Forthcoming in
William Day and Victor Krebs,
Seeing Wittgenstein Anew: New
Essays on Aspect Seeing
(Cambridge Univ. Press).

15

On Being Surprised

Wittgenstein on Aspect-Perception, Logic, and Mathematics

Juliet Floyd

Wittgenstein's remarks invoking aspect-perception mirror his overall development as a philosopher. While I do not want overly to geneticize the philosophical terrain connected with aspect-perception, I do think it worth emphasizing that the duck-rabbit of the *Philosophical Investigations* is only one kind of example of aspect-perception, and that some of the most vivid, natural, and compelling uses of the idea of seeing aspects, interpreting one system in another, or being struck by a new aspect of a diagram, word, or sentence – as well as the earliest, most frequent, and systematic appearances of these themes in his philosophy – occur in Wittgenstein's discussions of mathematics and logic.

After a few remarks about the constructive nature of Wittgenstein's preoccupation with pictures (Section 1), I consider the earliest passage in his writing invoking puzzle-pictures (Section 2), then consider *PI* §§523–25 in relation to his earliest thoughts (Section 3), and finally look at how his uses of aspect-perception bridge the evolution in his thought from earlier to later (Section 4).

1.

In his writings on logic and mathematics, Wittgenstein points recurrently toward cases of seeing aspects anew, not to maintain that mathematical objectivity is based upon intuition in anything like Kant's sense, but instead to transform Kantian ideas about how mathematics and logic structure our forms of perception and understanding.

Like Kant, Wittgenstein hoped to reorient the notion of "discovery" as it plays a role in discussions of logic, mathematics, and philosophy, critiquing the idea that the mathematician or philosopher uncovers surprising novel facts and objects. Unlike Kant, Wittgenstein replaces this with the idea that the mathematician allows us to "see the value of a mathematical train of thought in its bringing to light something that surprises us" (*RFM* I, Appendix II, §1).[1] Wittgenstein therefore points toward the importance of active arrangement of concepts and symbols, open-ended self-discovery, pleasure, and absorbed intuitive preoccupation with diagrams and symbols as ineradicable features of our philosophical, logical, and mathematical activities. And his ambition, from early on, was to replace the Kantian idea that the universality, objectivity, and necessity of mathematics is rooted in our *a priori* forms of sensation, with the idea that the role of mathematics, like that of logic and of philosophy, is to allow us to expand, rearrange, and interpret our expressive and representational powers.

There is nothing objectionable *per se*, on Wittgenstein's view, with relying on pictures, diagrams, and other visible symbolic and representational structures. Use of these properly informs our uses of language in the everyday – including, obviously and ineradicably, what we say and do in mathematics and logic. If a child or a teacher cannot produce, recognize, and become captivated by polygons and animal pictures that are "open to view" (not hidden or ambiguous, not so messy as to be hard to recognize, and so on), then there are questions about whether, and in what sense, she will be able to command concepts in the ways we do. If a person lacks the ability to take pleasure in the game of spotting pictures hidden in puzzle-pictures, or in playing the game I Spy (in which shared words are sought for what we together can see) then she may not be able to go on in language in ways our culture demands. Nothing Wittgenstein writes is intended to contradict or express skepticism about *this*.[2]

[1] For an excellent discussion of Wittgenstein's Appendix on the surprising in mathematics in *RFM*, see Felix Mühlholzer, "Wittgenstein and Surprises in Mathematics," in *Wittgenstein und die Zukunft der Philosophie: Eine Neubewertung nach 50 Jahren, Proceedings of the 24th International Wittgenstein Symposium* (Vienna: öbv & hpt, Verlag, 2002), 306–15.

[2] Compare Gordon Baker, *Wittgenstein's Method: Neglected Aspects*, ed. Katherine Morris (Oxford: Blackwell, 2004).

My view is that, in grammaticalizing our talk of the intuitive, Wittgenstein was proposing even in his early philosophy a new kind of criticism directed at our paths of interest when we discuss mathematics in philosophy. He was proceeding on the assumption that there are ways of speaking about experiences, interests, and activities within logic and mathematics – about what we *see* and *do* in them – that are part of the ordinary and everyday and that, therefore, may be repudiated, misapplied, and misunderstood when we philosophize.

His recurrent interest in aspect-perception in logic and mathematics is thus not metaphysical phenomenology, in which the existence of a certain kind of intentional object is at stake,[3] nor is it an uncritical obsession with freezing language as it is. He aims to do justice, instead, to ordinary experiences in mathematics and logic. This amounts to a philosophical alternative, both to formalism (which attaches no significance to the ways in which the patterns in symbol systems shape the ways we see extra-mathematical situations or ourselves), and to Frege's and Russell's ways of arguing against formalism (*WWK* 150ff.). Wittgenstein's focus on our immediate, unvarnished responses to proofs, equations, and diagrams – on our puzzlement, surprise, frustration, and pleasure – was designed to work through this expressive "raw material" in face of the idea that it might be made wholly irrelevant to debates about meaning in mathematics.[4] It is the idea that what we say about immediate mathematical experience *has* no cognitive or grammatical significance, but *merely* sensory or

[3] In what sense there are phenomenological strands in Wittgenstein's thinking is much debated. A useful range of essays includes Jean-Philippe Narboux, *Dimensions et Paradigms, Wittgenstein et le problème de l'abstraction* (Paris: Vrin Mathesis, forthcoming); Mathieu Marion, "Phenomenological Language: Thoughts and Operations in the *Tractatus*" and Jaakko Hintikka, "Reply to Marion," both in Randall E. Auxier and Lewis Edwin Hahn, eds., *The Philosophy of Jaakko Hintikka*, Library of Living Philosophers, vol. 30 (Chicago and La Salle: Open Court, 2006), 413–36; Wolfgang Kienzler, *Wittgensteins wende zu seiner Spätphilosophie 1930–1932* (Frankfurth am Main: Surhkamp Verlag, 1997); and Dagfinn Føllesdaal, "Ultimate Justification in Husserl and Wittgenstein," in Maria E. Reicher and Johann C. Marek, eds., *Experience and Analysis* (Vienna: öbv & hpt, 2005), 127–42.

[4] On what the mathematician throws off as "raw material" for philosophy, see *PI* §254; as we see from MS 124, page 35, the original version of this remark was directed at G. H. Hardy's *Apology of a Mathematician*, which I look at briefly below (see *CM*). Compare also Steven Gerrard's analogy between the quotation from Augustine at *PI* §1 and Hardy's remarks about mathematics in "Wittgenstein's Philosophies of Mathematics," *Synthese* 87, no. 1: 1991, 125–42.

aesthetic or psychological significance, that forms the target of much of his best writing on aspect-perception and mathematics. Aspect-perception is a way he has of calling attention to what interests us, to our voicing of what we take to be important (*RFM* III, §47).[5]

It remains a question whether, in the end, Wittgenstein's talk of what we see *in* a notation or proof or system of equations was intended by him to be merely transitional talk, prose to be worked through and replaced, ideally, by other, less perceptual sounding poetry. My sense, in attempting to make sense of the intersection of his remarks on mathematics and on aspect-perception, is that the answer is, "No".

2.

The first mention in Wittgenstein's writings of puzzle-pictures and the seeing of situations occurs in his wartime notebooks (8–9 November 1914):

> What can be confirmed by experiment, in propositions about probability, cannot possibly be mathematics.
>
> Probability propositions are abstracts [or "extracts," *Auszüge*] of scientific laws.
>
> They are generalizations and express an incomplete knowledge of those laws.
>
> If, e.g., I take black and white balls out of an urn I cannot say before taking one out whether I shall get a white or a black ball, since I am not well enough acquainted with the natural laws for that, but *all the same I do know* that if there are equally many black and white balls there, the numbers of black balls that are drawn will approach the number of white ones if the drawing is continued; I do know the natural laws as accurately as *this.*
>
> Now what I know in probability statements are certain general properties of ungeneralized propositions of natural science, such as, e.g., their symmetry in certain respects, and asymmetry in others, etc.
>
> Puzzle pictures and the seeing of situations. (*NB* 27–28)

The passage meditates on the distinction between probability as a purely logical notion unfolded in a system of calculations,

[5] On the issue of value and importance being central to aspect-perception I am indebted to Judith Genova's *Wittgenstein: A Way of Seeing* (New York: Routledge, 1995), to Stephen Mulhall's *On Being in the World: Wittgenstein and Heidegger on Seeing Aspects* (New York: Routledge, 1990), and to the other essays in this volume.

and probability applied to situations of everyday life. The technical details of Wittgenstein's effort to reduce probability to logical terms need not detain us here; they are by contemporary standards unsatisfactory because of their tie to his truth-functional conception of the logic of propositions.[6] What is of interest here instead is that by 1914 he already regarded the distinction between calculation and experiment – a distinction invoked and explored in every period of his writing about mathematics – not as a distinction between two disjoint domains of fact or necessity (e.g., *a priori* and *a posteriori*), but instead on analogy with parts of a puzzle-picture, a picture in which the name of the game is to spot one or more pictures or scenes in another.

Wittgenstein imagines a complete list or representation of all the balls in the urn, black and white, perhaps ordered by their names (lexicographically) – a numbering or a list of elementary propositions representing possible events of drawing from the urn (imagine, analogously, a representation of all the pairs of faces of two different die, presented in a sequence of pictures, diagramming possibilities for throwing them once). In any such (finite)[7] presentation of the total space of possible draws, we can see, "internally" to the presentation given by the list, that there are as many black balls as white

[6] For discussions that go into further detail, see G. H. von Wright, "Wittgenstein on Probability," in his *Wittgenstein* (Minneapolis: University of Minnesota Press, 1983); Brian McGuinness, "Probability," chapter 18 of his *Approaches to Wittgenstein: Collected Papers* (New York: Routledge, 2002), and M. C. Galavotti, *Philosophical Introduction to Probability* (Stanford: CSLI, 2005), chapter 6.5. I hope what I say in this essay might slightly assuage the sense of inarticulateness and unclarity Galavotti finds in Wittgenstein's discussion of the relation between *a priori* and *a posteriori* elements in probability, but there are nonetheless difficulties that Wittgenstein never did come to terms with (probability is practically not mentioned in *RFM*). Wittgenstein discussed probability frequently through about 1934; one may look at *PR* 289ff., *WWK* 93ff., and *BT* 98ff. (§33) for some relevant passages.

[7] Wittgenstein's remarks on aspect-perception have seemed to most (including himself) to fit the finite case most naturally. The issues surrounding the wider question whether he was committed, in principle, to a denial of the infinite are complex and I cannot go into them here; see, however, my "Wittgenstein on Philosophy of Logic and Mathematics" in Stewart Shapiro, ed., *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford: Oxford University Press, 2005), 75–128 for an accessible and brief survey of the theme, as well as my "Critical Study of Mathieu Marion, *Wittgenstein, Finitism, and the Philosophy of Mathematics*," *Philosophia Mathematica* 10, no. 1 (2002): 67–88.

ones. Situations are available for probabilistic modeling – that is, are subject to being seen in terms of the generalized calculus of probabilities – insofar as we compare and discern these internal ("logical," "grammatical") features of their arrangement (its "symmetries" and "asymmetries"). Finding these is, Wittgenstein suggests, like finding (how to see) a rabbit or a ship hidden within a larger picture or perceived scene. We can, for example, count, order, and rank the probabilities associated with choosing ten black balls in a row (or rolling two sixes with the die six times out of ten) by looking at the list and counting elements of the representation. Here we apply arithmetic *intramathematically*, within our application of probability to the "extract" of propositions.

It is important that we can draw out the "internal" features of the list (can count and calculate specific probabilities of cases) without actually drawing any balls or throwing the die. In fact, as Wittgenstein remarks, no matter how many draws from the urn we might try out empirically, these could never provide empirical confirmation of our probability calculations ("by experiment"). Probability is applicable to actual draws from the urn, but only insofar as we suppose that no intervening empirical biases or unknown factors, no interruptions of what we now take to be the physical and psychological laws governing draws from the urn, are relevant – insofar, that is, as we view our representation of the possibilities as an "extract" of a form of description that we apply to reality, *as if* it gives us a complete and correct description of our world.

So much must be counted as an ordinary understanding of probability. When I draw from the urn in any particular case, knowledge of the probabilities does not answer the question, "Is *this* draw going to produce a white or black ball?" Knowledge of probability does not have the function of speaking to this, as Peirce noted long ago.[8] Probability does not apply to any particular case individually, but only to a case conceived of within a represented system or domain of alternative (contrasting) possibilities, however they may be conceived and interpreted theoretically (in terms of draws in the long run, conditional probabilities, truth-functions, and so on). As

[8] Charles Sanders Peirce, "On the Doctrine of Chances, With Later Reflections," in *Philosophical Writings of Peirce*, ed. Justus Buchler (New York: Dover, 1955), 157–73.

the *Tractatus* says, "In itself, a proposition is neither probable nor improbable" (*TLP* 5.153).

Like Peirce before him, Wittgenstein is emphasizing that there are no unconditional facts or objects of probability (cf. *TLP* 5.1511: "There is no special object peculiar to probability propositions"). Wittgenstein's image of seeing probability in an extract of propositions relieves us of the need to say that probabilities are really out there in nature, apart from us, or even that they are merely "conventionally" or "imaginatively" applied to situations.[9] Probabilities are neither facts nor fictions. Empirical applications of the calculus of probability are instead parasitic on our taking ourselves to have represented a world of events accurately and completely enough, and on our ability to draw out (write down, symbolize, arrange) our representations concisely enough, that we can spot symmetries and asymmetries within *them* (and apply mathematics, in turn, to them and with them). To apply probability we are obliged to regard our powers of representation as successful but conditional in application, as open to further articulation and arrangement. Again: In answer to a question about what will happen on a particular draw from the urn on a particular occasion, probability says, "Don't ask me."[10]

None of this implies that I might not be overwhelmingly and vitally interested in the question of whether on the first pick I should get a black ball or a white one. (Peirce imagined a kidnapped man whose life would turn on the pick of one card, red or black). But if I am interested in *this* (and no other outcome) then the relevance of probability disappears from view. Under the aspect of the calculus of probability (as I might put it), whatever the particular outcome

9 As Brian McGuinness writes, "there is no *tendency* in things, no tendency, say, to fall out a certain way in 57.8 percent of all cases on average" ("Probability," *Approaches to Wittgenstein*, 210). Avner Baz stresses that seeing aspects is not the same as perceiving, that it need involve no representation, that is, does not concern "external world talk" and involves "a step beyond grammar" ("What's the Point of Seeing Aspects?" *Philosophical Investigations* 23, no. 2 [2000]: 97–121; cf. 120). On the importance of human gesture and body, alongside mental or intellectual construction, to aspect-perception, see Victor J. Krebs, ''The Subtle Body of Language and the Lost Sense of Philosophy," also in *Philosophical Investigations* 23, no. 2 (2000): 147–155.

10 See Charles Sanders Peirce, "Philosophy and the Conduct of Life," in *Reasoning and the Logic of Things*, ed. Kenneth Laine Ketner (Cambridge: Harvard University Press, 1992), 111.

of a particular draw from the urn, as long as I take my action to fall within the domain of possibilities represented I cannot really *be* surprised. Thus, picking from the urn of 100 balls, half white, half black, I might be very surprised to pull out 10 white balls in a row, having not expected such a turn of events. Reflecting on what I know of probabilities, that surprise might vanish in the thought that I should *not*, perhaps, have been surprised. But suppose I draw a *red* ball. This outcome would surprise me; at this point the system of situations connected with the urn would change. (I might look in the urn, alter my description, or look for the culprit who put in the red ball when I was not looking.) And I would be something other than surprised or interested – I would be stupefied or astonished or amazed – if the 100 balls suddenly disappeared for no apparent reason (compare *PI* §§80, 141–42; *OC* §§133–34).

Seeing an action in terms of a space of possibility, like seeing aspects generally, does not imply or require, in and of itself, that there is another way of seeing or regarding it.[11] But mathematics and philosophy shape, and are shaped by, our contingently given ways of looking at and experiencing events, actions and experiences. They involve *forms* or *arrangements* of facts, ways of intuiting (*Anschauungsarte*) the particular (*RFM* III, §12). As Wittgenstein often emphasizes in his later writings, mathematics shows us that the limits of empiricism lie "not in [*a priori*] assumptions guaranteed, but in the ways in which we make comparisons and in which we act" (*RFM* VII, §21).

The making of comparisons, the game of seeing one thing as like or in another, can be refused. *Can* I be surprised that in a draw of 25 balls from the box of 100 that are half white, half black, I get 13 white and 12 black ones? What if I am? What if I say, "But isn't it amazing, this confirmation in the empirical sphere of my draws of that *a priori* law"? Wittgenstein's remarks in his notebook are directed against the notion that a direct answer to this question will help one make sense of probability. To think so is to subscribe to a fantasy that my experience and the totality of relevant necessities can meet one another on unconditionally given ground, apart from my acceptance of a particular form of representation. The notions of possibility and necessity

[11] Compare Baz, "What's the Point of Seeing Aspects?" 114, note 17.

are *contrastive*, finding their place within multi-dimensional modes of representation. Probability can change the aspect under which we regard a particular experience only because we already regard that experience as an element *in* a structured order of possible experiences, that is to say, as a form, something displaying possible structure (in the early Wittgenstein's way of phrasing it).

What if I nevertheless insisted that I *did* find it, the very fact of the thisness of *this* draw from the urn – this draw itself, and not its consequences for my life – intrinsically surprising? This should be compared with Wittgenstein's remark to Engelmann about being in a state in which one *cannot get over* the existence of a fact.[12] What I may need here is a new way of looking at things, a way that allows me to change the quality of my attachment to this particular fact, word, or description, to see that what may be as interesting or important as the way things have gone here, now, is their place in a wider train of thought or experience or action to which I might also become attached. Such new articulations allow me to appreciate something anew, something I missed in my original way of seeing things, just as I might miss a figure hidden in a puzzle-picture. (This is not to deny that the original way of looking at things is just as much there, perhaps available still for my focus.)

It follows that our ways of regarding situations *as* probable, improbable, possible, or necessary are not themselves perceptions or single facts, but instead reflect our acceptance and experience of domains of possibilities that we have ourselves articulated and understood. We see one system of experience *in* another when we employ the calculus of probability. "Seeing-in" implies that there is nothing intrinsically necessary that requires us to apply a concept to a particular situation, and that we therefore bear some responsibility for the application of a structure (here, a mathematical one) to the interpretation of experience. We may have good psychological, emotional, ethical, or philosophical reasons for feeling that probability's way of seeing a situation does not speak to our lives or interests at all. The grammar of what to say then is, so far as the calculus of probability goes, open. This is where philosophy may step in.

[12] See Paul Engelmann, *Letters from Ludwig Wittgenstein: With a Memoir* (Oxford: Blackwell, 1967), 33–34 (Wittgenstein to Engelmann, 21 June 1920).

3.

The probability example shows one way in which Wittgenstein concerned himself with the human drive toward the symbolical, including in it the drive, familiar enough in mathematics, logic, and philosophy, *not* to engage only in descriptions of particular facts (assertions true and false), but instead to seek and find perspectives from which the specific content of what is true or false can take a back seat to our absorption in aspects we can draw from (find or see in) a scheme of interpretation or arrangement. I am stressing here that such finding – which may involve the discovery of emptiness or irrelevance, the vanishing of our interest (in, e.g., the calculus of probability, or in the outcome of *this* pick) – is itself constructive, sometimes pleasurable, and characteristic of certain kinds of significance we find and create in our lives.

Shifting attachment to, and surrender of, a particular kind of situation or word are involved in the aspect examples Wittgenstein explores in Part II, Section 11 of the *Investigations*. Memorableness, vividness, and ease of perception characterize these examples in comparison with the more detailed investigations of proof pictures, notations, diagrams, and formulae in his writings on mathematics, those writings that he set aside from the last two major drafts of the *Investigations*.[13] But it was the open-ended, familiar experiences of multi-dimensional interweaving in the examples from mathematics and logic, pure and applied, that brought Wittgenstein himself face to face with the idea that these experiences are part and parcel of our open-ended experience of, in, and with language.

The double cross (*PI* 207c) is an example for which no particular concept seems necessary to invoke the experience of change in the figure. Tracing the figure may help. Seeing it in terms of perspective, foreground versus background, may help or hinder seeing its doubleness, for two crosses may be experienced as something *in* the one figure. The duck-rabbit figure (*PI* 194b), also exhibiting a bivalent contrast in what we can see in it, calls forth, by contrast, a pair

[13] On the composition of *PI*, see G.H. von Wright, "On the Origin and Composition of *Philosophical Investigations*," *Wittgenstein*, 111–36; compare P. M. S. Hacker, Preface to *Wittgenstein: Mind and Will* (Oxford: Oxford University Press, 1996), xiv–xviii.

of mutually exclusive concepts of animals. This goes along with the fact that we cannot see both sides *in* the figure at once, one *in* the other – although, if both duck and rabbit are spotted, their appearance to the viewer may be varied at will, by desire, instantaneously, partly because it is easy to hold both concepts in mind (if not in sight) at once.

Our experience of these figures seems complete even if complex, bivalent or trivalent. In other cases of aspect-perception there is a more open-ended range of significance: What is to be discerned is not an object or fact or concept, but a world, a human being, an expression or gesture, a total field of significance. The distinction between object and world is an old one in Wittgenstein; in his wartime notebooks he contrasted seeing the stove before him as just one object among many and seeing it as an open-ended world, his life for the moment, beside which everything else is by contrast "colorless" (*NB* 83 [8 October 1916]). So we might contrast the picture-face (*PI* 194c), seen as a schematic picture of a particular emotion (happiness, sadness, surprise), with the kind of absorption involved in seeing a world of possibilities: children playing with dolls or hearing fairy tales, seeing my friend smiling down on me from the wall, and so on. More open-ended yet is a case like a figure of a triangle (*PI* 200c): Here the variety of possible contexts into which the figure may be imagined fitting is even wider (a blueprint, a paradigm for a child, a decorative motif, an illustration in a textbook, a diagram in a geometrical proof …), so that what seems to engage us is less the representation itself than the words and activities with which we surround it.

Each of these cases of aspect-perception replaces an idea of accuracy (isomorphic depiction) with an idea of interest and relevance; the contrast drawn is between the perception of a field of possibility or significance for the applications of concepts, and an application of a concept. What holds these cases together is a sense that seeing necessity or possibility requires us *not* to imagine that we have seen *all* possibilities – that, as Kant put it, the modality of judgments "contributes nothing to the content of the judgment … but concerns only the value of the copula in relation to thought in general."[14]

[14] Immanuel Kant, *Critique of Pure Reason*, trans. Norman Kemp Smith (London: Macmillan, 1929), A74/B100.

Remarks 522–24 of the *Investigations* comment on variety in picturing:

If we compare a proposition to a picture, we must think whether we are comparing it to a portrait (a historical representation) or to a genre-picture. And both comparisons have point.

When I look at a genre-picture, it "tells" me something, even though I don't believe (imagine) for a moment that the people I see in it really exist, or that there have really been people in that situation. But suppose I ask: "*What* does it tell me, then?"

I should like to say "What the picture tells me is itself." That is, its telling me something consists in its own structure, in *its* own lines and colors. (What would it mean to say "What this musical theme tells me is itself"?)

Don't regard it as obvious, but as a remarkable fact [*merkwürdiges Faktum*], that pictures and fictitious narratives give us pleasure, occupy our minds.

("Don't regard it as obvious [*als selbstverständlich*]" means: wonder over it [*Wundere dich darüber*], as you do some things which disturb you. Then what is problematic in the latter will disappear, by your accepting this fact as you do the other.).

((The transition from patent nonsense to something which is disguised nonsense.))[15]

These remarks invoke the concepts of the interesting, the pleasurable, and the remarkable in asking us to allow ourselves be struck by the complexity in our uses of pictures in everyday settings. The closing line of *PI* §524, a parenthetical reversal of the earlier §464 ("My aim is: to teach you to pass from a piece of disguised nonsense to something that is patent nonsense"), raises the prospect of a certain pattern in Wittgenstein's understanding of his philosophical methods, a back and forth movement from latent to patent nonsense. The idea that nonsense may require attractive articulation to be seen aright alludes, I take it, to his own development: The interlocutor's patently odd remark at *PI* §523 ("I should like to say 'What the picture tells me is itself'") is implicitly contrasted with the latently nonsensical remark at *TLP* 2.1: "We make to ourselves pictures of facts."[16] From the point of view of the *Investigations* this latter remark, perfectly grammatical and even jejunely true, was misplaced, forced into

[15] I have slightly altered Anscombe's translation of this remark, deleting her use of the verb "surprising." My reasons for doing so will become clearer in what follows.

[16] Trans. C. K. Ogden.

a setting in which it occurred wrongly punctuated, inviting the nonsensical idea that all our ways of picturing rest, ultimately, upon our ability to represent facts truly or falsely.[17] But illustrations of poems and fairy tales, diagrams in textbooks of engineering and mathematics, also form "a complicated amalgam of … words and pictures" (*BT* 69 (§22)) no less important to us. There was an overarching confusion at work in the *Tractatus* between the isomorphic character of portraits of states of affairs and the more complexly saturated modeling involved in diagrams and illustrations.[18] We have seen from scrutiny of his notebook remark on probability that even the early Wittgenstein struggled to emphasize much beyond fact-depiction in his discussions of mathematics, untangling such confusions. But as the probability example also shows, his early view was explicated in terms of a vision of "ungeneralized" elementary sentences, a collection of pictured facts treated as ultimate.

The *Investigations*' repeated revisiting of the fictional and the imaginative questions this idea of primary versus secondary forms of sense (cf. *PI* §232, II.xi), asking us to release philosophy from its ancient task of ranking the literal depiction of objects and truths above the fancies of poetry and fiction. It is striking that in *PI* §525 Wittgenstein explicitly ties the notion of wonder, philosophy's classic prompt since the time of Plato and Aristotle, to the idea of surrendering a disturbance or disquieted puzzlement about the possibility of picturing. In raising the prospect of such a "disappearance" of disquiet, aside from voicing his ambition to speak of philosophy in its ancient sense, Wittgenstein alludes to *PI* §133, in which it is remarked (evidently paradoxically, given the unending stream of questions with which the reader of the *Investigations* is confronted) that "the philosophical problems should *completely* disappear," and that "the real discovery is the one that makes me capable of stopping doing philosophy when I want to.—The one that gives philosophy peace, so that it is no longer tormented by questions which bring *itself* in question." While some have rejected this remark as trivializing of philosophy, as too "quietist," it should be noted that Wittgenstein is

of these remarks

[17] Here I concur with Hacker's discussion ~~in~~ *Wittgenstein: Mind and Will.*

[18] Jean-Philippe Narboux, "Diagramme, Dimensions et Synopsis," *Théorie, Littérature, Enseignement* 22 (Saint-Denis: Presses Universitaires de Vincennes, 2005): 115–41.

here suggesting how philosophy might be *defended*, protected from exposure to questions about its ultimate worth. The idea that philosophy's origins lie in our capacity for wonder over phenomena that surprise or puzzle or frighten us, that its end lies in a certain tranquility and satisfaction uniquely its own, and that it is an art of ordering the imagination, proceeding from surprise to wonder to admiration by drawing unforeseen connections (especially among phenomena that appear at first to be familiar and uninteresting), is a quite traditional one, familiar from Plato through Nietzsche.[19] Adam Smith, in his "History of Astronomy," explicitly articulates such a view:

> When something quite new and singular is presented, we feel ourselves incapable of [referring to some known species or class of things]. … Imagination and memory exert themselves to no purpose, and in vain look around all their classes of ideas in order to find one under which it may be arranged. … It is this fluctuation and vain recollection, together with the emotion or movement of the spirits that they excite, which constitute the sentiment properly called *Wonder*. … What sort of a thing can that be? What is that like? are the questions which, upon such an occasion, we are all naturally disposed to ask. … Upon the clear discovery of a connecting chain of intermediate events [Wonder] vanishes altogether. …
>
> Philosophy is the science of the connecting principles of nature … [and] endeavours to introduce order into this chaos of jarring and discordant appearances, to allay this tumult of the imagination, and to restore it, when it surveys the great revolutions of the universe, to that tone of tranquility and composure, which is both most agreeable in itself, and most suitable to its nature. Philosophy, therefore, may be regarded as one of those arts which address themselves to the imagination. …[20]

What is new in Wittgenstein is less his vision of the aims and purposes of philosophy than a post-Kantian, post-Fregean, post-Russellian preoccupation with releasing the imagination from its domination, in the empiricist tradition, by too unimaginative a conception of how words and ideas associate with one another.

[19] For Wittgenstein and Nietzsche, see Gordon C. F. Bearn, *Waking to Wonder: Wittgenstein's Existential Investigations* (Albany, N.Y.: State University of New York Press, 1997).

[20] Adam Smith, "The History of Astronomy," *Essays Philosophical and Literary*, (London: Ward, Lock & Co., 1880), 330–31, 333–34, 336.

New line here

4.

In the (ordinary) sentence "There are three plums on the table," the *Tractatus* asks us to see the word "three" not as just another name, concept, or adjectival word, but instead as a representational aspect, a space of form located within a larger picture (here, the sentence expressing a proposition, true or false). This may be seen in the idiosyncratic rewriting of the sentence that the *Tractatus* proposes, in which a separable grammatical term for the number three vanishes:

$(\exists x,y,z)(Px \,\&\, Py \,\&\, Pz)$.[21] What Frege or Russell would have treated as an identity assertion about the concept "plum" (*viz.*, that the number of objects falling under it is identical with the number three), Wittgenstein asks us to see *in* our depiction of the situation, like a part of a puzzle-picture. Three is part of (the grammar of) how we view this sentence as a picture of reality, an "internal" feature of our thought to be drawn out; we can count the variables to see this. This does not mean that Wittgenstein held a substitutional view of quantification on which numerals, as opposed to objects, are what we quantify over when we do mathematics; he is not siding with the formalists. Instead, the vanishing of the term "three" from his rewriting of the sentence asks us to see number words as an aspect of (a way of using) a symbolism. Such aspects are themselves subject to further, different forms of articulation. The *Tractatus* goes on to concoct specific terms for each number, placing these within an open-ended series of forms: 0, 1+1, 1+1+1, 1+1+1+1 … (*TLP* 6.02). This is not a "meta" form, standing outside the standpoint of the original space; instead, the point is to see the number three *in* it, and so to do something new with "3."

Strange though it may sound to say so, Wittgenstein's conception offered him a way of recovering (rearranging, making sense of) some of the everyday ways of speaking about mathematics with which he

[21] One must interpret the variables "x," "y," and "z" here as referring to distinct objects, and not in the way we are used to from ordinary first-order logic; Wittgenstein assumes that identity has been eliminated by the device of taking different names to refer to different objects. For more detail, see my "Number and Ascriptions of Number in Wittgenstein's *Tractatus*," in Juliet Floyd and Sanford Shieh, eds., *Future Pasts: Perspectives on the Analytic Tradition in Twentieth Century Philosophy* (Oxford: Oxford University Press, 2001), 145–92.

was familiar from his days as an engineer. One is the idea that logic and mathematics consist essentially of calculations; another is the idea that it is what we can discern in the signs themselves, as we calculate with them, that contains their essential interest and importance (*TLP* 6.21, 6.2331). One may find nearly the same words about mathematics in, for example, Whitehead's *An Introduction to Mathematics*, a work Wittgenstein knew:

In mathematics, granted that we are giving any serious attention to mathematical ideas, the symbolism is invariably an immense simplification. It is not only of practical use, but is of great interest. For it represents an analysis of the ideas of the subject and an almost pictorial representation of their relations to each other. If any one doubts the utility of symbols, let him write out in full, without any symbol whatever, the whole meaning of the following equations which represent some of the fundamental laws of algebra:

(1) $x + y = y + x$
(2) $(x + y) + z = x + (y + z)$
(3) $x \times y = y \times x$
(4) $(x \times y) \times z = x \times (y \times z)$
(5) $x \times (y + x) = (x \times y) + (x \times z)$. …

By the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain.

It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them.

One very important property for symbolism to possess is that it should be concise, so as to be visible at one glance of the eye and to be rapidly written. Now we cannot place symbols more concisely together than by placing them in immediate juxtaposition. In a good symbolism, therefore, the juxtaposition of important symbols should have an important meaning. This is one of the merits of the Arabic notation for numbers.[22]

Compare with this what Wittgenstein writes about equations in the *Tractatus*:

[22] Alfred North Whitehead, *An Introduction to Mathematics* (New York: H. Holt, 1911), 60. My "Number and Ascriptions of Number in Wittgenstein's *Tractatus*" discusses in more detail the issue of Wittgenstein, formalism, and Whitehead.

> If two expressions are combined by means of the sign of equality, that means that they can be substituted for one another. But it must be manifest in the two expressions themselves whether this is the case or not. …
>
> Frege says that the two expressions have the same meaning but different senses.

lässt

> But the essential point about an equation is that it is not necessary in order to show that the two expressions connected by the sign of equality have the same meaning, since this can be seen [learned, *ersehen ~~laesst~~*] from the two expressions themselves.
>
> And the possibility of proving the propositions of mathematics means simply that their correctness is seen [understood, *einzusehen ist*] without our having to compare what they express with the facts as regards correctness. …
>
> An equation only marks the point of view from which I consider the two expressions: it marks from the point of view of their equivalence in meaning. (*TLP* 6.23–6.2323)[23]

For Frege, an equation is a logical identity, true or false, in which sameness of reference (*Bedeutung*) is said to hold between two names. The informativeness of an equation resides, Frege also holds, in the contrasting, complex senses (*Sinne*) of the numerical terms involved, not in their references. The numerals are proper names insofar as Frege was wont to emphasize that each number is, in and of itself, unique, having its own identity and unique set of properties. In the *Tractatus* and ever after, Wittgenstein explicitly rejects Frege's conception of equations as logical identities; with this he rejects Frege's sense/reference distinction and his account of what it is to understand the specific content of a mathematical truth. As in the probability case, Wittgenstein assumes that what is of interest *within* arithmetic and algebra is not that we reach this *particular* number or form at the end of a calculation, but instead *how* we reach it and what we do *having* reached it, , through which comparisons, substitutions, and arithmetical calculations we go on to see and apply it. Truth, at least if it is conceived of in terms of a "comparison with reality," has no primary role in the *doing* of mathematics or logic, on Wittgenstein's view. Instead, "a number is what it does";[24] its primary significance and interest for

[23] I've altered slightly the Pears and McGuinness translation in the last two paragraphs (6.2321–6.2323), inspired in part by Ogden's.

[24] Timothy Gowers, *Mathematics: A Very Short Introduction* (Oxford: Oxford University Press, 2002), 18.

us lie in the characteristic "internal" features of the expressions we draw out, uncovering the multiple standpoints from which we are able to view, arrange, see, and manipulate terms within the process of calculating. Here – as in logic and philosophy – punctuation, syncopation, arrangement of notation, simile, and emphasis in expression are everything: "Process and result are equivalent" (*TLP* 6.1261; cf. *NB* 42 [24 April 1915], *RFM* I, §§80–84). Logical and mathematical operation signs are "punctuations" (cf. *TLP* 5.4611), that is, what they articulate are not special objects, but ~~necessities, perspectives~~ ways of arranging and communicating
~~on~~ what we feel the need to write and say. These expressions include the parentheses, brackets, circlings, underlinings, index-marks, and other surrounding signs that we use in logic and mathematics to help us to see. These expressions of emphasis and alteration of emphasis are not, as Frege might have held, inessential because they express the origin of our thoughts (cf. *BT* 277); instead, they show us what thinking in such cases *is*.

Though Wittgenstein's thought evolved, these ideas never left him; instead they were rearranged, repunctuated, seen anew, in hundreds of examples that he explored, some from fairly advanced mathematics, some from the most ~~basic~~ elementary mathematics. We can see the very same lines of thought addressed in the opening remark of *Remarks on the Foundations of Mathematics*. Suppose we ask, as Wittgenstein does: Are there two variables at work in

$$y = (x^2 + z)^2 - z(2x^2 + z)?$$

Well, obviously, Yes. Just *look at* the expression. (We see "x" and "z" before our eyes.)

But we can see the question *differently* if we work the equation out (i.e., calculate with it). Rewriting (expanding the notation) we get first

$$(x^2 + z)(x^2 + z) - z(2x^2 + z)$$

And then

$$x^4 + zx^2 + zx^2 + z^2 - 2zx^2 - z^2.$$

By adding and canceling, we see that this is equivalent to x^4. We see now also how the ~~larger~~ algebraic formula e {formulae} ~~instantiates a single variable~~ express the concept of x raised to a fourth power. In compacting our signs, we enlarge (can make more general) our point of view. We also see that the answer to the original question really is now (was?), "No". For there are not two variables essentially at work in the original equation.

This may be a surprise for those who have not yet seen the variable "z" vanish. But the vanishing – and hence the *mathematical* surprise – evinces itself only in the course of working out the equation. Once it is worked out, we *see* both equations *in a new way*. The seeing or coming to see in a new way is, in fact, both process and result of the process: the activity of puzzling through to a solution is both the problem and the solution. For without this working out, this *coming to see*, the mathematical interest of the problem, its mathematical content or point, cannot be seen – just as the interest of a puzzle-picture cannot be seen by one unable or unwilling to (try to) see several pictures in it. But with the working out, the interest vanishes from *inside* the problem, shifting it, so to speak, to its outside (*RFM* I, Appendix II, §2), , to the fact that it can be a puzzle at all, though not for the one who knows how to solve it.

It is the same within logic, according to Wittgenstein, from the *Tractatus* onward. Consider a sentence structure of the following form:

$$(p \mathbin{\&} \neg r) \vee (p \mathbin{\&} q \supset r)$$

This looks like a structure that would express a sentence with sense, a picture of reality, assuming that the elementary components of the sentence themselves have sense. But if we rewrite it in the form of a truth-table, as the *Tractatus* says we can, we see it anew. For in this diagram we can see the tautologousness of the original sentence form *in* the final column, which contains only T's; the sentence's apparent sense, its ruling in and ruling out of states of affairs, vanishes. It may surprise us to see, when written out this way, that the sentence yields a tautology. But once we see the sentence written *in* this diagram, we change our way of viewing what we might have regarded as a sentence representing a state of affairs, true or false. We also see *in* the senseless sentence the general form of tautology, a so-called "proposition" of logic.

Now in the *Tractatus*'s notion of what is proper to logic alone we do meet a kind of claim to completeness or maximality with respect to our capacity to see one system in another. I have so far contrasted the idea of a "meta" stance on logic or language with that of the ability to see one system *in* another. But in the *Tractatus*, Wittgenstein claims to have found, in his general form of proposition, a diagram expressing the entire grammatical space of all propositions, a scheme expressing ~~the three~~ words beyond which everything else is just roaring and booming (*viz.*, "So it goes," "*Es verhält sich so*") (*TLP* 4.5, 6).

By way of a single logical operation, operator N, a generalized version of the Sheffer stroke of joint denial, Wittgenstein lays bare what he thinks of as a complete systematization or diagram in which we can *see* the system (the one and only system) of logical operations. But he goes even further, writing that "It is possible … to give at the outset a description of all 'true' logical propositions. Hence there can *never* be surprises in logic" (*TLP* 6.125–26.1251). Wittgenstein is claiming to have found the most concise possible symbolic way of depicting, not merely the essence of the proposition, but the essence of logic itself. All so-called logical propositions are sentences without sense – tautologies or contradictions. In them the portrayal of reality (sense) vanishes. Since on Wittgenstein's view a new proposition is nothing but the coordination of truth values to the whole on the basis of the truth values of the elementary parts, all logically equivalent sentences share the same general form of tautologousness. So in the end there is ultimately only one kind of tautology, only one form of such *Sinnlosigkeit.* It is as if we have reached the limit of our ability to see one system in another, and can see no further, though we see darkly and partially. For only a being like Leibniz's God, capable of seeing an infinite number of calculations at a glance, would be able to see every instance of the one form of tautology for what it really is. For Leibniz's God alone could there *never* be (even apparent) surprises *within* logic.

The general form of proposition is a scheme whose physiognomy is fixed, not open ended, not subject to elaboration of new aspects. Wittgenstein does not take into account, for example, the (later discussed) possibility of a multi-valued logic, or proof methods which do not rely on the law of the excluded middle. Thus in logic even more than in mathematics, there can *never* be surprises – and more than just in the sense in which "process and result are equivalent" within calculations. Wittgenstein's philosophical task, as he understood it in the *Tractatus*, was to examine his own uses of language with an eye toward seeing them *in* the general form of proposition. What he strove for was a perspective that would transcend the limitations of any particular notation or symbolism, while at the same time encompassing, , diagramming, the logical aspect of any possible notation or language. This was a struggle to design a particular notational method (the truth-tables, the a–b notation) that could operationalize – that is to say, fully formalize or mathematize – the intuitive notion of one sentence's following from another *by pure logic alone*, in such a way that the method would be complete (in applying to any and every possible notation), yet unbiased with respect to the particular notation chosen for any language.[25]

By 1929 Wittgenstein had surrendered this aspiration to completeness. Although the notation of truth-tables was all right in its place, it worked for only a fragment or one aspect of language, not the whole: one could not see *in* the general propositional form ~~*the*~~ logic of language. The *Tractatus*'s recursive specification s of the general propositional form~~,~~ and of the grammar of number words~~, was~~ were too "nebulous" (*PR* 131 [§109]). As Russell pointed out in his introduction to the *Tractatus*, the mathematics of the higher infinite had not been diagrammed, but only gestured at, in Wittgenstein's remarks on mathematics. As Ramsey emphasized, the method of truth-tables could not help with the more fine-grained needs of mathematical logicians. Ordinary statements of color, measurement, degree, and continuity could not be seen in the method of truth-table diagramming either. And the idea that the needs of natural science, perhaps of cosmology, would be decisive in determining the particular choice of notational system came to seem to Wittgenstein a cop out. It was both too much of a concession to promissory scientism, and too little engaged with the task of seeing aspects of grammar and notation in the small. ~~It also held philosophy hostage to the deliverances of the empirical as it would be understood in physics.~~

[25] For an elaboration of this reading, see my "Wittgenstein and the Inexpressible," in Alice Crary, ed., *Wittgenstein and the Moral Life: Essays in Honor of Cora Diamond* (Cambridge: MIT Press, 2007).

In this, as Wittgenstein explained to Waismann and Schlick, he had unwittingly made of philosophy something “dogmatic”:

One fault you can find with a dogmatic account is, first, that it is, as it were, arrogant. But that is not the worst thing about it. There is another mistake, which is much more dangerous and also pervades my whole book, and that is the conception that there are questions the answers to which will be found at a later date. It is held that, although a result is not known, there is a way of finding it. Thus I used to believe, for example, that it is the task of logical analysis to discover the elementary propositions. I wrote, we are unable to specify the form of elementary propositions, and that was quite correct too. It was clear to me that here at any rate there are no hypotheses and that regarding these questions we cannot proceed by assuming from the very beginning, as Carnap does, that the elementary propositions consist of two-place relations, etc. Yet I did think that the elementary propositions could be specified at a later date. Only in recent years have I broken away from that mistake. At the time I wrote in a manuscript of my book (this is not printed in the *Tractatus*), “The answers to philosophical questions must never be surprising.” In philosophy you cannot discover anything. I myself, however, had not clearly enough understood this and offended against it.

The wrong conception which I want to object to in this connection is the following, that we can hit upon something that we today cannot yet see, that we can discover something wholly new. That is a mistake. The truth of the matter is that we have already got everything, and we have got it actually present; we need not wait for anything. We make our moves in the realm of the grammar of our ordinary language, and this grammar is already there. Thus we have already got everything and need not wait for the future. (*WWK* 182–83)

Wittgenstein replaced his reliance on the idea of the independence of the elementary propositions, as well as the primacy of the truth-table notation as part of a specification of a complete general form of proposition, with an image of *Satzsysteme*, systems of propositions exhibiting grammatical variety, autonomy, and distinctive internal character or physiognomy. While he continued to emphasize the importance of the calculational aspect of mathematical activity, everywhere we see aspect-perception and the dawning of new ways of seeing systems lifting his account beyond the limits of this way of seeing logic. Like Peirce, he seems to have regarded our ability to shift our way of seeing a given diagram, projecting it into a new dimension, as a mark of what makes human mathematical reasoning distinct from anything codifiable in deductive formal logic alone, or solvable by mechanical means.[26]

In leaving behind part of his perspective, Wittgenstein did not surrender his reliance on aspect-perception; he instead increased and intensified it, precisely so as to retain the underlying idea that in philosophy there are no (deeper than aspectual) surprises (necessities, possibilities). He extended and refined his appeals to the seeing (and failing to see) of one system *in* another, applying them to a wide range of mathematical and logical examples – including the Sheffer Stroke itself (*BT* 477–78). Aspect-perception lay behind not only his idea that proofs by induction in arithmetic are schematic pictures, rather than proofs consisting of sequences of sentences with sense, but also his idea that consistency and impossibility proofs for systems are similarly a matter of embedding one system inside another, as well as his idea that because proofs of elementary sums written out in the prose of *Principia Mathematica* would require us to apply arithmetic to the formalism – counting variables to check the proofs – the claim that Russell's foundation of arithmetic provided a substantial epistemic foundation is like the claim that the painted rock is the foundation of the painted tower (again, an analogy with aspects of ~~puzzle~~ pictures; cf. *RFM* VII, §16). This allowed Wittgenstein to retain and deepen his earlier idea that in logic and mathematics there are no surprises – no discovery of facts or of possibilities construed on the model of properties or facts – but instead activities, trains of thought and arrangements of grammar that strike us.

The grammars of different "systems" can cross and so change our ways of looking at each of them. This forms a nascent but significant element in articulating what was to become a crucial theme in Wittgenstein's later philosophy: namely, his critique of the idea that human thought and language is everywhere governed by grammatical rules in the same way, his insistence that the evolution of language~~, in general,~~ and of mathematics and logic in particular, is both open-ended and *unforeseeable in general*. This makes itself felt in

[26] On Peirce's views see Judson C. Webb, "Hintikka on Aristotelean Constructions, Kantian Intuitions, and Peircean Theorems," in Auxier and Hahn, *The Philosophy of Jaakko Hintikka*, 195–301 (cf. 246ff).

the fact that Wittgenstein's discussions of figurative or "secondary" meaning, as Cavell puts it so well in *The Claim of Reason*, takes place in regions where "there is no antecedent agreement on criteria" and that "this is itself a grammatical remark."[27] Surprises are ineradicable in mathematics, in logic, and in philosophy. Part of what it is to command language is to incorporate into it, case by case, the unforeseen and the interesting. That is the beauty and the importance of looking at how to arrange it.[28]

[27] Stanley Cavell, *The Claim of Reason: Wittgenstein, Skepticism, Morality, and Tragedy* (Oxford: Oxford University Press, 1979), 355.

[28] I should like to thank Avner Baz, Robert Bowditch, Robert Briscoe, Akihiro Kanamori, Wolfgang Kienzler, Matthias Kross, Montgomery Link, Felix Mühlhölzer, Jean-Philippe Narboux, Norbert Schappacher, Peter Simons, Hartley Slater, and Anja Weiberg for helpful discussion of the ideas in this essay, as well as William Day and Victor J. Krebs for their patience and unwavering intellectual support in putting together this volume. Some of the ideas presented here received helpful responses from audiences at the Einstein Forum Potsdam, the University of Kent at Canterbury, and the University of Chicago.