Accuracy, limiters and approximate Riemann solvers

Donna Calhoun

Boise State University

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Outline

- ☑ We solved exactly the Riemann problem for the constant coefficient linear system case, e.g. linear shallow water equations
- ☑ We described what is involved in solving the Riemann problem for the nonlinear shallow water wave equations.
- ☐ How do these Riemann solvers make it into an actual code?
- ☐ Do we actually solve the non-linear problem at every grid cell interface?
- ☐ How accurate are these methods?

Riemann problem for linear systems

Solving the Riemann problem for linear problem

$$q_t + A\, q_x = 0$$

(1) Compute eigenvalues and eigenvectors of matrix A

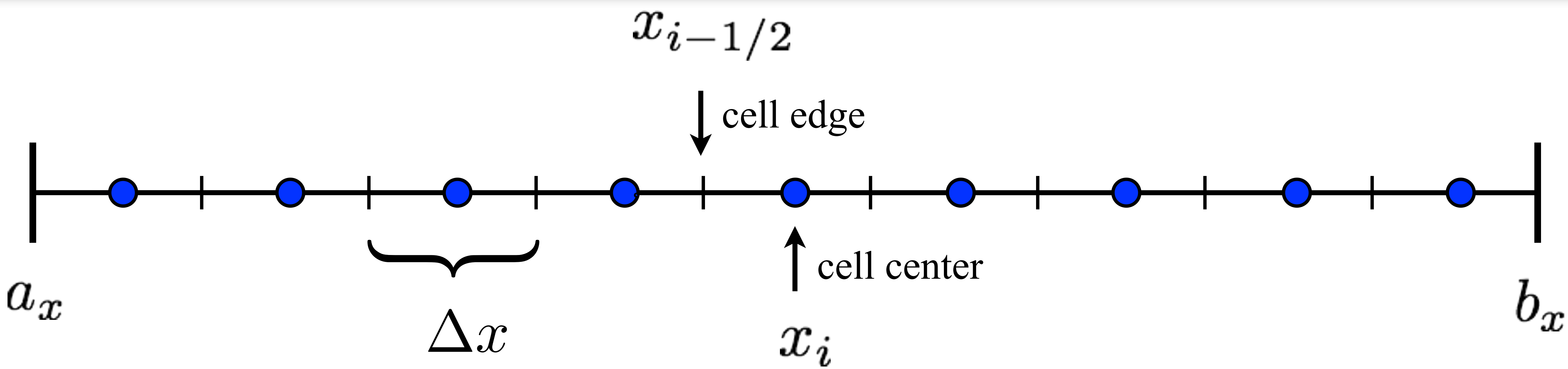
(2) Compute characteristic variables by solving

$$R\, \alpha = q_r - q_\ell$$

(3) Use eigenvalues or "speeds" to determine piecewise constant solution

$$\begin{aligned} q(x,t) &= q_\ell + \sum_{p\,:\,\lambda^p < x/t} \alpha^p\, r^p \\ &= q_r - \sum_{p\,:\,\lambda^p > x/t} \alpha^p\, r^p \end{aligned}$$

One dimensional Cartesian grid



Cell centers :

$$x_i = a_x + (i - 1/2)\Delta x, \quad i = 1, 2, \ldots, M_x$$

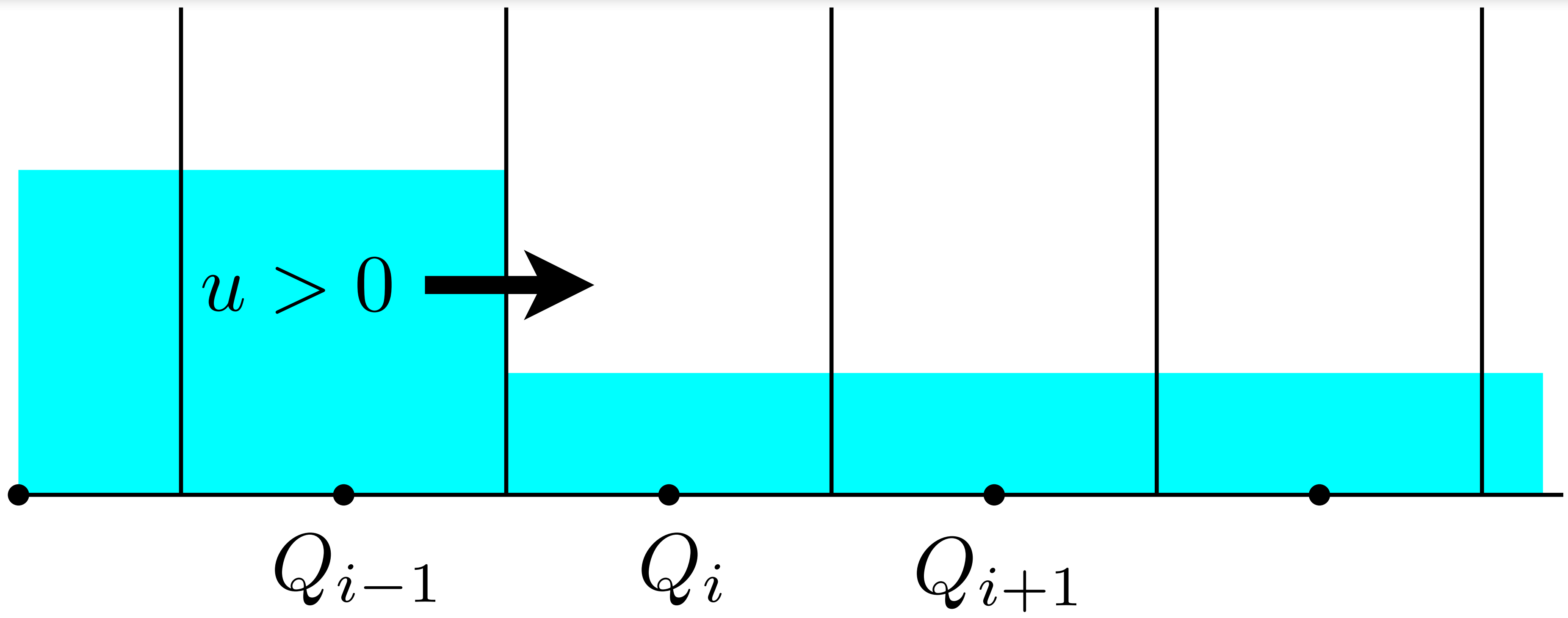
Cell edges :

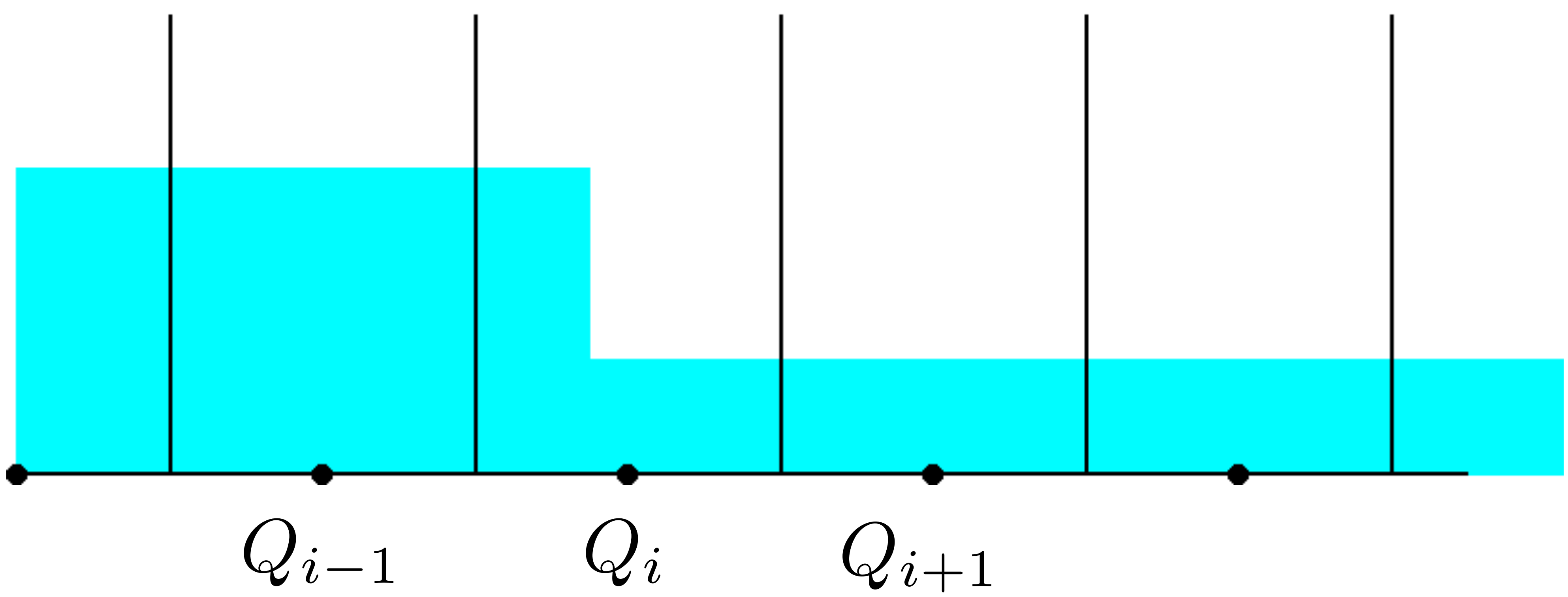
$$x_{i-1/2} = a_x + (i - 1)\Delta x, \quad i = 1, 2, \ldots, M_x + 1$$

Time step over interval $[0, T]$:

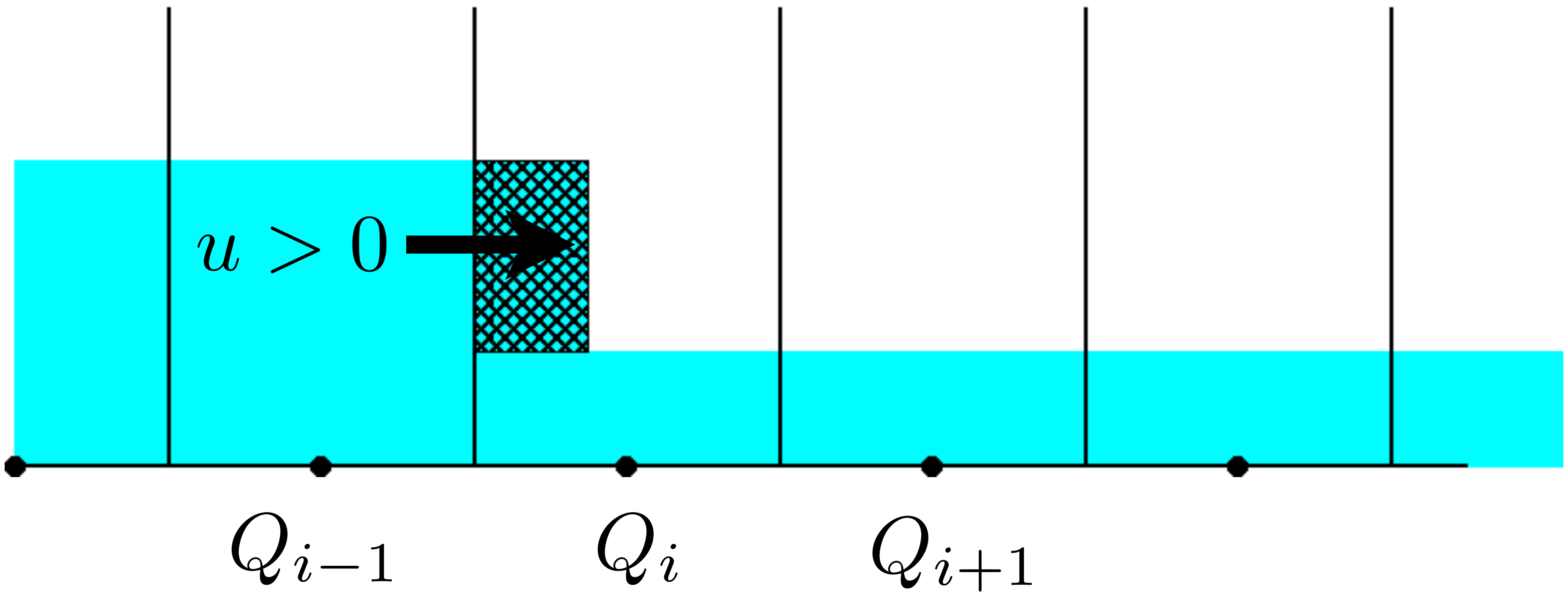
$$t_n = n\Delta t, \quad n = 1, 2, \ldots N_{out}$$

Update cell averages explicitly





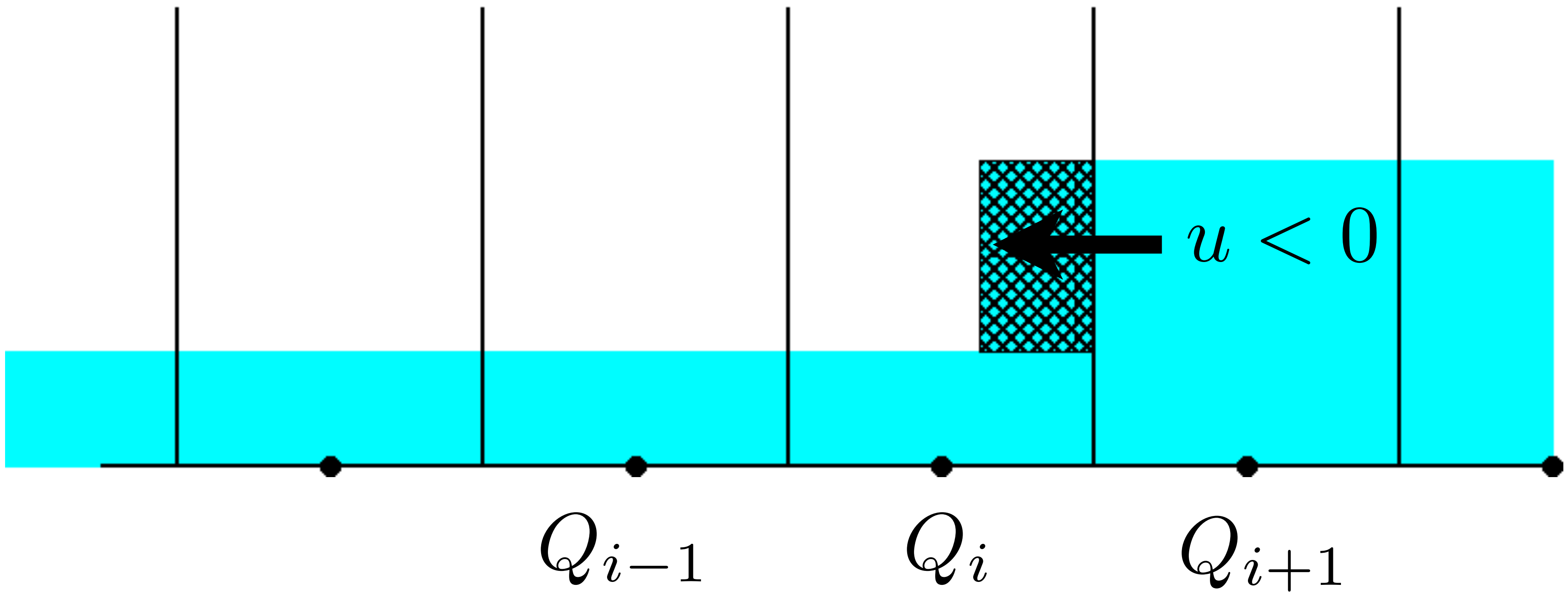
Update cell averages explicitly



In time Δt, mass in cell C_i increases by shaded area :

$$\Delta x Q_i^{n+1} = \Delta x Q_i^n - u\,\Delta t\left(Q_i^n - Q_{i-1}^n\right)$$

Update cell averages explicitly



In time Δt, mass in cell C_i increases by shaded area :

$$\Delta x Q_i^{n+1} = \Delta x Q_i^n - u\,\Delta t\left(Q_{i+1}^n - Q_i^n\right)$$

Wave propagation viewpoint - scalar equation

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u^+(Q_i^n - Q_{i-1}^n) + u^-(Q_{i+1}^n - Q_i^n)\right)$$

where

$$u^+ = \max(u, 0), \qquad u^- = \min(u, 0)$$

We can define *waves* at each interface as :

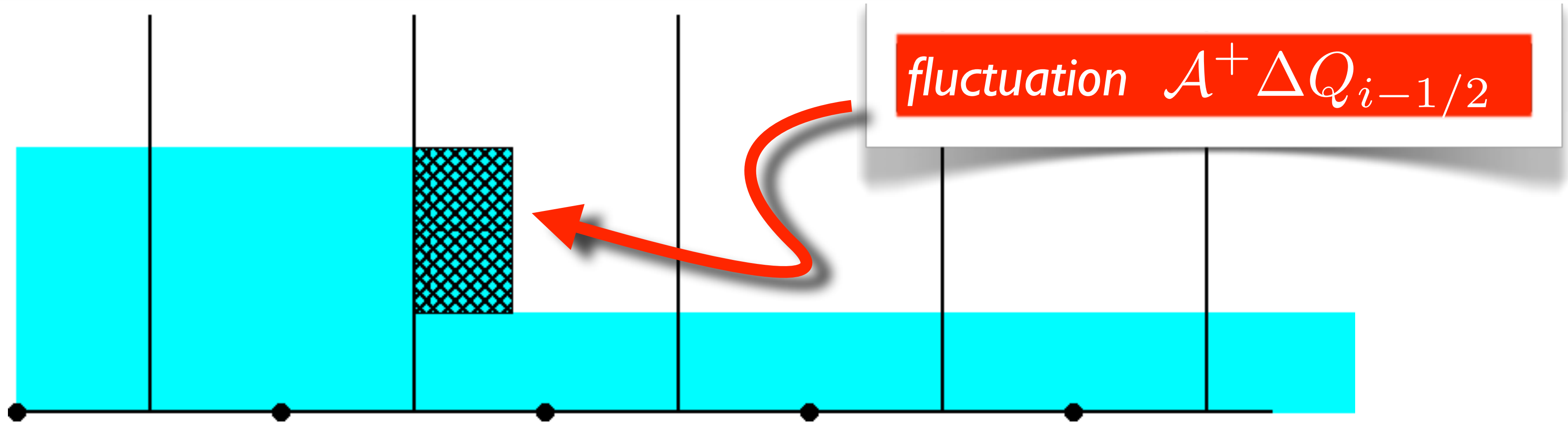
Waves : $$\mathcal{W}_{i-1/2} \equiv Q_i - Q_{i-1}$$

Our scheme might look like :

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u^+\mathcal{W}_{i-1/2} + u^-\mathcal{W}_{i+1/2}\right)$$

"wave propagation algorithm" (R. J. LeVeque)

Wave propagation viewpoint - scalar equation



$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(u^+\mathcal{W}_{i-1/2} + u^-\mathcal{W}_{i+1/2}\right)$$

We can write this in terms of *fluctuations* :

$$\mathcal{A}^+\Delta Q_{i-1/2} \equiv u^+\mathcal{W}_{i-1/2}$$
$$\mathcal{A}^-\Delta Q_{i+1/2} \equiv u^-\mathcal{W}_{i+1/2}$$

The first order term in the update used by Clawpack and GeoClaw

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

Wave propagation viewpoint - systems

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i+1/2}^p\right)$$

where the waves are now defined from an eigenvalue decomposition of the jump in value at each interface

Waves : $\quad \mathcal{W}_{i-1/2}^p \equiv \alpha^p r^p$

$$\boxed{R\alpha = q_r - q_\ell}$$

Written in terms of *fluctuations* :

$$\mathcal{A}^+\Delta Q_{i-1/2} \equiv \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2} \qquad (\lambda^p)^+ = \max(\lambda^p, 0)$$

$$\mathcal{A}^-\Delta Q_{i-1/2} \equiv \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i-1/2} \qquad (\lambda^p)^- = \min(\lambda^p, 0)$$

Clawpack - rp1ad.f

```
      sqrtgh = sqrt(grav*h_mean)
      do  i = 2-mbc,mx+mbc
         do m = 1,2
            delta(m) = qr(i,m) - qr(i-1,m)
         enddo

c        # Speeds
         s(i,1) = u_mean - sqrtgh
         s(i,2) = u_mean + sqrtgh

         a1 = (h_mean*delta(2) - sqrtgh*delta(1))/(2*grav*h_mean)
         a2 = (h_mean*delta(2) + sqrtgh*delta(1))/(2*grav*h_mean)

c        # Waves
         wave(i,1,1) = -a1*sqrtgh
         wave(i,2,1) = a1*grav
         wave(i,1,2) = a2*sqrtgh
         wave(i,2,2) = a2*grav

c        # Fluctuations
         do m = 1,meqn
            amdq(i,m) = 0
            apdq(i,m) = 0
            do mw = 1,mwaves
               amdq(i,m) = amdq(i,m) + min(s(i,mw), 0.d0) * wave(i,m,mw)
               apdq(i,m) = apdq(i,m) + max(s(i,mw), 0.d0) * wave(i,m,mw)
            enddo
         enddo
      enddo
```

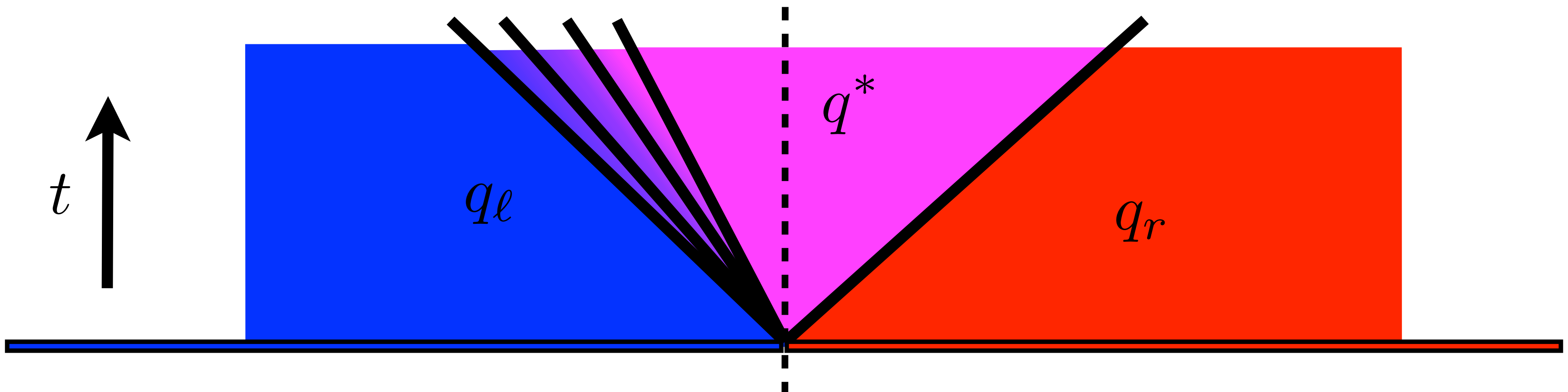
Speeds

Waves

Fluctuations

Nonlinear case

Given an exact solution, we can also construct waves, speeds and fluctuations



transonic case?

Waves :

$$\mathcal{W}^1 \equiv q^* - q_\ell$$

$$\mathcal{W}^2 \equiv q_r - q^*$$

Fluctuations :

$$\mathcal{A}^+\Delta Q_{i-1/2} \equiv f(q_r) - f(q^*)$$

$$\mathcal{A}^-\Delta Q_{i+1/2} \equiv f(q^*) - f(q_\ell)$$

Speeds : Shock speed or average speed in a rarefaction

Connection to flux formulation

Numerical fluxes can be written in terms of fluctuations :

$$F_{i-1/2} \equiv f(Q_i) - \mathcal{A}^+ \Delta Q_{i-1/2}$$

or

$$F_{i-1/2} \equiv f(Q_{i-1}) + \mathcal{A}^- \Delta Q_{i-1/2}$$

Flux differences can be expressed in terms of left going and right going fluctuations :

$$F_{i+1/2} - F_{i-1/2} = \mathcal{A}^+ \Delta Q_{i-1/2} + \mathcal{A}^- \Delta Q_{i+1/2}$$

Roe linearization for the non-linear case

$$q_t + f(q)_x = 0$$

We solve a linearized system at each cell interface, at each time step

$$q_t + f'(\hat{q})q_x = 0 \quad \longleftrightarrow \quad q_t + A(\hat{q})q_x = 0$$

for "Roe averaged" values $\widehat{q}$.

For conservation, we need $\hat{q}$ to satisfy :

$$f(q_r) - f(q_\ell) = f'(\hat{q})(q_r - q_\ell)$$

P. Roe (JCP, 1981) showed an approach for many important systems.

Roe averaged values for SWE

- Compute Roe averaged values :

$$\hat{h} = \frac{h_\ell + h_r}{2}, \qquad \hat{u} = \frac{\sqrt{h_\ell}u_\ell + \sqrt{h_r}u_r}{\sqrt{h_\ell} + \sqrt{h_r}}$$

- Evaluate eigenvalues and eigenvectors at these values.

- Compute waves, speeds and fluctuations as in the linear case. Does not require the nonlinear root-finder.

Roe averages are also available for the Euler equations and other important physical systems.

Other approximate Riemann solvers are available.

Roe solver in Clawpack

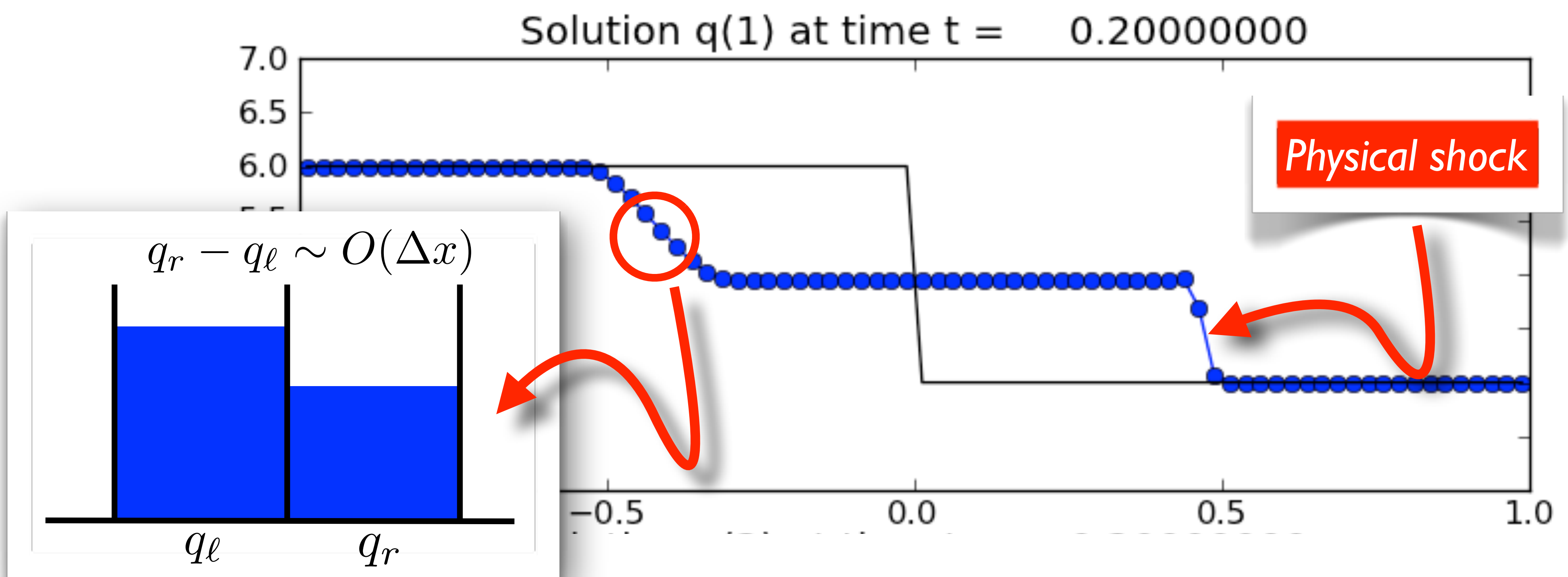
```
      do i = 2-mbc,mx+mbc
         ......................................................
c        # compute  Roe-averaged quantities:
         u_roe = (ur/sqrt(hr) + ul/sqrt(hl))/(sqrt(hl) + sqrt(hr))
         h_mean = (hl + hr)/2.d0
         sqrtgh_roe = sqrt(grav*h_mean)

c        # wave speeds
         s(i,1) = u_roe - sqrtgh_roe
         s(i,2) = u_roe + sqrtgh_roe

c        # compute coeffs in the evector expansion of delta(1),delta(2)
         a1 =(-delta(2) + (u_roe + sqrtgh_roe)*delta(1))/(2*sqrtgh_roe)
         a2 = (delta(2) - (u_roe - sqrtgh_roe)*delta(1))/(2*sqrtgh_roe)

c        # finally, compute the waves.
         wave(i,1,1) = a1
         wave(i,2,1) = a1*(u_roe - sqrtgh_roe)
         wave(i,1,2) = a2
         wave(i,2,2) = a2*(u_roe + sqrtgh_roe)

         ..................................................
            do  mw=1,mwaves
               amdq(i,m) = amdq(i,m) + min(s(i,mw), 0.d0) * wave(i,m,mw)
               apdq(i,m) = apdq(i,m) + max(s(i,mw), 0.d0) * wave(i,m,mw)
            enddo

      enddo
```

Speeds

Waves

Fluctuations

Using Riemann solvers

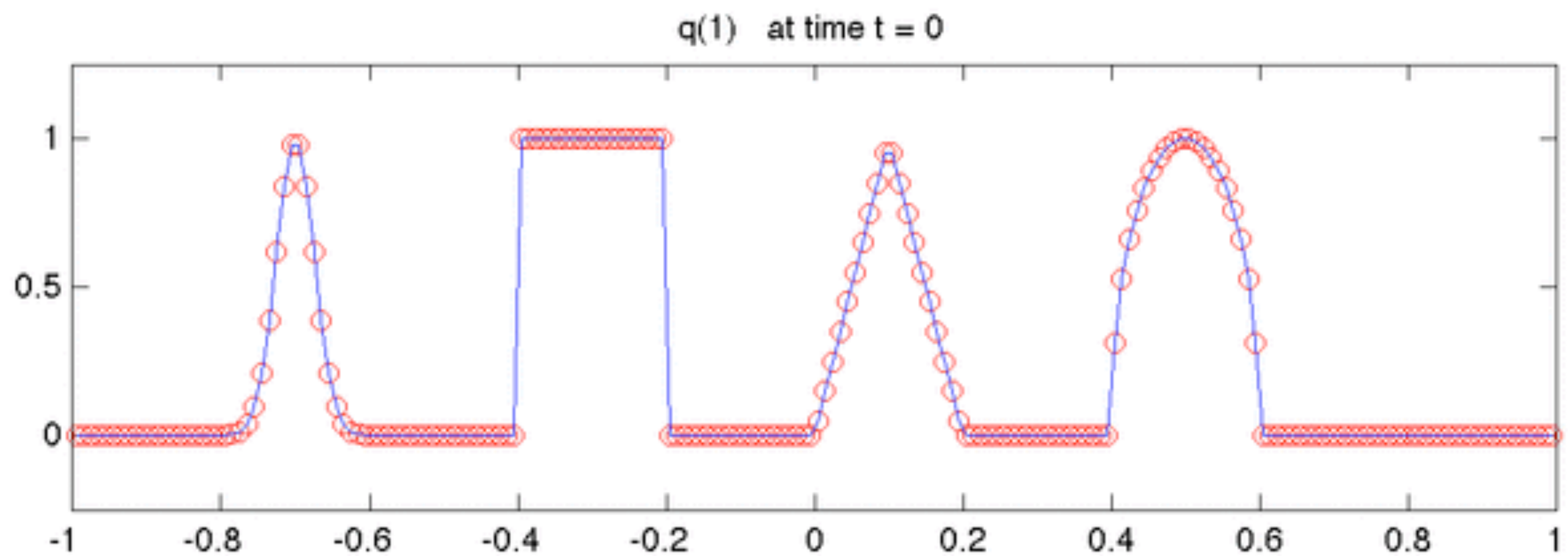


Question : There is only one shock and one rarefaction, but we solve a Riemann problem (either exactly, or approximately) at each cell interface. What happens in the smooth regions?

Answer : In smooth regions, shocks/rarefactions are *weak*. They only have strength on the order of the mesh cell size, i.e.

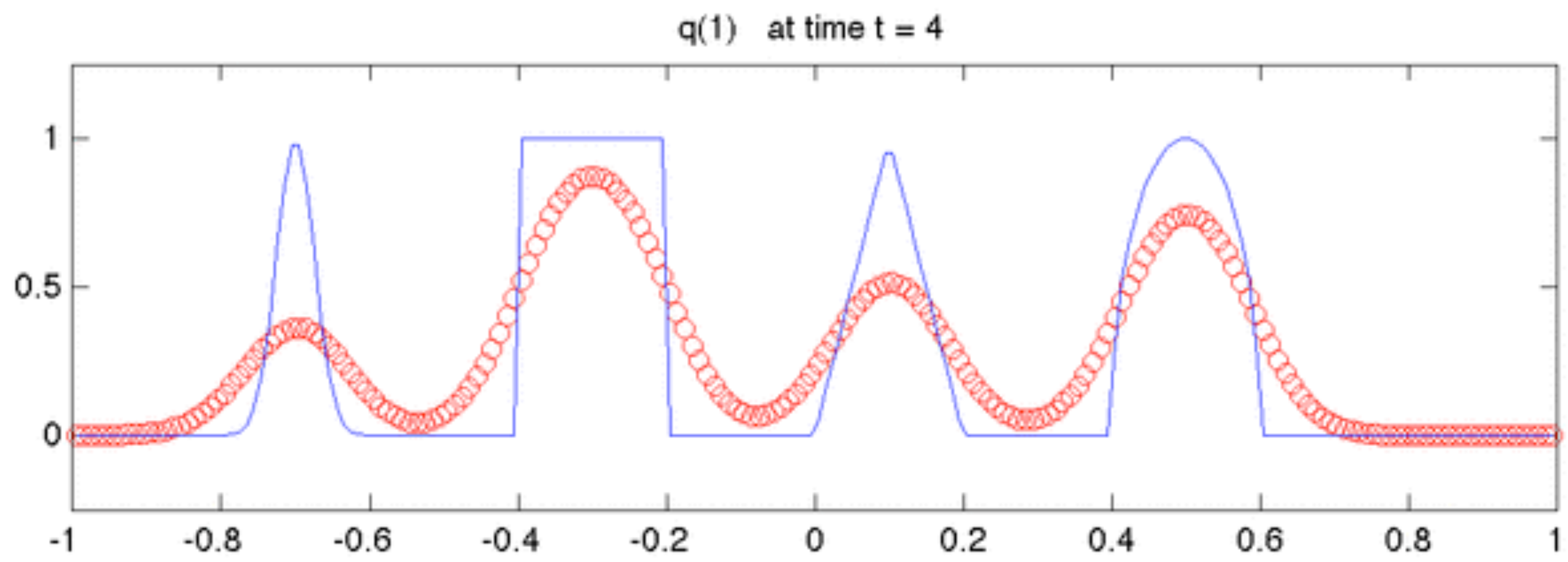
$$q_r - q_\ell \sim O(\Delta x)$$

Upwind method



First order scheme (200 points)

Upwind method



First order scheme (200 points)

The upwind method

The upwind method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$
$$= Q_i^n - \frac{\Delta t}{\Delta x}\bar{u}(Q_i^n - Q_{i-1}), \quad \text{for} \quad \bar{u} > 0$$

is only a first order approximation, but gives a good second order approximation to the equation

modified PDE

$$q_t + \bar{u}q_x = \frac{\bar{u}\Delta x}{2}\left(1 - \frac{\bar{u}\Delta t}{\Delta x}\right) q_{xx}$$

The "diffusion term" is proportional to the mesh spacing and the Courant number

Improved accuracy

Why not include these "diffusion" terms in the numerical scheme to get better accuracy?

$$q_t + u q_x = \frac{u\Delta x}{2}\left(1 - \frac{u\Delta t}{\Delta x}\right) q_{xx} \qquad u > 0$$

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta t} + u\frac{Q_i^n - Q_{i-1}^n}{\Delta x} = \frac{u\Delta x}{2}\left(1 - \frac{u\Delta t}{\Delta x}\right)\left(\frac{Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n}{(\Delta x)^2}\right)$$

Re-arranging terms, we get the second order "Lax-Wendroff" method.

The Lax-Wendroff method

$$Q_i^{n+1} = Q_i^n - \underbrace{\frac{u\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right)}_{\text{Upwind term}} - \underbrace{\frac{1}{2}\frac{u\Delta t}{\Delta x}\left(1 - \frac{u\Delta t}{\Delta x}\right)\left(\left(Q_{i+1}^n - Q_i^n\right) - \left(Q_i^n - Q_{i-1}^n\right)\right)}_{\text{Second order correction}}$$

The Lax-Wendroff method gives a third order approximation to the modified equation

$$q_t + uq_x = -\frac{u(\Delta x)^2}{6}\left(1 - \left(\frac{u\Delta t}{\Delta x}\right)^2\right) q_{xxx}$$

Errors are *dispersive*

Lax Wendroff method

Waves

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(1 - \frac{u\Delta t}{\Delta x}\right)\left(\left(Q_{i+1}^n - Q_i^n\right) - \left(Q_i^n - Q_{i-1}^n\right)\right)$$

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}\left(Q_i^n - Q_{i-1}^n\right) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(\Delta x - u\Delta t\right)\left(\sigma_i^n - \sigma_{i-1}^n\right)$$

slopes

Slopes $\longleftrightarrow$ Waves

$$\sigma_i^n = \frac{Q_{i+1}^n - Q_i^n}{\Delta x}$$

$$\mathcal{W}_{i-1/2}^n = Q_i^n - Q_{i-1}^n$$

Wave propagation viewpoint

For $u > 0$:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(u\, \mathcal{W}_{i-1/2}^n) - \frac{1}{2}\frac{u\Delta t}{\Delta x}\left(1 - \frac{u\Delta t}{\Delta x}\right)\left(\mathcal{W}_{i+1/2}^n - \mathcal{W}_{i-1/2}^n\right)$$

In wave propagation form, we can write this as :

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right)$$

where *second order correction terms* are defined as

$$\mathcal{F}_{i-1/2} \equiv \frac{1}{2}|u|\left(1 - \frac{|u|\Delta t}{\Delta x}\right)\mathcal{W}_{i-1/2}^n$$

Wave propagation algorithm for systems

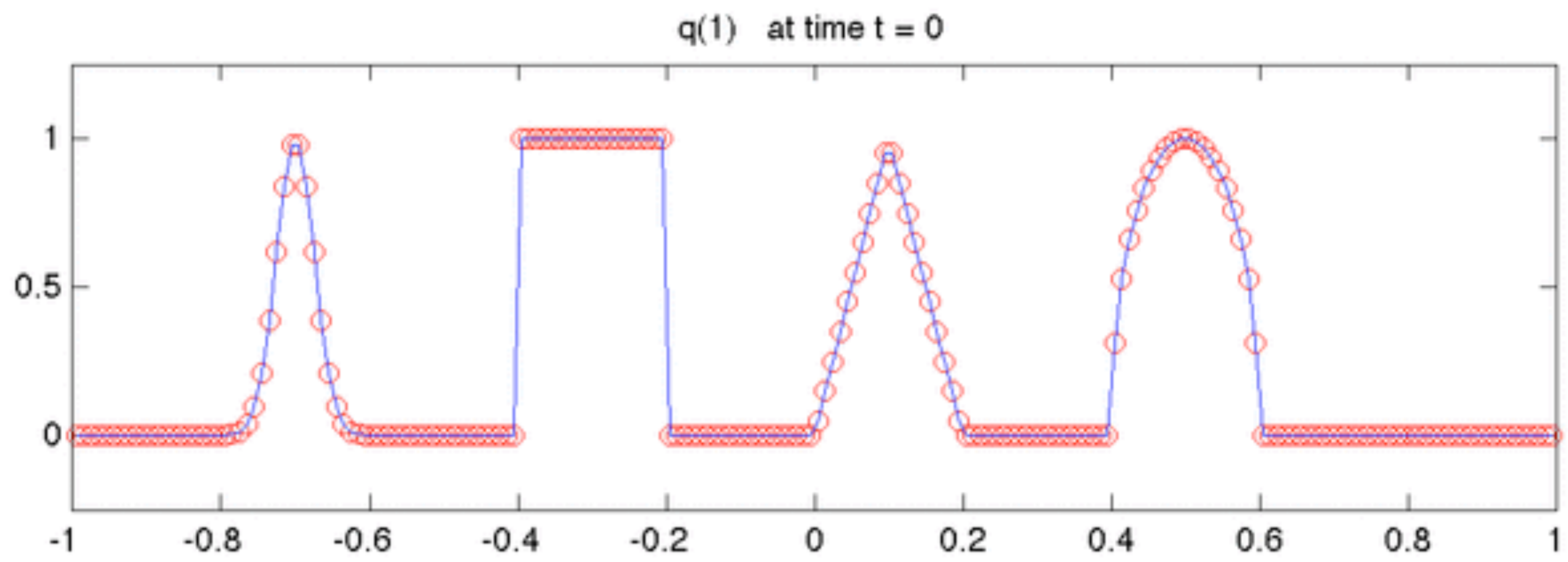
For systems (both linear and nonlinear), we have the update formula

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}) - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right)$$

where the second order corrections are defined as

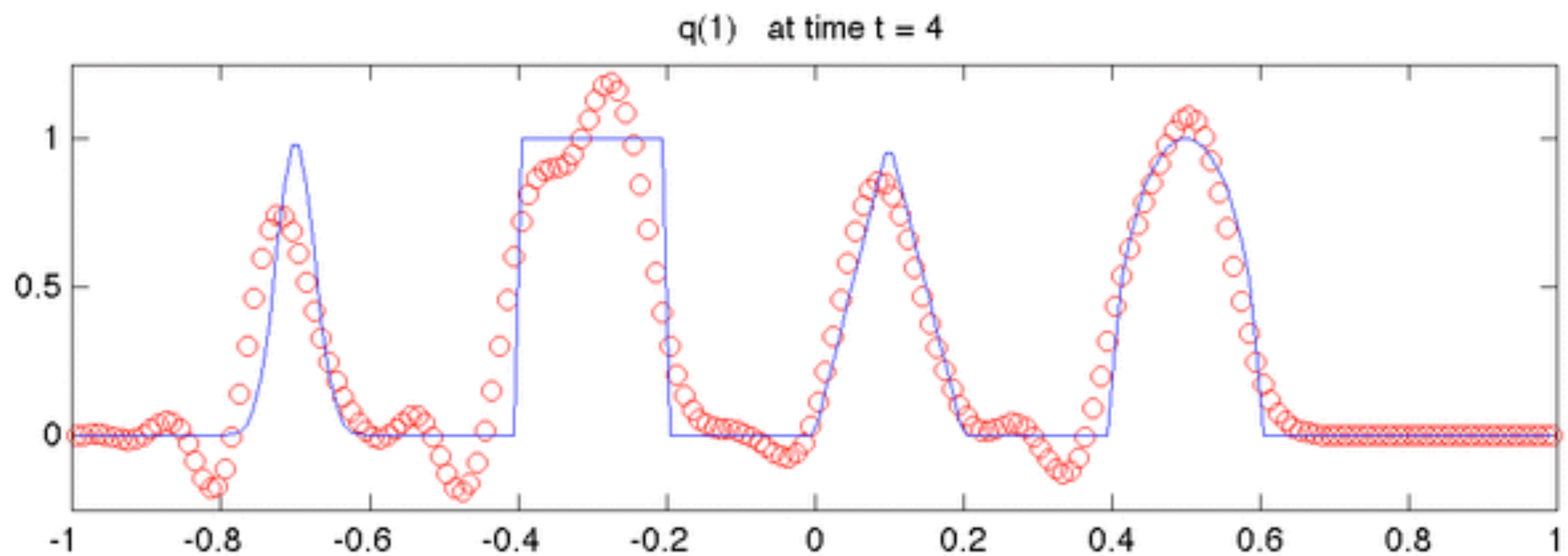
$$\mathcal{F}_{i-1/2} \equiv \frac{1}{2}\sum_{p=1}^{m} |\lambda^p| \left(1 - \frac{\Delta t}{\Delta x}|\lambda^p|\right) \mathcal{W}_{i-1/2}^p$$

The Lax-Wendroff method



Second order terms included (200 points)

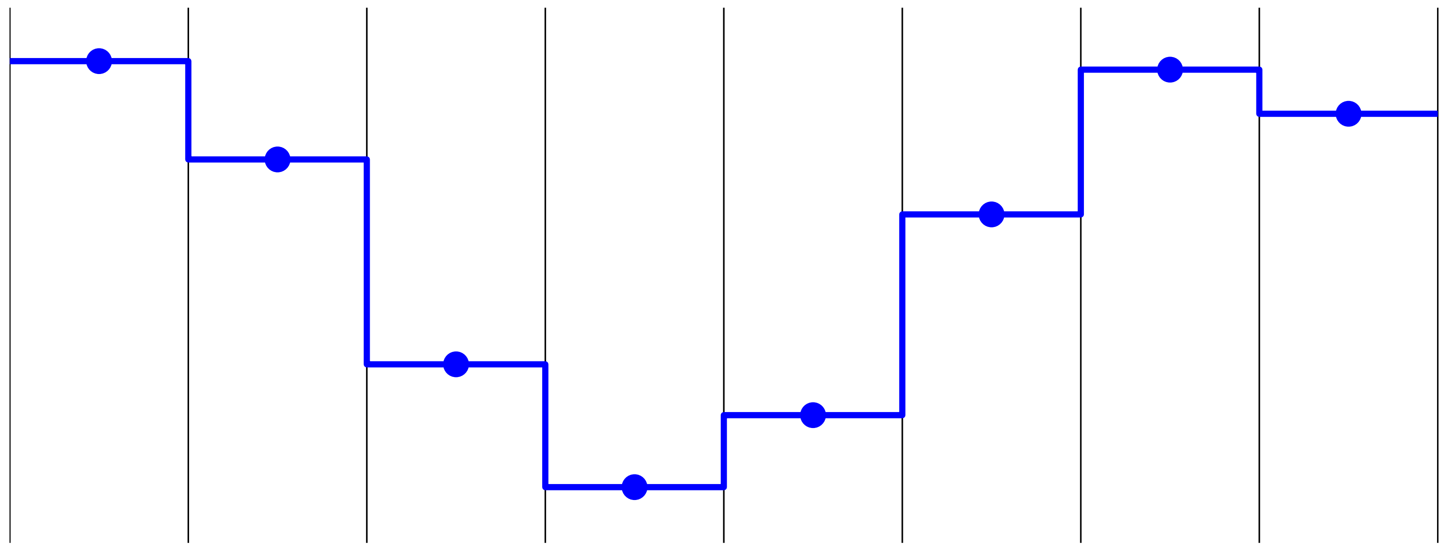
The Lax-Wendroff method



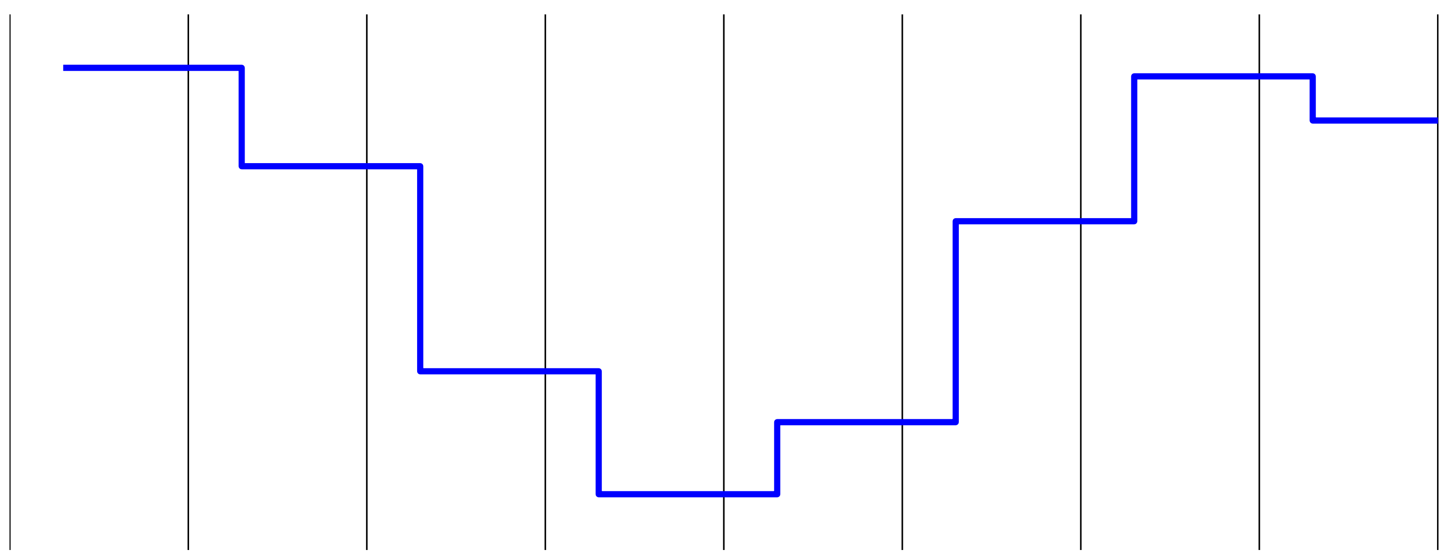
Second order terms included (200 points)

First order REA algorithm

Cell averages and piecewise constant reconstruction:

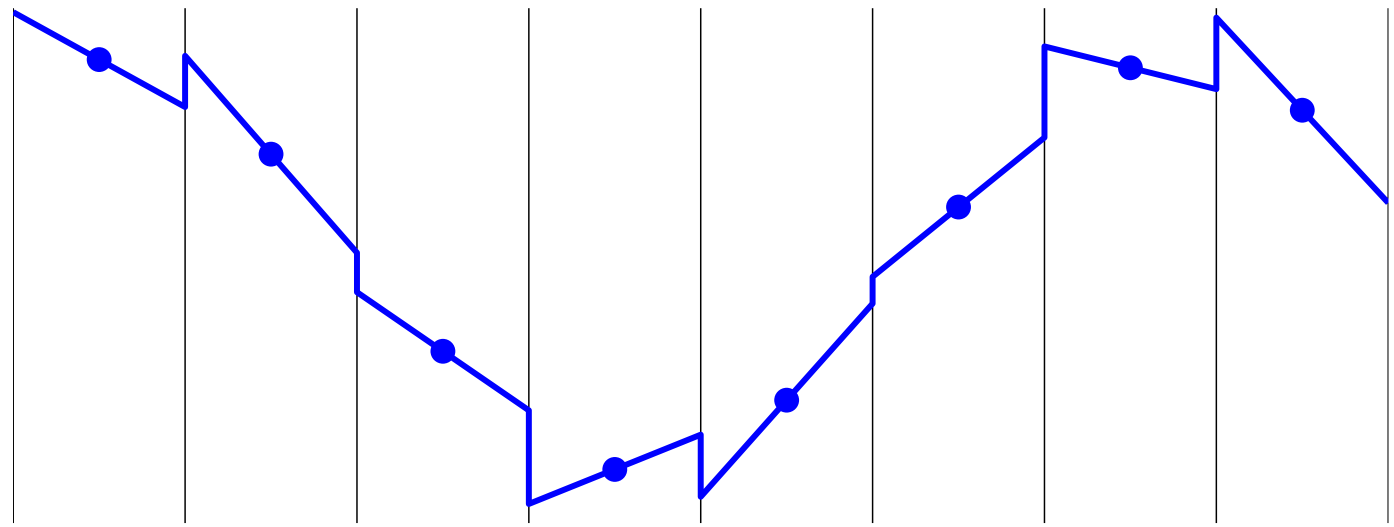


After evolution:

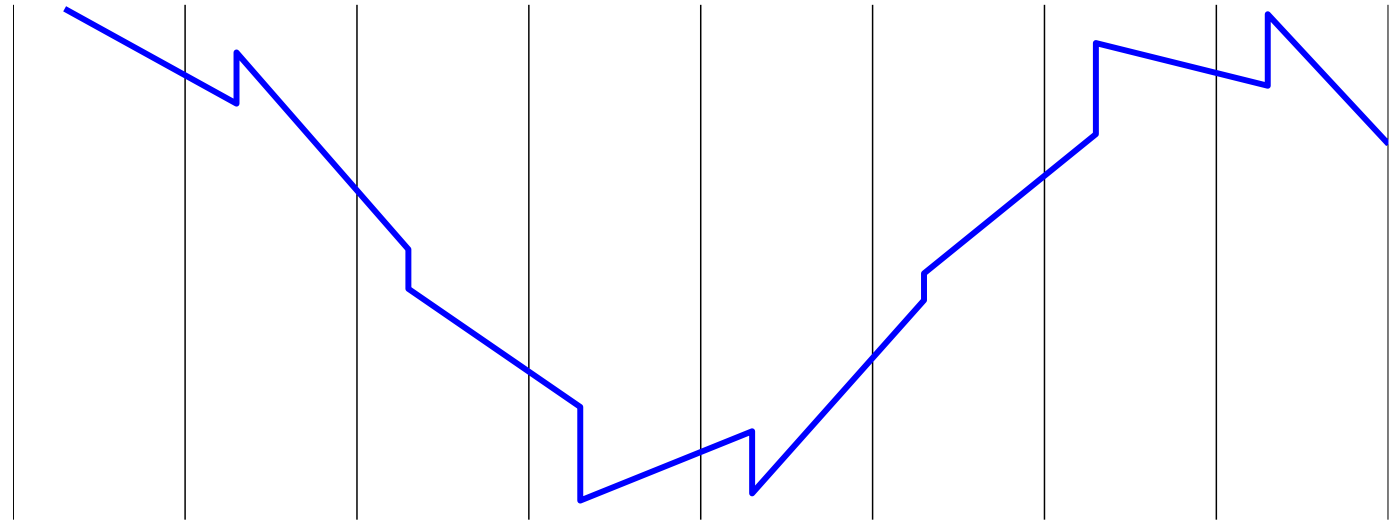


Second order REA algorithm

Cell averages and piecewise linear reconstruction:



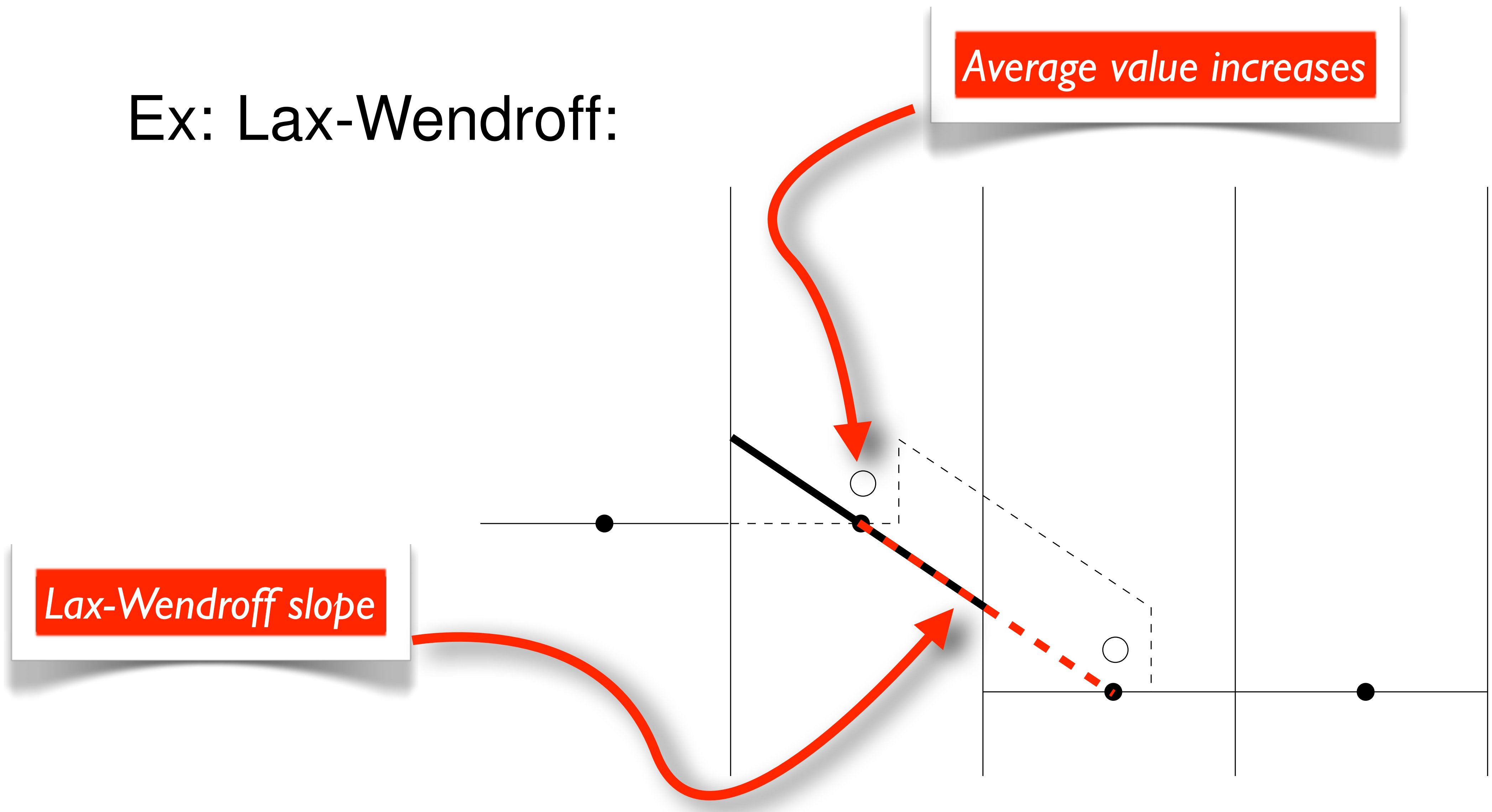
After evolution:



Oscillations

Any of these slope choices will give oscillations near discontinuities.

Ex: Lax-Wendroff:



High resolution methods

Want to use slope where solution is smooth for “second-order” accuracy.

Where solution is not smooth, adding slope corrections gives oscillations.

Limit the slope based on the behavior of the solution.

$$\sigma_i^n = \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\Phi_i^n.$$

$\Phi = 1 \implies$ Lax-Wendroff,

$\Phi = 0 \implies$ upwind.

Minmod slope

$$\text{minmod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |b| < |a| \text{ and } ab > 0 \\ 0 & \text{if } ab \le 0 \end{cases}$$

Slope:

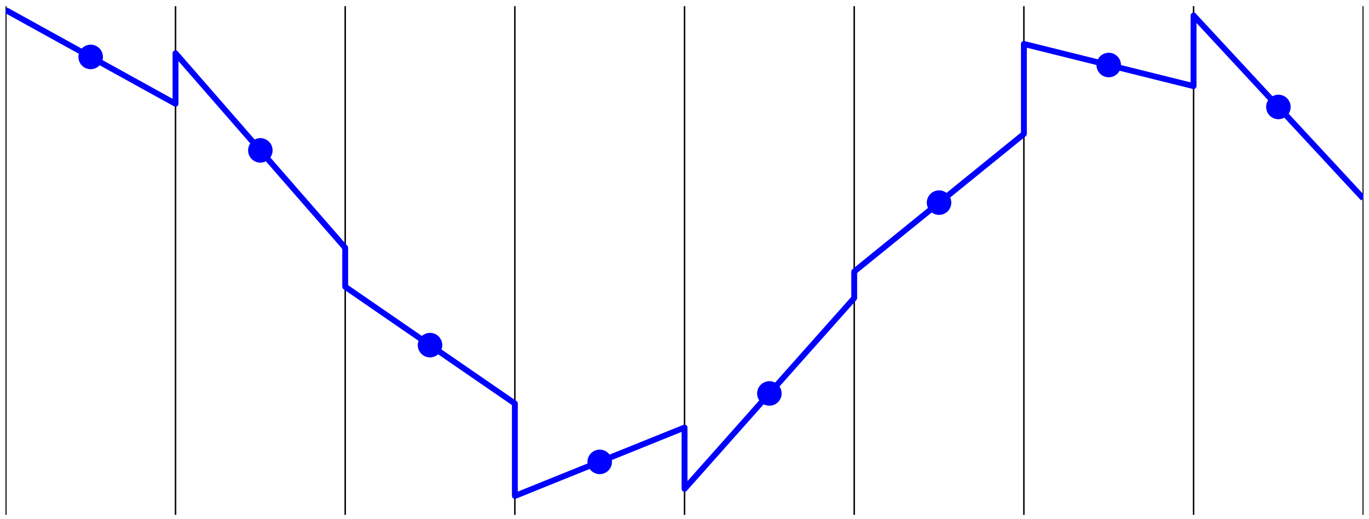
$$\begin{aligned} \sigma_i^n &= \text{minmod}((Q_i^n - Q_{i-1}^n)/\Delta x,\ \ (Q_{i+1}^n - Q_i^n)/\Delta x) \\ &= \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\Phi(\theta_i^n) \end{aligned}$$

where

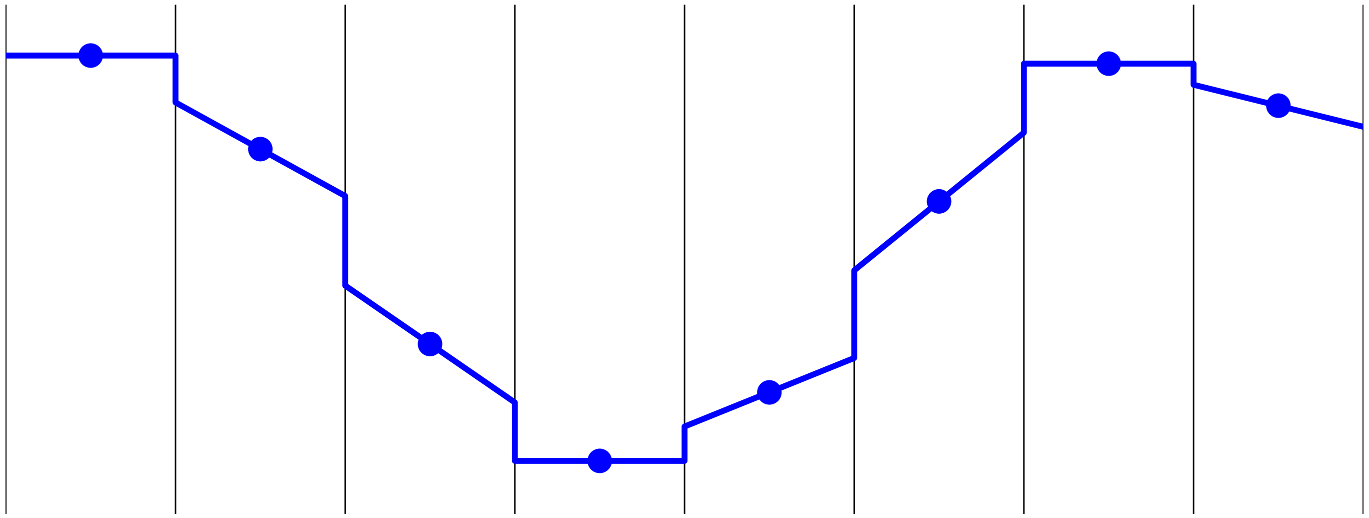
$$\begin{aligned} \theta_i^n &= \frac{Q_i^n - Q_{i-1}^n}{Q_{i+1}^n - Q_i^n} \\ \Phi(\theta) &= \text{minmod}(\theta, 1) \qquad {\color{blue} 0 \le \Phi \le 1} \end{aligned}$$

Minmod reconstruction

Lax-Wendroff reconstruction:



Minmod reconstruction:



Limiters

Linear methods:

$$\begin{aligned} \text{upwind}: &\quad \phi(\theta) = 0 \\ \text{Lax-Wendroff}: &\quad \phi(\theta) = 1 \\ \text{Beam-Warming}: &\quad \phi(\theta) = \theta \\ \text{Fromm}: &\quad \phi(\theta) = \frac{1}{2}(1+\theta) \end{aligned}$$

Examples of limiters available in Clawpack

High-resolution limiters:

$$\begin{aligned} \text{minmod}: &\quad \phi(\theta) = \text{minmod}(1,\theta) \\ \text{superbee}: &\quad \phi(\theta) = \max(0,\ \min(1,2\theta),\ \min(2,\theta)) \\ \text{MC}: &\quad \phi(\theta) = \max(0,\ \min((1+\theta)/2,\ 2,\ 2\theta)) \\ \text{van Leer}: &\quad \phi(\theta) = \frac{\theta+|\theta|}{1+|\theta|} \end{aligned}$$

Hierarchy of methods for systems

Waves $\mathcal{W}^p_{i-1/2} \equiv \alpha^p r^p$

Fluctuations $\mathcal{A}^+\Delta Q_{i-1/2} \equiv \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}_{i-1/2}$ $\quad \mathcal{A}^-\Delta Q_{i-1/2} \equiv \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}_{i-1/2}$

Upwind

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right)$$

Lax Wendroff

$$Q_i^{n+1} = Q_i^n - (\text{upwind}) - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right)$$

High Resolution

$$Q_i^{n+1} = Q_i^n - (\text{upwind}) - \frac{\Delta t}{\Delta x}\left(\widetilde{\mathcal{F}}_{i+1/2} - \widetilde{\mathcal{F}}_{i-1/2}\right)$$

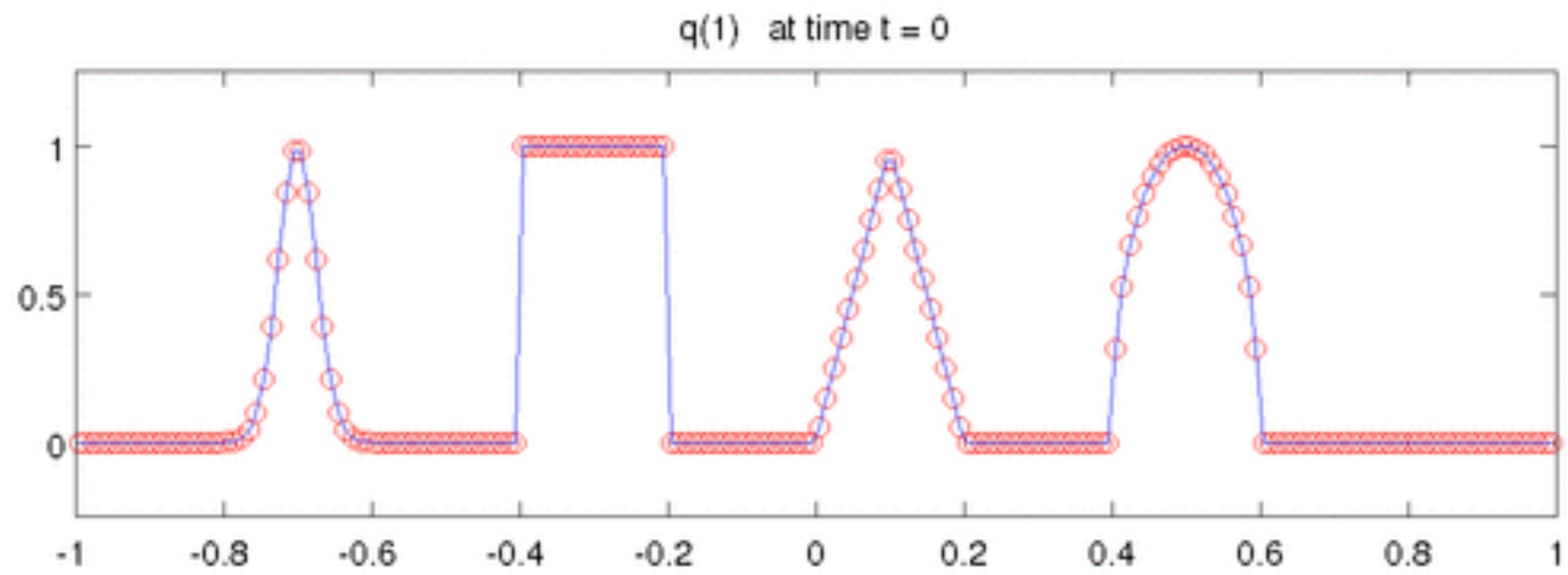
$$\mathcal{F}_{i-1/2} \equiv \frac{1}{2}\sum_{p=1}^{m} |\lambda^p| \left(1 - \frac{\Delta t}{\Delta x}|\lambda^p|\right) \mathcal{W}^p_{i-1/2} \qquad \widetilde{\mathcal{F}}_{i-1/2} \equiv \frac{1}{2}\sum_{p=1}^{m} |\lambda^p| \left(1 - \frac{\Delta t}{\Delta x}|\lambda^p|\right) \Phi\mathcal{W}^p_{i-1/2}$$

limited wave

High resolution methods

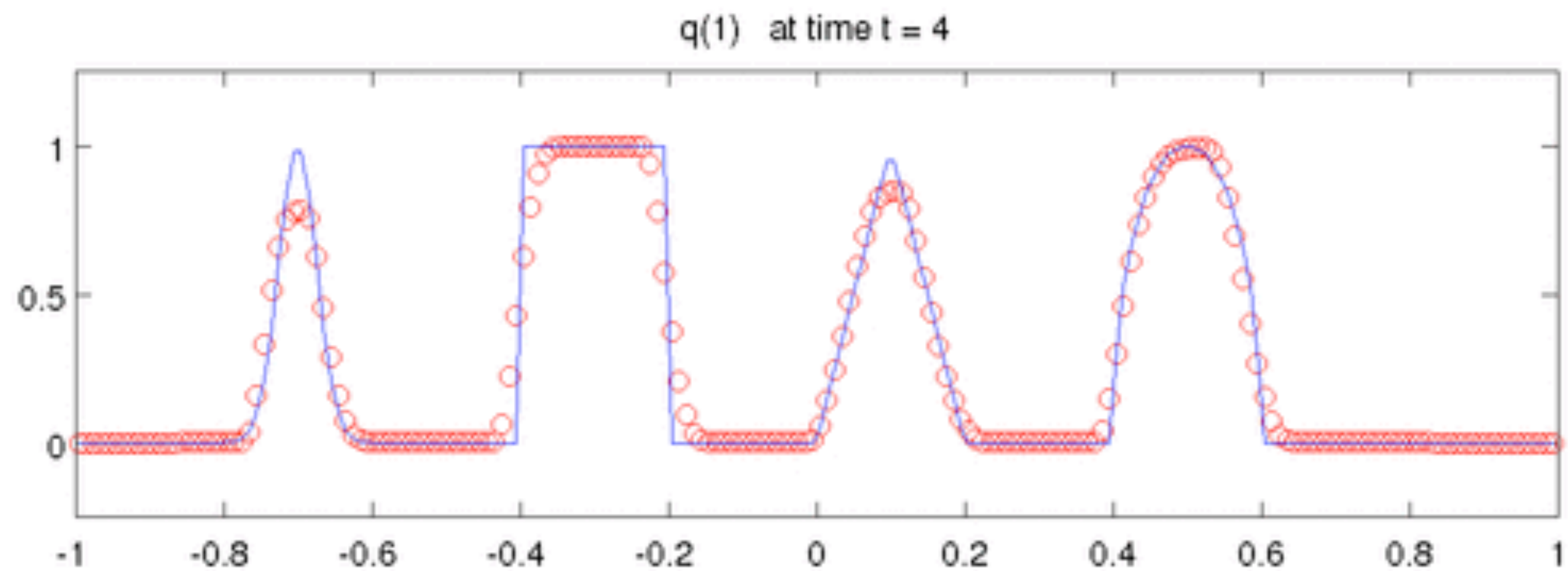
- Wave limiters reduce oscillations near discontinuities, but preserve second order accuracy in smooth regions
- Methods are no longer formally second order accurate, but are "high resolution".
- Often magnitude of the error is reduced by the use of limiters,
- Useful even for linear problems such as advection
- Clawpack has several limiters available
- Limiters only affect second order correction terms; first order method does not use limiters

High resolution methods



Second order method with limiter (200 points)

High resolution methods



Second order method with limiter (200 points)

What next?

- [x] How do these Riemann solvers make it into an actual code? Clawpack is based on solving Riemann problems.
- [x] Do we actually solve the non-linear problem at every grid cell interface? No! One can use approximate Riemann solvers.
- [x] How accurate are these methods? (second order, or high resolution with limiters)
- [ ] Well-balancing
- [ ] What do we do in two-dimensions?
- [ ] Adaptive mesh refinement?