nichts Erstaunliches*, Wittgenstein's own emphasis], because you have as yet no idea whatsoever what this apparently utterly clear prose-sentence says.

Until you go through the proof from A to Z, down to the last detail, you can't see what it proves. To one who would wonder at the fact that two opposite sentences are provable I would say: look at [*schau an*] the proof and then you will see "in which sense" the one and "in which sense" the other is proved. And before you have studied the entire proof precisely you have no reason to wonder. All that you can learn from "my teachings" is that about such a proof [with its result] *nothing* can be said before you have investigated the determinate proof. That is: the philosopher is always wrong who wants to prophesy a quasi-something in mathematics and say, "*that* is impossible", "*that* cannot be proved". Why not? That which is supposed to be proved is nothing but a word expression and the proof gives it its particular sense; and with how much warrant we then call this proof the proof of this prose- sentence is partly a matter of taste; that is, it is a matter of our judgment [*Ermessens*] and our inclination whether we want to apply the structure expressed here in this prose-sentence, or not. How the matter of our inclination is, whether we want to speak of imaginary *points* or not; or of invisible light, or not. -- The proper [*genau*] investigation of a complicated proof is extraordinarily difficult. That is, it is extraordinarily difficult to organize the structure of [*gestalten*] the proof *perspicuously* [*durchsichtig*] and to obtain complete clarity about its relation to other proofs, its position in certain systems, and so on. You have only to try properly to investigate a proof such as that of the sentence that $\sqrt{2}$ is irrational and you will persuade yourself of this. This does not however mean that there is something mystical in this proof before this investigation, but only that we have not yet clearly *taken in* [*überschauen*] the proof and especially its position among other proofs. --You are on the

> wrong track if you say, you feel in spite of my teachings completely helpless before such and such a proof.[43]

This remark might be taken to express an early version of what has been lately called "naturalism" about mathematics (i.e., philosophers' deference to what mathematicians say and do), and it certainly elaborates upon Wittgenstein's early remarks that in logic, mathematics and philosophy "there are no surprises" (see footnote 9). But in its original context, it is not simply expressing a kind of "wait and see" attitude, or anti-*a priorism* about Gödel's result. In it we can see something more.

Wittgenstein is distancing himself sharply from what Schlick, the father of Vienna positivism as a philosophical movement, seems to think about Wittgenstein's own conception of philosophy. Schlick seems to think that Wittgenstein was offering him a philosophical method, a general characterization of notions like *proof*, *truth* and *meaning*, as well as a philosophy oriented toward scientific method as a subject matter.[44] Wittgenstein is saying in 1935 that he is not doing this. That is why he ends the letter by saying that "you are in the grip of a false conception if you believe that you have to prick up your ears whenever you are told about a new proof". Wittgenstein is not saying that mathematics in general, and Gödel's proof in particular, is *a priori* irrelevant to philosophy. He is also not simply saying that if a philosophical remark contradicts mathematical practice, then the philosophy must always give (compare Maddy 1997). Something more complicated is going on: he is stressing how difficult it is, how much a matter of taste and judgment and context, to really appreciate the significance of a mathematical result.

---

[43] Wittgenstein to Schlick, 31.7.1935, in Wittgenstein 2004.

[44] To be fair, Schlick may also have been in mind of what Wittgenstein had written in the *Tractatus* at 4.442, viz., "A proposition cannot possibly assert of itself that it is true." He may be forgiven for having wondered how that remark might be made compatible with Gödel's proof of the incompleteness theorem.

And he is insisting that that appreciation requires a special kind of activity, what he calls an "investigation", something that requires time and patience and is "very difficult". Part of what it involves is working through the details of a proof and its relation to other proofs in order to separate misleading "prose" from the genuine, inner core of the result. This requires developing a nose for which notions are rigorized, and which are not, in the context of an argument, which forms of "prose" are "gas" (potentially misleading) and which are not, which affect our extra-mathematical, ordinary notions, and which do not. This for Wittgenstein is very much something to which a philosopher, rather than a mathematician, is and should be specially attuned.[45]

In the course of his reply to Schlick Wittgenstein gives as one example to be "investigated" the proof of the irrationality of the square root of two. We now know that he wrote many comments on this proof, some of which were apparently drafted as annotations to the margins of his own copy of Hardy's *A Course in Pure Mathematics* (1941).[46] These apparent annotations are striking indeed, and remarkably odd at first glance, a kind of midrashic editorial commentary, strewn around the edges of the book, circling, castigating, rewording, crossing out

[45] Cf. item 118 (1937) pp. 113ff, on the topic of "hidden" contradictions and proof as a "forcible" ground:

> People have only sometimes said [or said occasionally] that they couldn't judge such and such, for they hadn't studied philosophy. This is an irritating piece of nonsense, (for) it pretends that philosophy is a science. And one talks of philosophy as something like medicine. -- One can say, however, that people who have never taken up an investigation of a philosophical kind--like e.g. most mathematicians--are not equipped with the right organ of sight for such [a kind of] investigation or test. Just as one who is not used to hunting in the woods for bears [flowers, bears or herbs] will find none, because his eyes are not trained for such an investigation, and he doesn't know where in particular one should look for them. In this way someone unpracticed in philosophy passes by all the places where difficulties lie hidden in the grass, while someone who has practiced philosophy stops and stands there, and feels that here is a difficulty, although he doesn't see it yet. --And no wonder, if one knows how long even the well-practiced person, who notices that a difficulty lies here, must search in order to find it.
>
> When something is well hidden, it's hard to find.

[46] Copies of (at least some of) these annotations were kindly provided to me by Michael Nedo, of the Cambridge Wittgenstein Archives. Compare the Introductory volume of WA and Nedo 2008 for discussions and published images. I have not been able to verify the authenticity of these annotations, because according to Nedo the original copy of Wittgenstein's book has been lost. The remarks do appear to be written in Wittgenstein's hand and interweave in direct and interesting ways with MS 126.

and substituting individual words, especially those connected with generality. What they seem to be are philosophical editings of Hardy's textbook "prose": they enact and carry through precisely what Wittgenstein was telling Schlick to do in 1935 for himself. In 1942 Wittgenstein apparently transferred some of these remarks into a manuscript, and connected them explicitly with his remarks on Gödel's theorem--returning, as we might say, to the most important themes of this 1935 letter (MS 126 14 December 1942). In his remarks he contrasted the "determinate proof machinery" in Hardy's demonstration of the irrationality of the square root of two with Gödel's remarks in the introduction to his 1931 incompleteness paper, which struck him as purporting, misleadingly, to belong to an "eternally valid form of proof", instead of to a determinate proof context (or "machinery").[47]

What is important here is that Wittgenstein *has* no general stance on the notions of *truth* and *meaning* and *proof*. His whole idea is to try to philosophize *without* this. And his key philosophical terms of criticism involve the investigation of terms that are, ultimately, purposive terms of art, terms like "surprising" and "astonishing".

[47] In Floyd 2001 I touch upon the issue of Wittgenstein's attitude toward the introductory remarks of Gödel's 1931 paper. Shanker usefully reconstructed Wittgenstein's likely attitude toward these in his (1988). By 1942, when the remarks on Gödel I refer to here were written, I believe Wittgenstein had read through the whole of Gödel's paper (there are references in 1941 to κ-provability and ω-consistency at MS 163 pp 32, 41 respectively). We may therefore assume, I think, that in 1942 he was contrasting the Introductory remarks, where Gödel discusses the paradoxes in general, with the "determinate proof machinery" involved in the actual proof. The reader should see Rodych 2002 for a discussion of the *Nachlass* passages involving Gödel—though Rodych adopts in general a very different reading of Wittgenstein's philosophy of mathematics from mine.

References

Abbreviations of Wittgenstein references:

BT Big Typescript, TS 213
MN "Notes Dicated to G.E. Moore in Norway, April 1914", Appendix II in NB
NB *Notebooks 1914-16*
DWL *Wittgenstein's Lectures, Cambridge 1930-32*
AWL *Cambridge Lectures: 1932-1935*
LA *Lectures and Conversations on Aesthetics, Psychology, and Religious Belief*
LE Lecture on Ethics
LFM *Wittgenstein's Lectures on the Foundations of Mathematics*, *Cambridge 1939*
LW I *Last Writings on the Philosophy of Psychology, vol. I*
LW II *Last Writings on the Philosophy of Psychology, vol. II*
PG *Philosophical Grammar*
PI *Philosophical Investigations*
RFM *Remarks on the Foundations of Mathematics*
RPP I *Remarks on the Philosophy of Psychology, Volume I*
RPP II *Remarks on the Philosophy of Psychology, Volume II*
TLP *Tractatus Logico-Philosophicus*
WA *Wiener Ausgabe*
WVC *Ludwig Wittgenstein and the Vienna Circle*
MS/TS Manuscript and typescript numbers according to von Wright's standard referencing; see Wittgenstein (2000) for details.

Barker-Plummer, D., et.al. (eds) (2002), *Words, Proofs, and Diagrams*. Stanford, CA: CSLI.

Baz, A. (2000). "What's the Point of Seeing Aspects?" Philosophical Investigations **23**(2): 97-121.

Baz, A. (forthcoming). Aspect Perception. Ludwig Wittgenstein: Key Concepts. K. Jolley. Durham, U.K., Acumen.

Bearn, G.C.F. (1997), *Waking to Wonder: Wittgenstein's Existential Investigations*. Albany, New York: State University of New York Press.

Brown, R.W. (1999), *An Introduction to the World of Proofs and Pictures*. New York: Routledge.

Budd, M. (1989), *Wittgenstein's Philosophy of Psychology*. New York: Routledge.

Cavell, S. (1976), "Aesthetic Problems of Modern Philosophy", in *Must We Mean What We Say?* New York: Oxford University Press.

Diamond, C. (1991), *The Realistic Spirit*. Cambridge, MA: MIT Press.

Diamond, Cora (1997), 'Wittgenstein, Mathematics, and ethics: resisting the attractions of realism', in Hans and Stern Sluga, David (ed.), *The Cambridge Companion to Wittgenstein* (New York: Cambridge University Press), 226-60.

Dreben, B. and Floyd, J. (1991), “Tautology: How Not to Use A Word”, *Synthese* 87 1: 23-50.

Dummett, M. (1978), “The Justification of Deduction”, in *Truth and Other Enigmas*. Cambridge, MA: Harvard University Press.

Dummett, M. (1959), "Wittgenstein's Philosophy of Mathematics", *Philosophical Review,* 63:324-348; reprinted in *Truth and Other Enigmas*, pp. 166-185.

Ekman, P. (2003), *Emotions Revealed.* New York: Henry Holt.

Floyd, J. (2001), "Prose versus Proof: Wittgenstein on Gödel, Tarski and Truth”, *Philosophia Mathematica* 3, 9: 901-928.

Floyd, J. (2005). Wittgenstein on Philosophy of Logic and Mathematics. The Oxford Handbook to the Philosophy of Logic and Mathematics. S. Shapiro. New York, Oxford University Press**:** 75-128.

Floyd, J. (2008). Wittgenstein über das Überraschende in der Mathematik. <<Ein Netz von Normen>>: Wittgenstein und die Mathematik. M. Kross. Berlin, Parerga Verlag**:** 41-78.

Floyd, J. (forthcoming). On Being Surprised: Wittgenstein on Aspect Perception, Logic and Mathematics. Seeing Wittgenstein Anew: New Essays on Aspect Seeing. V. Krebs and W. Day. New York, Cambridge University Press.

Fogelin, R. (1987), *Wittgenstein*, 2nd ed. New York, Routledge & Kegan Paul

Frege (1980), *Translations from the Philosophical Writings of Gottlob Frege*, P. Geach and M. Black (eds). 3rd edition, Totowa, New York: Rowman and Littlefield.

Gerrard, S. (1991) “Wittgenstein's Philosophies of Mathematics", *Synthese* 87 1: 125-142.

Glock, H.-J. (1996), *A Wittgenstein Dictionary*. Cambridge, MA: Blackwell Publishing.

"*Das Überraschende*: Wittgenstein on the Surprising in Mathematics", in Jon Ellis and Daniel Guevara, eds., *Wittgenstein and the Philosophy of Mind* (Oxford University Press, 2010).

Gödel, K. (1944), "Russell's Mathematical Logic", Gödel 1944 in S. Feferman et.al. (eds), *Kurt Gödel Collected Works Volume II, Publications 1938-1974*. New York: Oxford University Press, pp. 119-141.

Gödel, K. (2003), *Kurt Gödel: Collected Works Volume IV: Correspondence A-G* S. Feferman et.al. (eds). New York: Oxford University, Clarendon Press.

Grialou, P., Longo, G., Okada, M. (eds) (2005), *Images and Reasoning*, *Proceedings of an Interdisciplinary Conference Series on Reasoning Studies.* Tokyo: Keio University Centre for Integrated Research on the Mind.

Grimaldi, R.P., *Discrete and Combinatorial Mathematics* 5th ed. Boston: Pearson/Addison-Wesley.

Grosholz, E. (2007). Representation and Productive Ambiguity in Mathematics and the Sciences. New York, Oxford University Press.

Hardy, G.H. (1940), *A Mathematician's Apology*. Reprint edition, New York: Cambridge University Press, Canto, 1967.

Hardy, G.H. (1941), *A Coursebook in Pure Mathematics*, 8th edition. London: Cambridge University Press.

Kanamori, A. (2005), *The Higher Infinite*. New York: Springer Verlag, 2nd ed.

Kennedy, J. (manuscript), "On Reading Mathematical Constructions As Works of Art".

King, J. "Recollections of Wittgenstein", in R. Rhees (ed), *Recollections of Wittgenstein*. New York: Oxford University Press, pp. 68-75.

Kremer, M. (2002). "Mathematics and Meaning in the Tractatus." Philosophical Investigations **25**(3): 272-303.

Kripke, S. (1981), *Wittgenstein on Rules and Private Language*. Cambridge, MA: Harvard University Press.

Maddy, P. (1997), *Naturalism in Mathematics*. Oxford: The Clarendon Press.

Mancosu, P., Jørgensen, K. and Pedersen, S. (eds.) (2005), *Visualization, Explanation and Reasoning Styles in Mathematics* New York: Springer Verlag.

Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics* New York Oxford University Clarendon Press.

Marion, M. (2009). "Radical anti-realism,Wittgenstein and the length of proofs." Synthese **171**: 419-432.

Monk, R. (1990), *Wittgenstein: The Duty of Genius*. New York: Free Press.

Moore, A.W. (1990), *The Infinite.* New York: Routledge, 1990.

Mühlhölzer, F. (1997), "Mathematik ohne Metaphysik", in Julian Nida-Rümelin (Hrsg.), *Rationalität, Realismus, Revision* (*Vorträge des 3. internationalen Kongresses der Gesellschaft für Analytische Philosophie vom 15. bis zum 18. September 1997 in München*). Berlin: Walter de Gruyter, 1999, pp. 416-423.

Mühlhölzer, F. (2002). Wittgenstein and Surprises in Mathematics. Wittgenstein und die Zukunft der Philosophie: Eine Neubewertung nach 50 Jahren, Proceedings of the 24th International Wittgensein Symposium, 2001. K. Puhl, Rudolf Haller eds. Kirchberg am Wechsel, Austria, öbver hpt Verlag**:** 306-315.

Mülhölzer, F. (2006). "'A mathematical proof must be surveyable': What Wittgenstein meant by this and what it implies." Grazer Philosophische Studien **71**: 57-86.

Mülhölzer, F. (forthcoming a). Braucht die Mathematik eine Grundlegung? Ein Kommentar des Teils III von Wittgensteins _Bemerkungen über die Grundlagen der Mathematik. Frankfurt am Main, Vittorio Klostermann.

Mühlhölzer, F. (forthcoming b). „Wittgenstein and Metamathematics". Wittgenstein: Philosophie und Wissenschaften. P. Stekeler-Weithofer, ed.. Hamburg, Verlag Felix Meiner.

Mulhall, S. (1990), *On Being in the World: Wittgenstein and Heidegger on Seeing Aspects*. New York: Routledge.

Nedo, M. (2008). Anmerkungen zu Wittgensteins Bemerkungen uber die Grundlagen der Mathematik. <<Ein Netz von Normen>>: Wittgenstein und die Mathematik. M. Kross. Berlin, Parerga Verlag**:** 80-103.

Norman, J. (2006), *After Euclid: Visual Reasoning and the Epistemology of Diagrams*. Stanford, CA: CSLI.

Potter, M. (2000), *Reason's Nearest Kin: Philosophies of Arithmetic from Kant to Carnap*. New York, Oxford University Press.

Putnam, H. (1994), "Philosophy of Mathematics: Why Nothing Works" in *Words and Life*, Conant (ed.). Cambridge, MA: Harvard University Press, pp. 499-512.

Raïd, L. (2009). Les Probabilités ont-elles un Objet? La conception logique des probabilitiés selon le Tractatus. Lire le Tractatus logico-philosophicus de Wittgenstein. C. Chauviré. Paris, Vrin**:** 171-184.

King, J. (1984), "Recollections of Wittgenstein", in Rhees, R. (ed) *Recollections of Wittgenstein*, New York: Oxford University Press, 1984, pp. 68-75.

Roberts, F.S. (1984), *Applied Combinatorics*. Englewood Cliffs, NJ: Prentice Hall.

Robinson, J. (1995), "Startle", *Journal of Philosophy* 92,2 (1995): 53-74.

Rodych, Victor. "Wittgenstein on Gödel: The Newly Published Remarks." *Erkenntnis* 56, no. 3 (2002): 379-97.

Rota, G. (1997), "The Phenomenology of Mathematical Beauty", *Synthese* 111: 171-182.

Russell, B. (1936), "The Limits of Empiricism", *Proceedings of the Aristotelian Society* pp. 131-150.

Shanker, S.G. (1988), 'Wittgenstein's Remarks on the Significance of Gödel's Theorem', in S.G. Shanker (ed.), *Gödel's Theorem in Focus* (New York/London: Routledge), 155-56.

Shanker, S. (1998), *Wittgenstein's Remarks on the Foundations of AI*. New York: Routledge.

Shanker, S. and Greenspan, S. (2004), *The First Idea: How Symbols, Language and Intelligence Evolved from Our Primate Ancestors to Modern Humans*. Cambridge, MA: Da Capo Press.

Smith, A. (1795), "The History of Astronomy", in *Essays on Philosophical Subjects*, W.P.D. Wightman, J.C. Bryce (eds). Indianapolis: Liberty Classics 1980 reprint of the Glasgow Edition of the Works and Correspondence of Adam Smith, Vol. III.

Steiner, M. (1996), "Rules, "Wittgenstein: Mathematics, Regularities and Rules" in *Benacerraf and His Critics*, A. Morton, S. Stich (eds.). Cambridge, Mass.: Blackwell Publishing Company, pp. 190-212.

Stillwell, J. (1989), *Mathematics and Its History*. New York: Springer Verlag.

Van Heijenoort, J. (1967), *From Frege to Gödel*. Cambridge, MA: Harvard University Press.

Wilson, M. (2006). Wandering Significance: An Essay on Conceptual Behavior. New York, Oxford University Press.

Wittgenstein, L. (1921), "Logische-Philosophische Abhandlung", final chapter, Ostwald's *Annalen der Naturphilosophie*; English trans. *Tractatus Logico-Philosophicus* by C.K. Ogden. New York: Routledge and Kegan Paul, 1922.

Wittgenstein, L. (1929), "A Lecture on Ethics", in *Ludwig Wittgenstein: Philosophical Occasions 1912-1951*, J. Klagge and A. Nordmann (eds). Indianapolis, Hackett Press, 1993.

Wittgenstein, L. (1958), *Philosophical Investigations*, 3rd ed., G.E.M. Anscombe, R. Rhees, (trans). Oxford, Blackwell.

Wittgenstein, L. (1970), *Lectures and Conversations on Aesthetics, Psychology, and Religious Belief*. Berkeley, CA: University of California Press.

Wittgenstein, L. (1973), *Ludwig Wittgenstein and the Vienna Circle*, shorthand notes recorded by F. Waismann, eds. B. McGuinness, J. Schulte, trans. B. McGuiness, Oxford, Blackwells.

Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, revised ed., eds. G.H. von Wright, R. Rhees G.E.M. Anscombe (eds). Oxford: Blackwells.

Wittgenstein, L. (1979), *Cambridge Lectures: 1932-1935*, from the Notes of Alice Ambrose and Margaret Macdonald, A. Ambrose (ed), University of Chicago Press.

Wittgenstein, L. (1980), *Wittgenstein's Lectures, Cambridge 1930-32, from the notes of John King and Desmond Lee*, D. Lee (ed). Oxford:Blackwells.

Wittgenstein, L. (1983). Remarques sur les fondements des mathématiques. Paris, Gallimard.

Wittgenstein, L. (1989), *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939*, C. Diamond (ed). Chicago, IL: The University of Chicago Press.

Wittgenstein, L. (1993-), *Wiener Ausgabe*, Nedo, M. (ed). Vienna/New York: Springer-Verlag.

Wittgenstein, L. (2000), *Wittgenstein's Nachlass, The Bergen Electronic Edition* (CD-Rom). Oxford University Press.

Wittgenstein, L. (2004), *Ludwig Wittgenstein: Briefwechsel*, Innsbrucker elektronische Ausgabe (CD-Rom), M. Seekircher, B. McGuinness, A. Unterkircher, A. Janik and W. Methlagl eds., Intelex.

"*Das Überraschende*: Wittgenstein on the Surprising in Mathematics", in Jon Ellis and Daniel Guevara, eds., *Wittgenstein and the Philosophy of Mind* (Oxford University Press, 2010).

Wittgenstein, L. (2005). Big Typescript, TS 213. Malden, MA, Wiley-Blackwell Publishing.

———. " Remarques Sur Les Fondements Des Mathématiques." Paris: Gallimard, 1983.

Wright, Crispin. "Rule-Following without Reasons: Wittgenstein's Quietism and the Constitutive Question." *Ratio* 20, no. 4 (2007): 481-502.

Wright, C. (1980), *Wittgenstein on the Foundations of Mathematics*. Cambridge, MA: Harvard University Press.

Wright, C. (1993), "Strict Finitism", in *Realism, Meaning and Truth*, 2nd ed. Oxford: Blackwell.

Wright, C. (2007). "Rule-Following Without Reasons: Wittgenstein's Quietism and the Constitutive Question." Ratio **20**(4): 481-502.