Tsunami modeling

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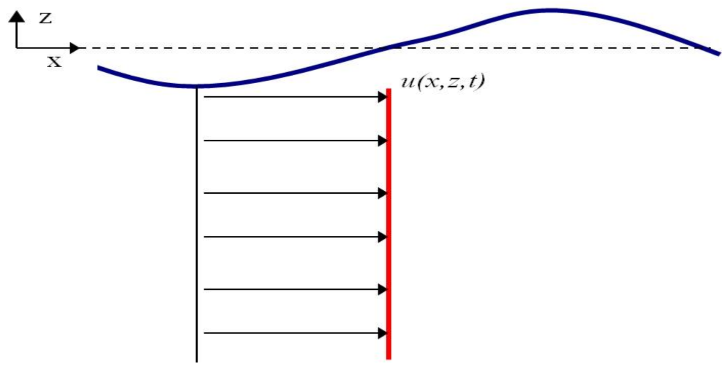
PASI 2013: Tsunamis and storm surges
Valparaiso, Chile
January 2-13, 2013

Mathematical models for tsunami waves

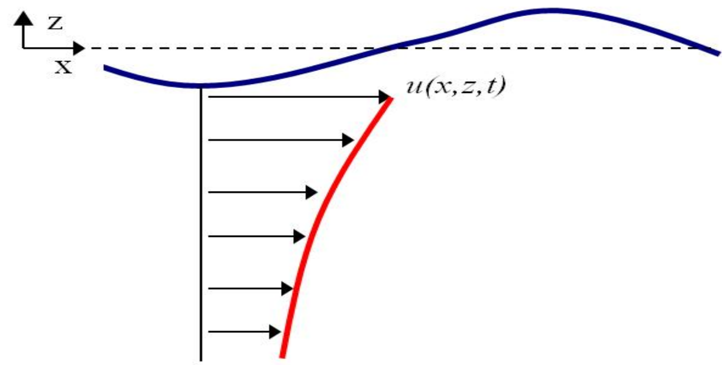
Model Types

- Nonlinear Shallow Water Equations

– Linear Shallow Water Equations

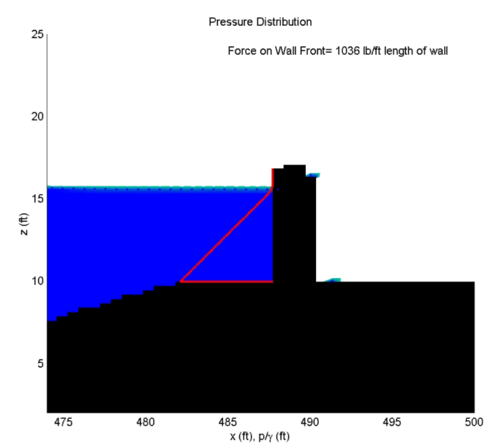


- Boussinesq-type Equations



- Computational Fluid Dynamics

– RANS, SPH, LES, DNS



CFD **models**

- Navier-Stokes equations

$$\nabla \cdot \mathbf{u} = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{g}$$

where $(\mathbf{u} = (u, v, w))$ is the velocity vector and *p* the pressure.

- Boundary conditions and initial conditions

1. On the free surface, $z = \quad (x, y, t)$ the kinematic boundary condition

$$\frac{D(z \qquad)}{Dt} = 0, \text{ or } \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = w$$

2. On the free surface, the dynamic boundary condition requires:

$$p = p_{atm} = 0, \text{ on } z = \quad (x, y, t)$$

3. On the seafloor, $z = \quad h(x, y, t)$, the kinematic boundary condition

$$\frac{D(z \quad h)}{Dt} = 0, \text{ or } \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = \quad w$$

If the time history of the seafloor displacement is prescribed, this boundary condition is linear.

o Initial conditions

1. If the time history of the seafloor displacement is prescribed, the initial conditions can be given as:

$$= 0,\ \mathbf{u} = 0, \text{ at } t = 0.$$

2. If seafloor is stationary, the bottom boundary condition become

$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = \quad w, \text{ on } z = h(x, y)$$

The initial free surface displacement needs to be prescribed,

$$(x, y, t = 0) = \quad {}_0(x, y) \quad \text{with} \quad \mathbf{u}(x, y, z, t = 0) = 0,$$

where ${}_0(x, y)$ mimics the final seafloor displacement after an earthquake.

Approximated long-wave models

- Long-wave assumptions
- Depth-integrated equations (reduce 3D to 2D)
    - Boussinesq-type equations
    - Shallow-water equations

Brief Review on Tsunami Models

- N-S model (3-D) *O(months-years)*
  - Based on Navier-Stokes Equations (or Euler Equations for potential flow)

    e.g., Truchas (Los Alamos Lab, USA)
- Boussinesq Model (H2D) *O(days-months)*
  - Based on Boussinesq-type Equations for weakly dispersive waves, e.g.

    FUNWAVE (Wei and Kirby, 1995; Kirby et al, 1998)

    CoulWave (Lynett and Liu, 2004)
- SWE Model (H2D) *O(hours)*
  - based on Shallow Water Equations for non-dispersive waves, e.g.,

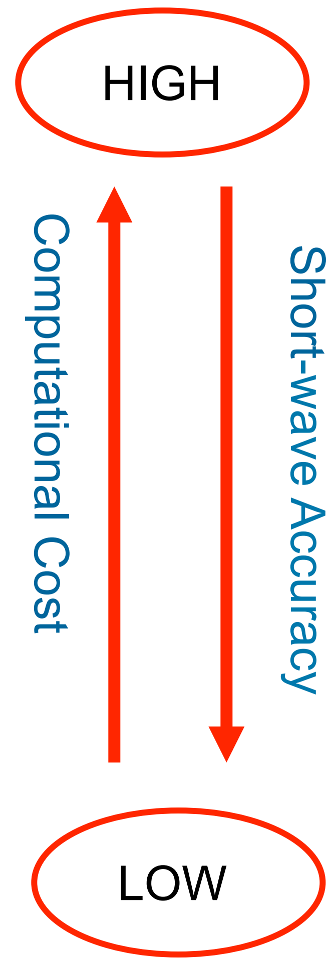
    COMCOT (Liu et al,1995),

    MOST (Titov et al, 1997)

    Anuga (Australia)

    TSUNAMI (ARSC, UAF)

Dispersion is neglected!!



Long wave approximations

Consider small amplitude single harmonic progressive wave:

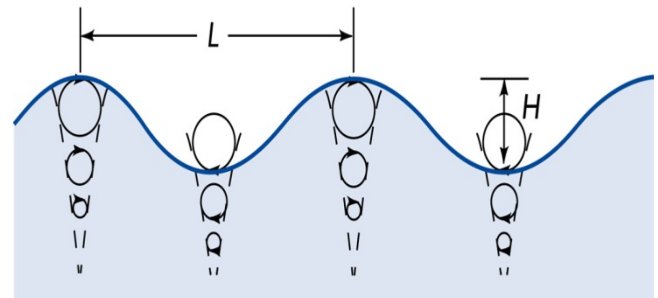
$$= Ae^{i(kx \quad t)}$$

$$u(x,z,t) = \frac{gkA}{\quad} \frac{\cosh k(z+h)}{\cosh kh} e^{i(kx \quad t)},$$

$$w(x,z,t) = \quad \frac{igkA}{\quad} \frac{\sinh k(z+h)}{\cosh kh} e^{i(kx \quad t)},$$

$$p = \quad g \quad \frac{\cosh k(z+h)}{\cosh kh}$$

$$C = \frac{\quad}{k} = \sqrt{\frac{g}{k} \tanh kh}$$



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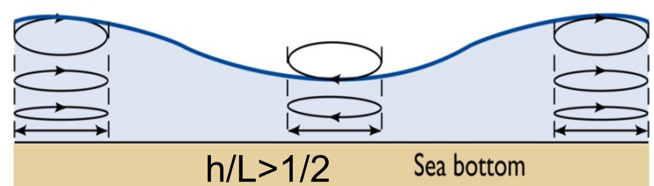


$$\cosh k(z+h) = 1 + \frac{1}{2}[k(z+h)]^2 + \frac{1}{24}[k(z+h)]^4 +$$

$$\sinh k(z+h) = [k(z+h)] + \frac{1}{6}[k(z+h)]^3 +$$

Shallow water wave equations

- Assumptions:

1. Horizontal scales (i.e., wavelength) are much longer than the vertical scale (i.e., the water depth). Accurate only for very long waves, $kh < 0.25$ (wavelength > 25 water depths)
2. Ignore the viscous effects.

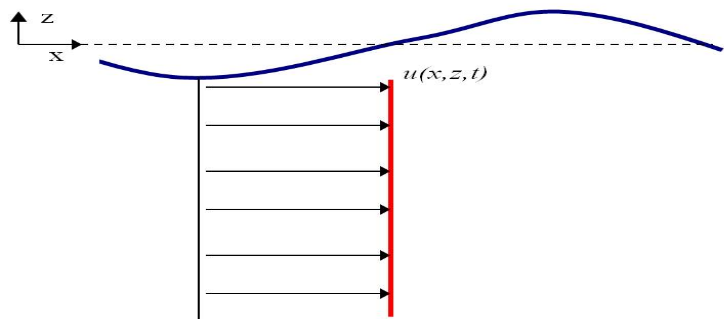
- Consequences:

1. The vertical velocity, w, is much smaller than the horizontal velocity, (u,v).
2. The pressure is hydrostatic, $p = \quad g(\quad z)$
3. The leading horizontal velocity is uniform in water depth.

- Nonlinear shallow-water wave equations

Conservation of mass:

$$\frac{\partial}{\partial t}[\quad +h]+\frac{\partial}{\partial x}\left(\quad +h\right)u\quad+\frac{\partial}{\partial y}\left(\quad +h\right)v\quad=0$$



Momentum equations:

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\mathrm{g}\quad\mathbf{u}+g\qquad=0$$

- Linear shallow–water wave equations

$$\frac{\partial}{\partial t}[\quad +h]+\quad \mathrm{g}[\mathbf{u}h]=0$$

$$\frac{\partial \mathbf{u}}{\partial t}+g\qquad=0$$

Special case: One-dimensional and constant depth

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\left(\quad + h \right) u \quad = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial}{\partial x} = 0.$$

which can be rewritten as

$$\frac{\partial}{\partial t} + \left(u + C\right)\frac{\partial}{\partial x} \quad \left(u + 2C\right) = 0,$$

$$\frac{\partial}{\partial t} + \left(u \quad C\right)\frac{\partial}{\partial x} \quad \left(u \quad 2C\right) = 0.$$

where $C = \sqrt{g\left(\quad + h\right)}$

$u \pm C$ are Riemann invariants along charateristics along $\frac{dx}{dt} = u \pm 2C.$

Linear shallow water wave equation

$$\frac{\partial}{\partial t} + h\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + g\frac{\partial}{\partial x} = 0.$$

which can be combined into a wave equation

$$\frac{\partial^2}{\partial t^2} \quad gh\frac{\partial^2}{\partial x^2} = 0$$

The general solution is $\quad = \quad (x \pm Ct), \quad C = \sqrt{gh}$

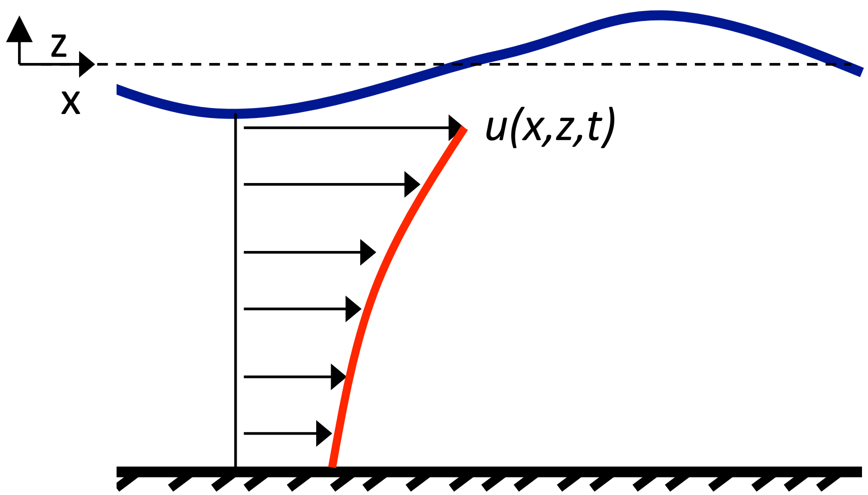
Wave form does not change! NO dispersion!

Boussinesq equations

(Peregrine, 1967; Ngowu, 1993)

- Pressure field is not hydrostatic: quadratic
- Horizontal velocities are not vertically uniform

$$u(x,z,t) = A(x,t) + \underbrace{z\;B(x,t) + z^2\;C(x,t)}_{\text{should be small compared to } A(x,t)}$$



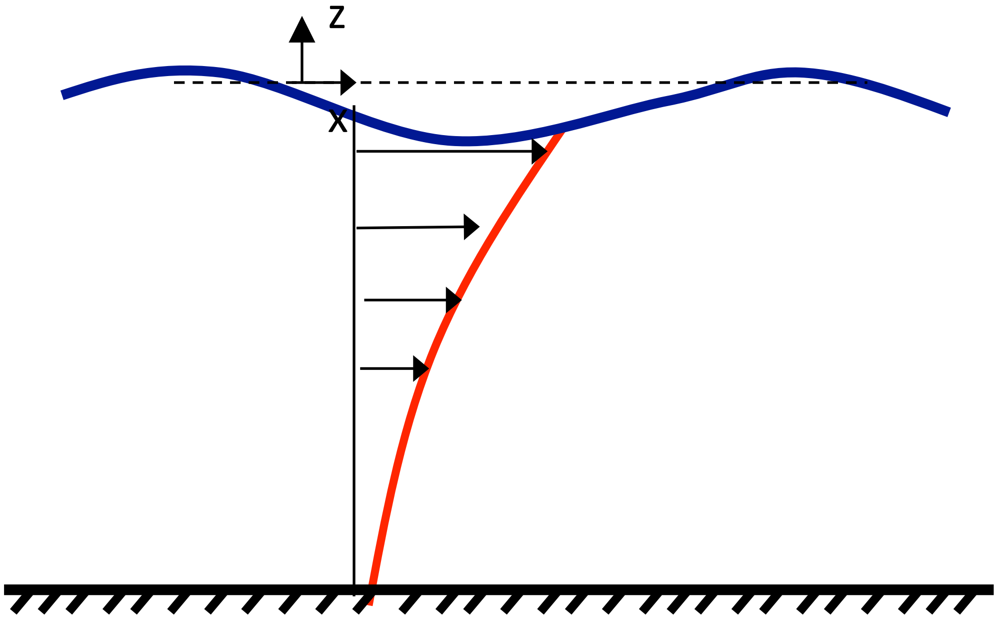
Accurate for long and intermediate depth waves, kh < 3 (wavelength > 2 water depths)

Functions *B, C* lead to 3rd order spatial derivatives in model

High-Order Boussinesq Equations (Gobbi *et al.*, 2000)

$$u(x,z,t) = A(x,t) + \; z * B(x,t) + z^2 * C(x,t)$$

$$+ \; \underbrace{z^3 * D(x,t) + z^4 * E(x,t)}_{\text{Should be small compared to } B,C \text{ group}}$$



- Accurate for long, intermediate, and moderately deep waves, kh < 6 (wavelength > 1 water depth)
- Functions *D, E* lead to 5th order spatial derivatives in model

Derivation of depth-integrated wave equations

- Normalize the conservation equations and the associated boundary conditions

$$(x, y) = (x', y')/L_0, \ \ z = z'/h_0, \ \ t = t'/(L_0/\sqrt{gh_0}), \ \ h = h'/h_0$$

$$\mathbf{u} = \mathbf{u}'/\left(\sqrt{gh_0}a_0/h_0\right), \ \ w = w'/\left(\sqrt{gh_0}a_0/L_0\right), \ \ p = p'/\rho g a_0, \ \ \eta = \eta'/a_0$$

- Perturbation analysis

$$f = \sum_{n=0}^{\infty} \mu^{2n} f^{(n)}, \quad f = \mathbf{u}, \ w, \ p, \ \eta\,; \qquad = \frac{h_0}{L_0} = \ 1$$

- Integrate the governing equations and employ
  - o the **irrotational condition**
  - o the boundary conditions

Boussinesq-type equations

$O(\ )=1, O(\ ^{2}) = \ 1$

$$\frac{1}{\epsilon}\frac{\partial h}{\partial t}+\frac{\partial \eta}{\partial t}+\nabla\cdot\left[\left(\epsilon\eta+h\right)\mathbf{u}\right]$$
$$-\mu^2\nabla\cdot\left\{\left[\frac{\epsilon^3\eta^3+h^3}{6}-\frac{(\epsilon\eta+h)k^2}{2}\right]\nabla S\right\}=\mathcal{O}\left(\mu^4\right)$$

$$\frac{\partial \mathbf{u}}{\partial t}+\epsilon\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\eta+\mu^2\frac{\partial}{\partial t}\left\{\frac{k^2}{2}\nabla S+k\nabla T\right\}$$
$$+\epsilon\mu^2\left[\left(\mathbf{u}\cdot\nabla k\right)\nabla T+k\nabla\left(\mathbf{u}\cdot\nabla T\right)+k\left(\mathbf{u}\cdot\nabla k\right)\nabla S+\frac{k^2}{2}\nabla\left(\mathbf{u}\cdot\nabla S\right)\right]$$
$$+\epsilon\mu^2\left[T\nabla T-\nabla\left(\eta\frac{\partial T}{\partial t}\right)\right]+\epsilon^2\mu^2\nabla\left(\eta ST-\frac{\eta^2}{2}\frac{\partial S}{\partial t}-\eta\mathbf{u}\cdot\nabla T\right)$$
$$+\epsilon^3\mu^2\nabla\left[\frac{\eta^2}{2}\left(S^2-\mathbf{u}\cdot\nabla S\right)\right]+\mathbf{F}_{\text{friction}}+\mathbf{F}_{\text{breaking}}=\mathcal{O}\left(\mu^4\right)$$

$$\epsilon=\frac{a_0}{h_0},\ \mu=\frac{h_0}{L_0},\ S=\nabla\cdot\mathbf{u},\ T=\nabla\cdot\left(h\mathbf{u}\right)+\frac{1}{\epsilon}\frac{\partial h}{\partial t},\ \mathbf{u}=\left.(u,v)\right|_{z=k(x,y,t)}$$

Boussinesq equations

$$O(\quad) = O(\quad ^2); k = \quad h$$

$$\frac{1}{\quad}\frac{\partial h}{\partial t} + \frac{\partial}{\partial t} + \quad (\quad + h)\mathbf{u} + \frac{1}{3} \quad ^2 \quad h^3 \left(\quad \mathbf{u} \right) = O(\quad ^4, \quad ^2, \quad ^2)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \quad \mathbf{u} \quad \mathbf{u} + \quad + \quad ^2 \frac{\partial}{\partial t} \frac{h^2}{2} \left(\quad \mathbf{u} \right) \quad h \left(\quad h\mathbf{u} \right) = O(\quad ^4, \quad ^2, \quad ^2)$$

Different Boussinesq equation forms can be derived by using different Representative velocity or by substituting the higher order terms by an Order one relation, e.g., $\frac{\partial \mathbf{u}}{\partial t} = \quad + O(\quad , \quad ^2)$

The resulting equations have slightly different dispersion characteristics.

Modeling the dispersion effects

- To apply the depth-integrated equations for shorter waves, higher order terms can be included. However, it more difficult to solve the high-order model
  - Momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + ... + C_1\frac{\partial^5 u}{\partial x^5} = 0$$

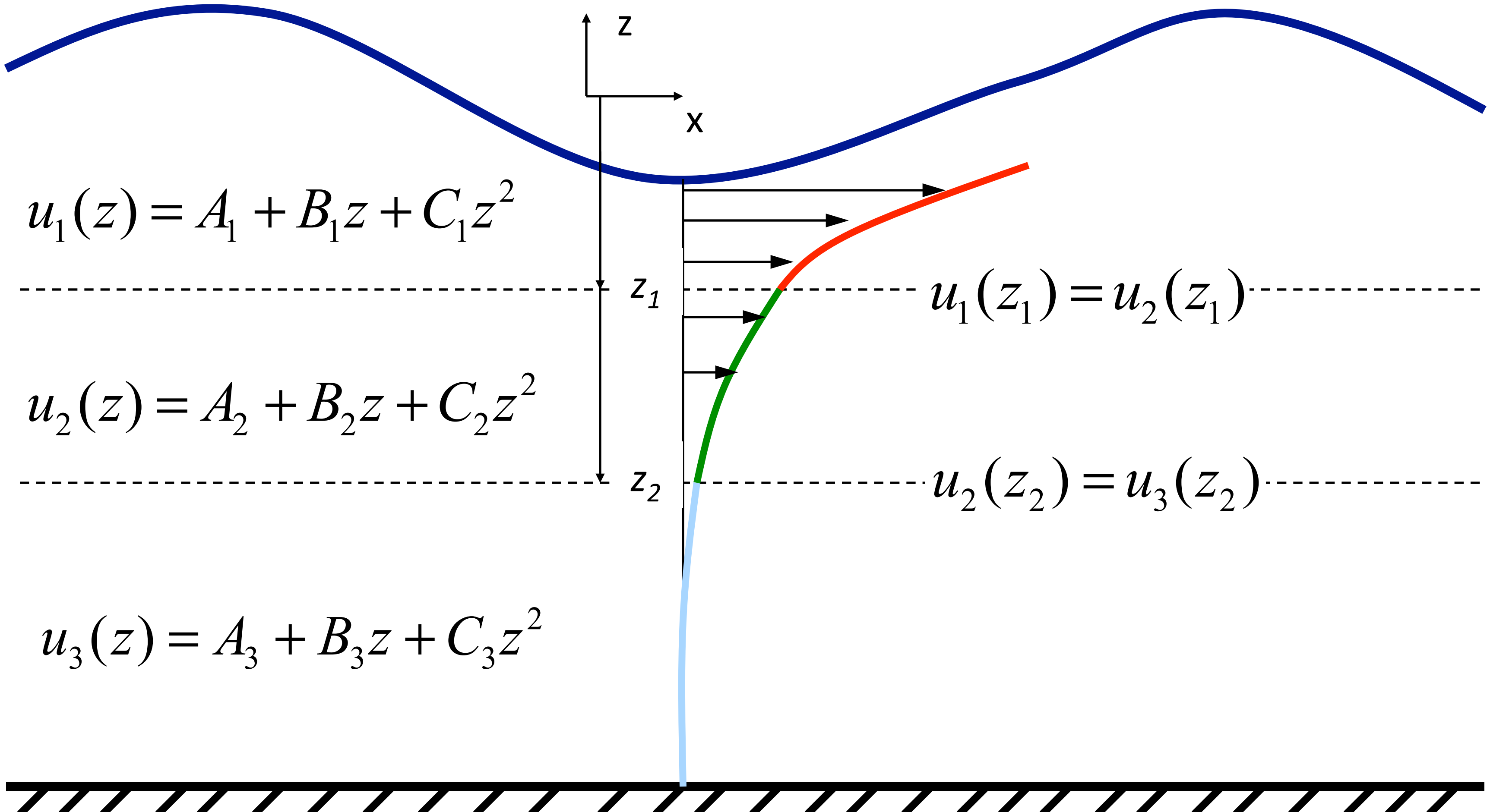
  - To solve consistently, numerical truncation error (Taylor series error) for leading term must be less important than included terms.
    - For example: 2nd order in space finite difference:

$$\frac{\partial u(x_o,t)}{\partial x} = \frac{u(x_o + \quad x,t) \quad u(x_o \qquad x,t)}{2 \quad x} \qquad \frac{x^2}{6}\frac{\partial^3 u(x_o,t)}{\partial x^3}$$

    - High-order model requires use of 6-point difference formulas (Dx^6 accuracy). Additionally, time integration would require a Dt^6 accurate scheme.
    - It is difficult to specify boundary conditions.

COULWAVE (Cornell University Long and Intermediate Wave Modeling)
A Multiple Layers Approach

- Divide water column into arbitrary layers
  - Each layer governed by an independent velocity profile, each in the same form as traditional Boussinesq models:



$$u_1(z) = A_1 + B_1 z + C_1 z^2$$

$$u_1(z_1) = u_2(z_1)$$

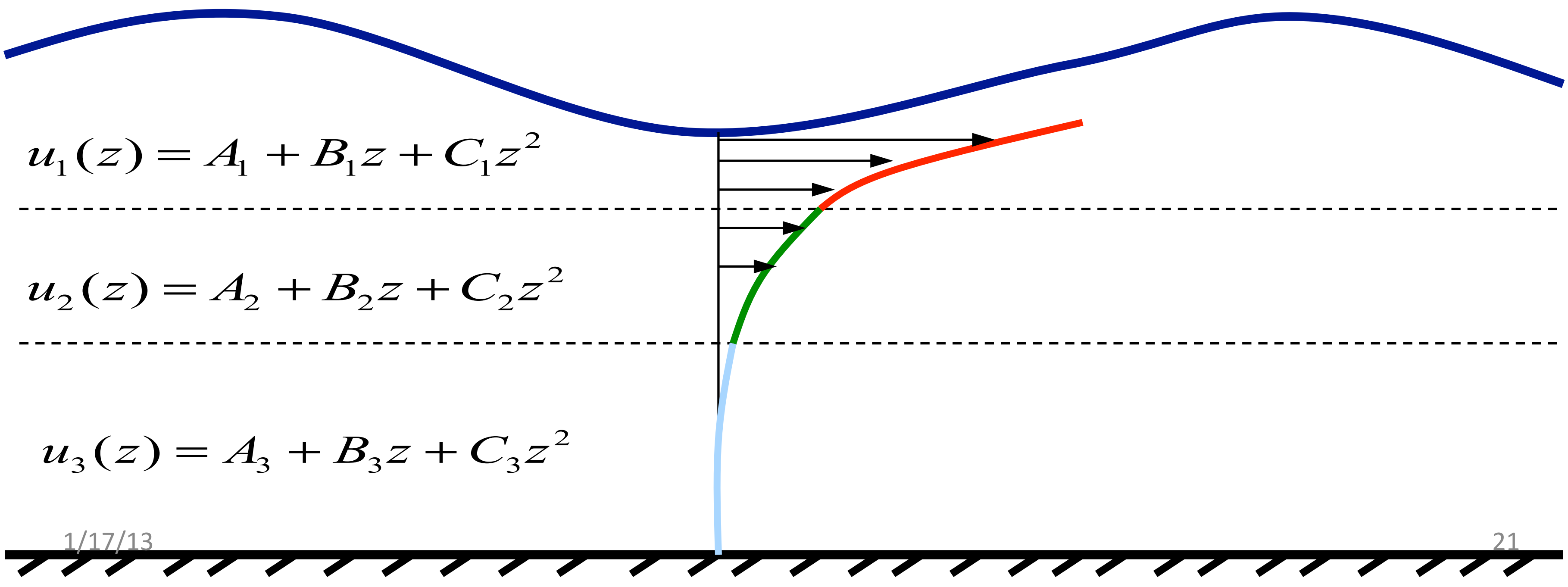
$$u_2(z) = A_2 + B_2 z + C_2 z^2$$

$$u_2(z_2) = u_3(z_2)$$

$$u_3(z) = A_3 + B_3 z + C_3 z^2$$

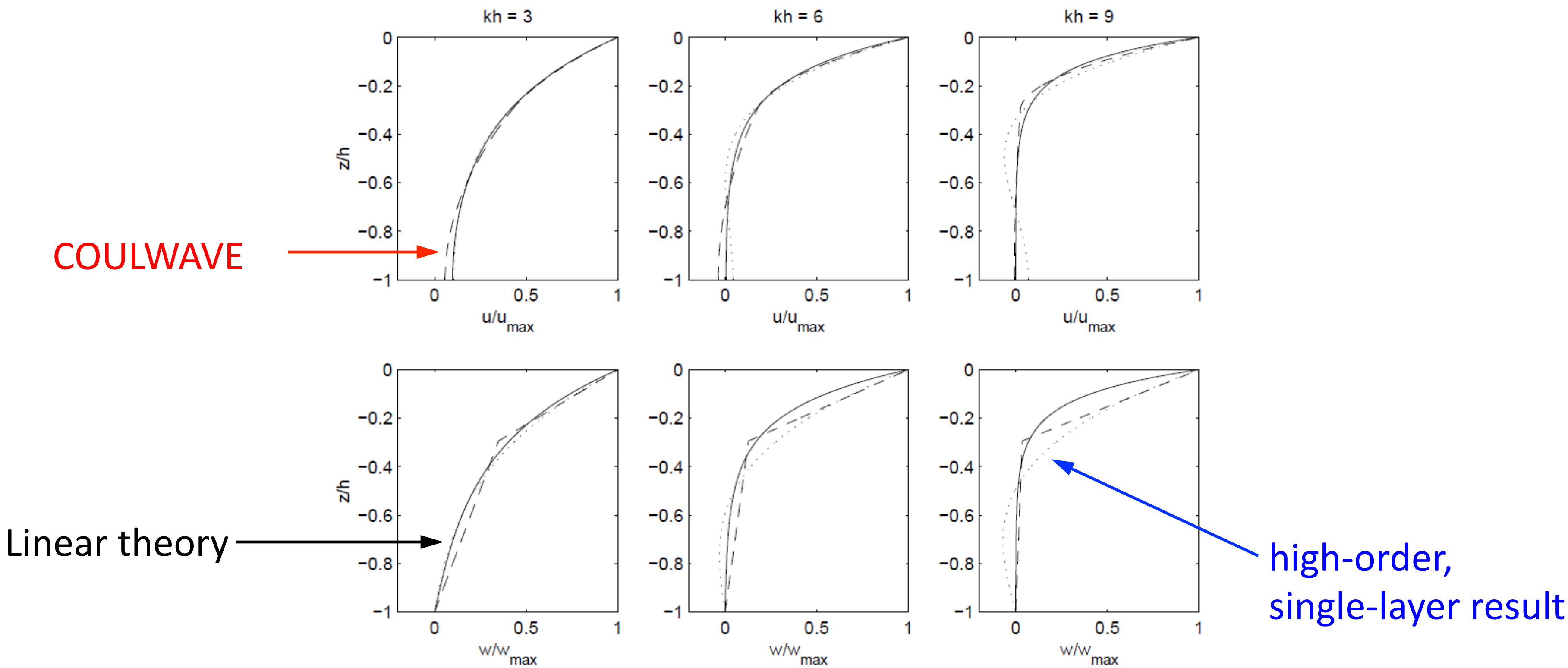
COULWAVE (Cornell University Long and Intermediate Wave Modeling)

- Regardless of # of layers, ***highest order of derivation is 3***
- The more layers used, the more accurate the model
- Any # of layers can be used
  - 1-Layer model = Boussinesq model
  - Numerical applications of 2-Layer model to be discussed
- Location of layers will be optimized for good agreement with known, analytic properties of water waves



COULWAVE

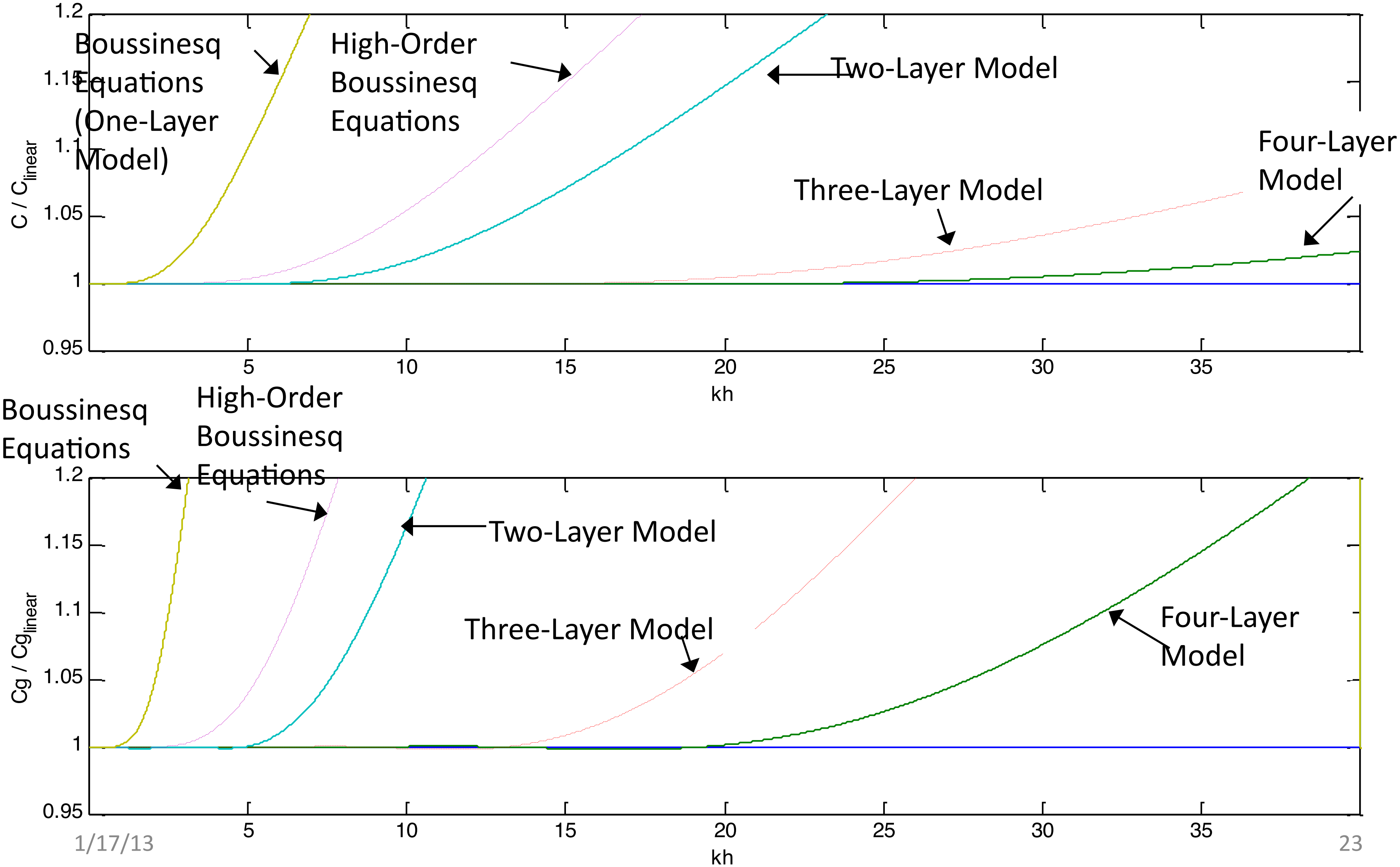
- Governing equations: multi-layer, fully-nonlinear



- Numerical scheme: predictor-corrector method
  - Predictor: explicit 3rd-order Adams-Bashforth
  - Corrector: implicit 4th-order Adams-Moulton

Linear Dispersion Properties of Multi-Layer Model

- Compare phase and group velocity with linear theory



General remarks on tsunami modeling by long-wave equations

- Add-hoc dissipation mechanism
  - Bottom friction: quadratic form
  - Wave breaking: parameterized model
- Controlling wave mechanism
  - Near the earthquake source: linear, dispersive or non-dispersive waves
  - Deep ocean: linear, non-dispersive waves
  - Close to the shore: nonlinear, non-dispersive waves

(Observation from the wave gauge data, and, the analytical study of one-dimensional problem)

Tsunami modeling packages

- Boussinesq-type equations model
  - COULWAVE (Cornell University Long and Intermediate Wave Modeling Package)
  - An improved multi-layer approach
- Shallow water equations model
  - COMCOT (Cornell Multi-grid Coupled Tsunami Model)
  - Covers tsunami generation, propagation, and wave runup/rundown on coastal regions
  - If desired, physical frequency dispersion can be mimicked by the numerical dispersion
  - This is a more practical choice for the tsunami simulation

Numerical simulation of tsunami waves

- Governing equations (COMCOT)

$$\frac{\partial \eta}{\partial t} + \frac{1}{R\cos\phi}\left\{\frac{\partial P}{\partial \psi} + \frac{\partial}{\partial \phi}(\cos\phi Q)\right\} = -\frac{\partial h}{\partial t}$$

$$\frac{\partial P}{\partial t} + \frac{1}{R\cos\phi}\frac{\partial}{\partial \psi}\frac{P^2}{\eta + h} + \frac{1}{R}\frac{\partial}{\partial \phi}\frac{PQ}{\eta + h} + \frac{g(h+\eta)}{R\cos\phi}\frac{\partial \eta}{\partial \psi} - fQ = -F_x$$

$$\frac{\partial Q}{\partial t} + \frac{1}{R\cos\phi}\frac{\partial}{\partial \psi}\frac{PQ}{\eta + h} + \frac{1}{R}\frac{\partial}{\partial \phi}\frac{Q^2}{\eta + h} + \frac{g(h+\eta)}{R}\frac{\partial \eta}{\partial \phi} + fP = -F_y$$

R : the radius of the Earth, $(\phi, \psi) =$ (latitude, longitude)

$(P, Q) = (uh, vh)$: the volume flux,

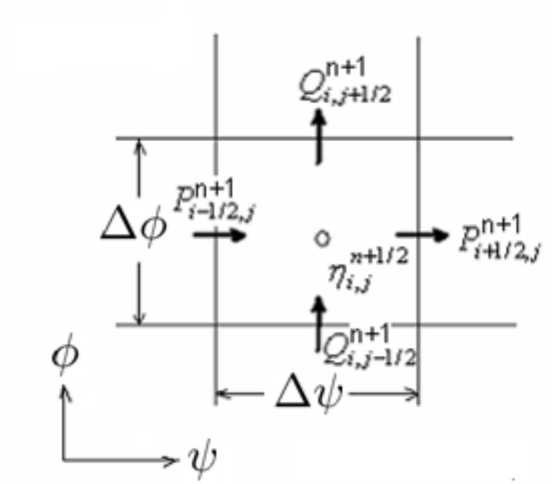
$f = \Omega \sin\phi$: the Coriolis force coefficient

$F_{x,y}$: bottom friction term (Manning's formula)

- Linear model: deep ocean
- Nonlinear model: shallower region

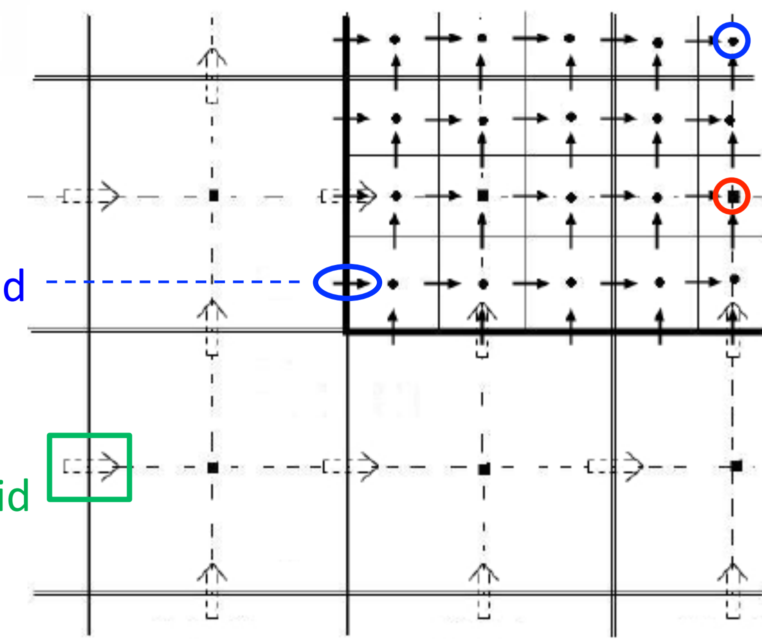
COMCOT: numerical scheme

- Explicit leap-frog Finite Differencing Method



$$\frac{\partial \eta}{\partial t} + \frac{1}{R\cos\phi}\frac{\partial P}{\partial \psi} \approx \frac{\eta^{n+1/2}_{i,j} - \eta^{n-1/2}_{i,j}}{\Delta t} + \left\{\frac{1}{R\cos\phi}\right\}_{i,j} \frac{P^{n}_{i+1/2,j} - P^{n}_{i-1/2,j}}{\Delta \psi}$$

- Nested grid



Inputs (initial conditions) for the tsunami wave modeling

- Earthquake-generated seafloor displacement
  - Impulsive model
    - The seafloor deforms instantaneously and the entire fault line ruptures simultaneously
    - The sea surface follows the seafloor deformation instantaneously
  - Transient model
    - The seafloor deformation and the rupture along the fault line are both described as transient processes
    - Require time histories of the horizontal and vertical seafloor displacements

Fault plane models

- Impulsive vs. transient models
  - Transient models are not always available
  - The speed of the rupture is orders of magnitude faster than the tsunami wave propagation speed
  - Advanced impulsive models can provide detailed spatial variations of the final seafloor displacement
  - No significant difference in the resulting tsunami wave height and propagation time from these two types of fault plane models
    - Analysis arguments: Kajiura (1963); Momoi (1964-5); Tuck and Hwang (1972)
    - Numerical experiments: Wang and Liu (2006)

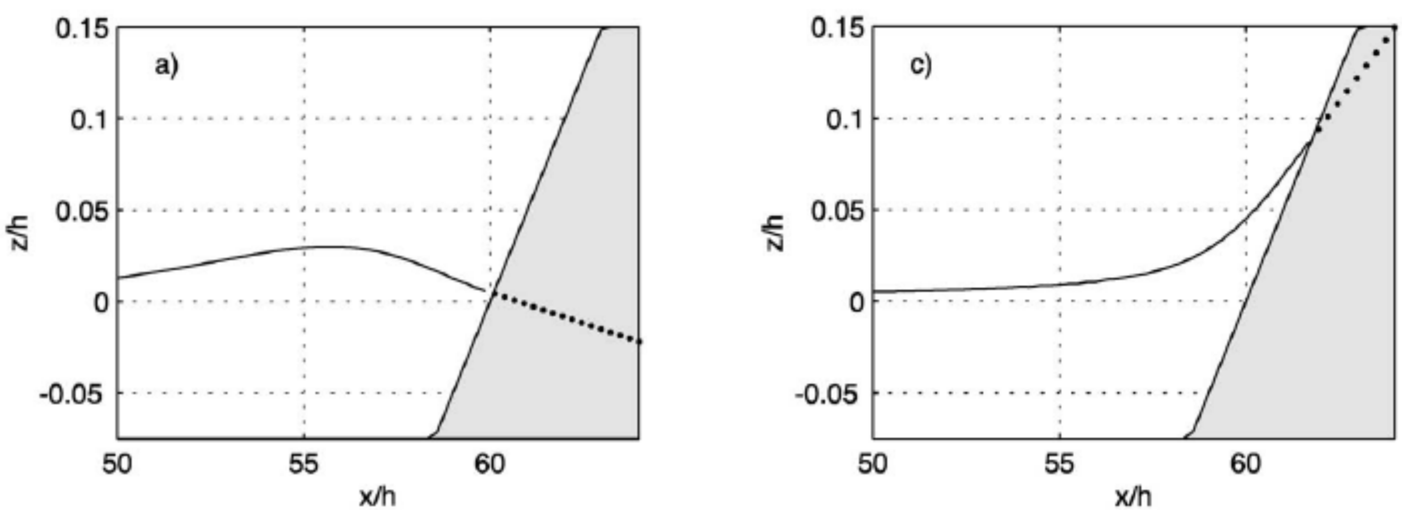
Numerical simulations of tsunami waves

- Source: impulsive model
- Deep ocean: linear shallow water equations
- Shallower region (continental shelf, coastal area): nonlinear shallow water equations
- Effects of frequency dispersion
  - Near the source region
  - Over a very long traveling distance
- Shoreline: moving boundary scheme
  - Various numerical techniques
  - Estimation on the runup (inundation)

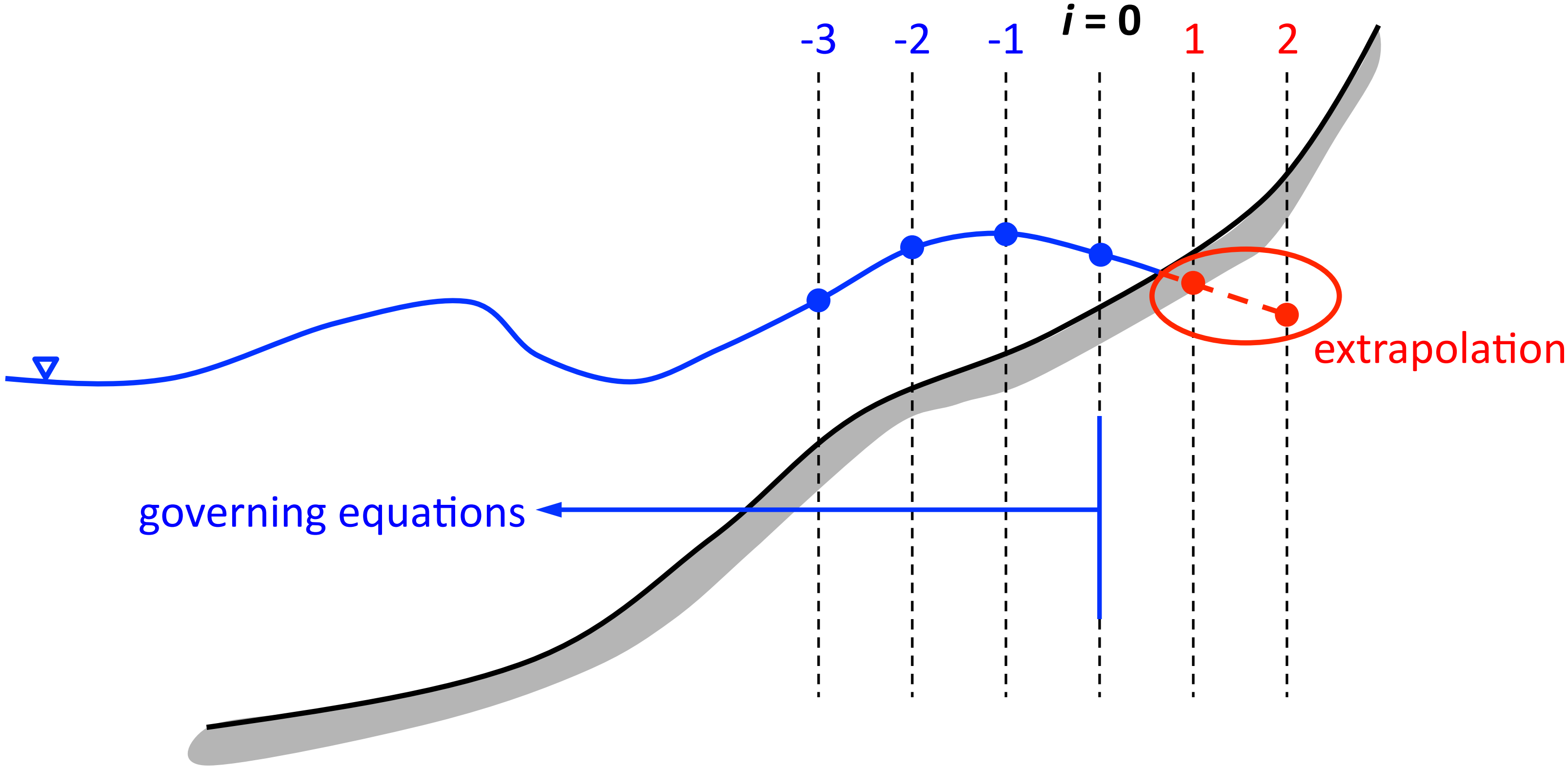
Moving boundary algorithms

- Staircase representation (COMCOT)

- Extrapolation model (COULWAVE)



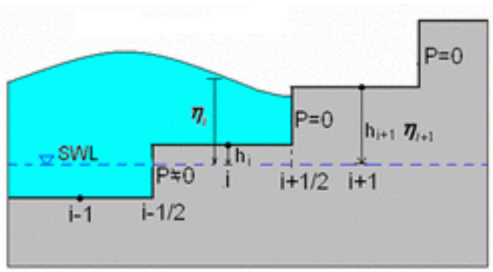
Extrapolation scheme for the moving shoreline



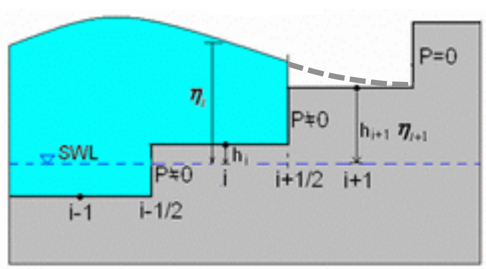
$$P_1 = 2P_0 - P_{-1}, \quad P = \eta,\ \mathbf{u}$$
$$P_2 = 3P_0 - 2P_{-1}$$

Staircase shoreline

$$\eta \ @ \ i-1,\ i,\ i+1; \quad P \ @ \ i-1/2,\ i+1/2$$



$$P_{i+1/2} = 0 \Rightarrow \text{ shoreline stays at } \eta_i$$



$$P_{i+1/2} \neq 0 \Rightarrow \text{ shoreline advances to } \eta_{i+1}$$

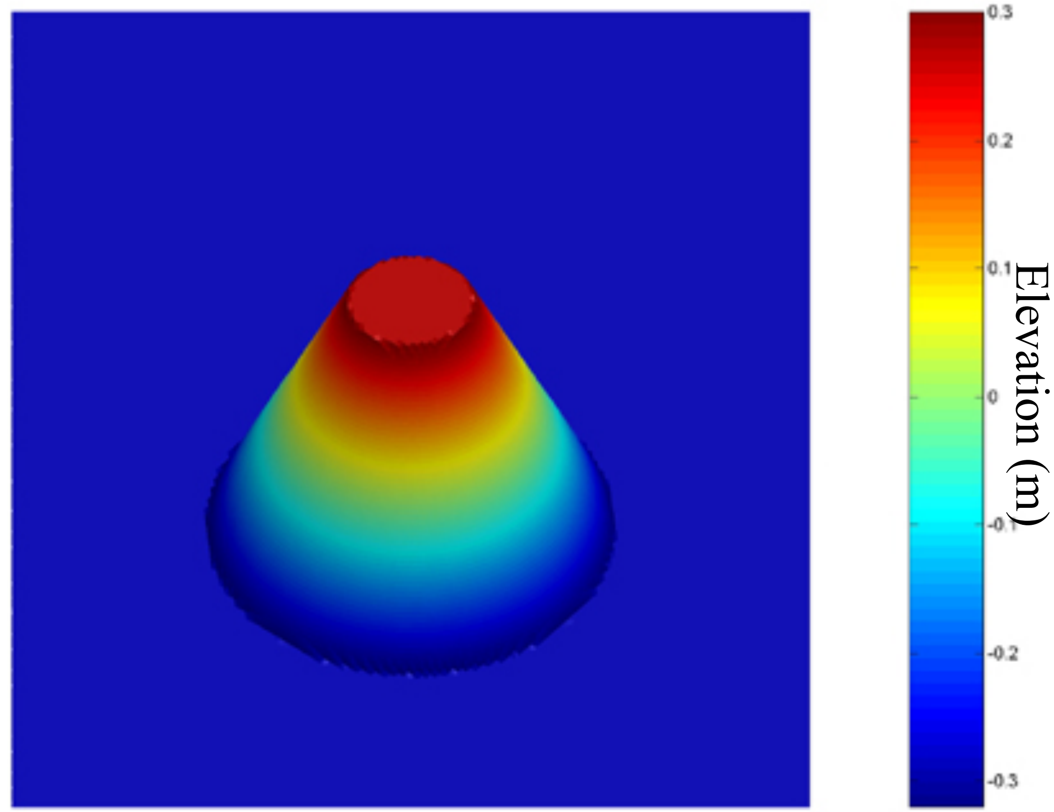
$$(h+\eta)_{i+1} > 0 \Rightarrow P_{i+1/2} \neq 0,\ P_{i+3/2} = 0 \Rightarrow \text{shoreline @ } i+1$$

$$(h+\eta)_{i+1} \leq 0 \begin{cases} h_{i+1} + \eta_i \leq 0 \Rightarrow P_{i+1/2} = 0 \Rightarrow \text{shoreline @ } i \\ \\ h_{i+1} + \eta_i > 0 \Rightarrow P_{i+1/2} \neq 0,\ P_{i+3/2} = 0 \Rightarrow \text{shoreline @ } i+1 \end{cases}$$

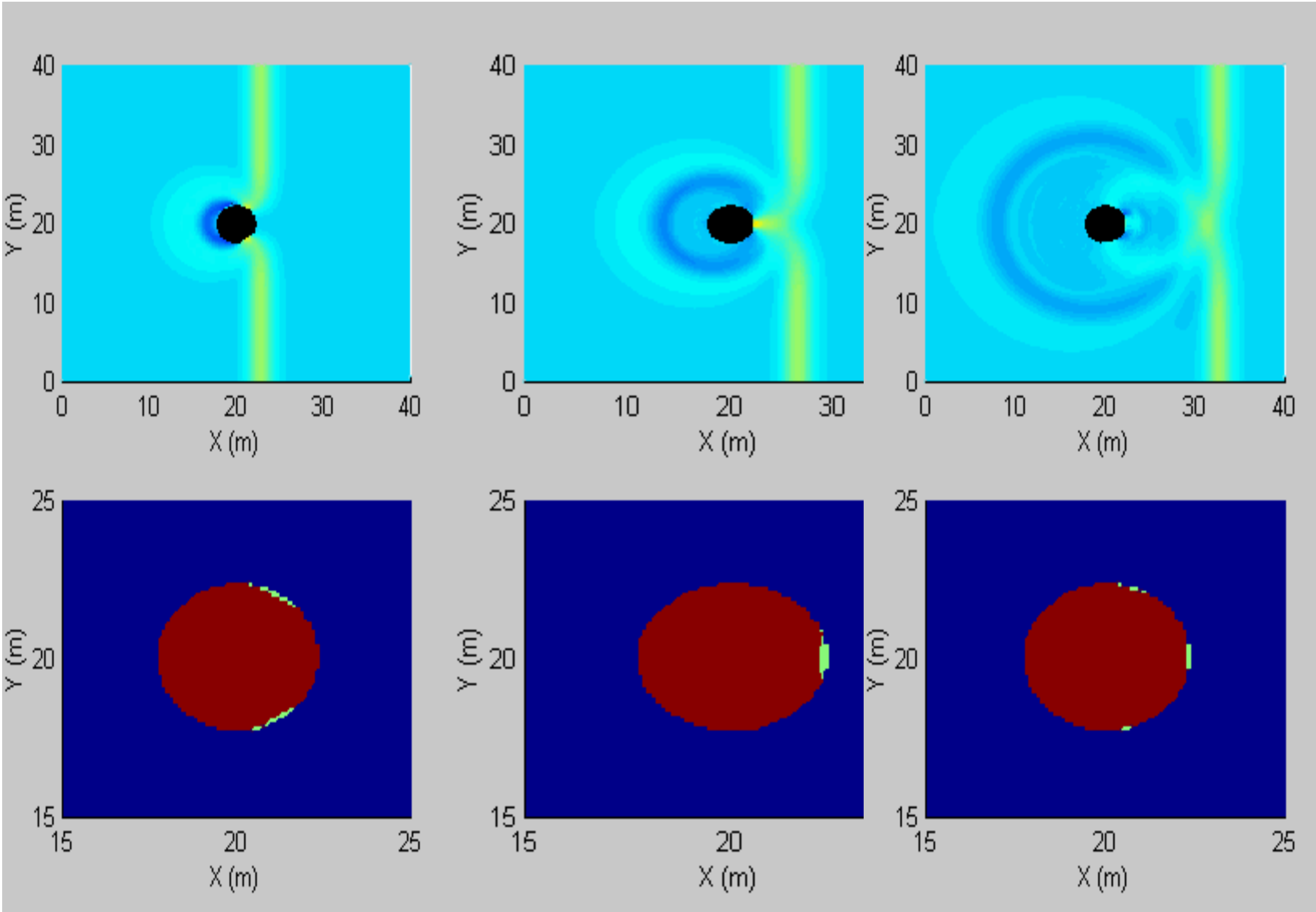
Validation of Runup Algorithm

(Lynett, Wu and Liu 2002, Coastal Engineering)

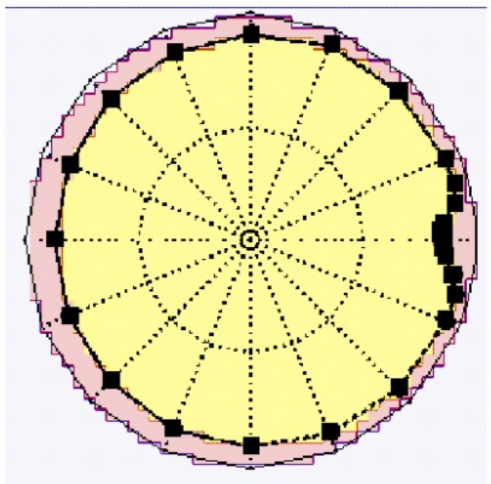
- Runup of solitary wave around a circular island
  - Experimental data taken from Liu *et al.* (*J.F.M.* 1995)
- Physical setup:
  - Still water depth = 0.32 m
  - Slope of side walls = 1:4
  - Depth profile →
- Numerical simulation of conical island runup:
  - Wave amplitude = 0.028 m
  - Still water depth = 0.32 m
  - Beach slope = 1:4
  - Dx = 0.1 m



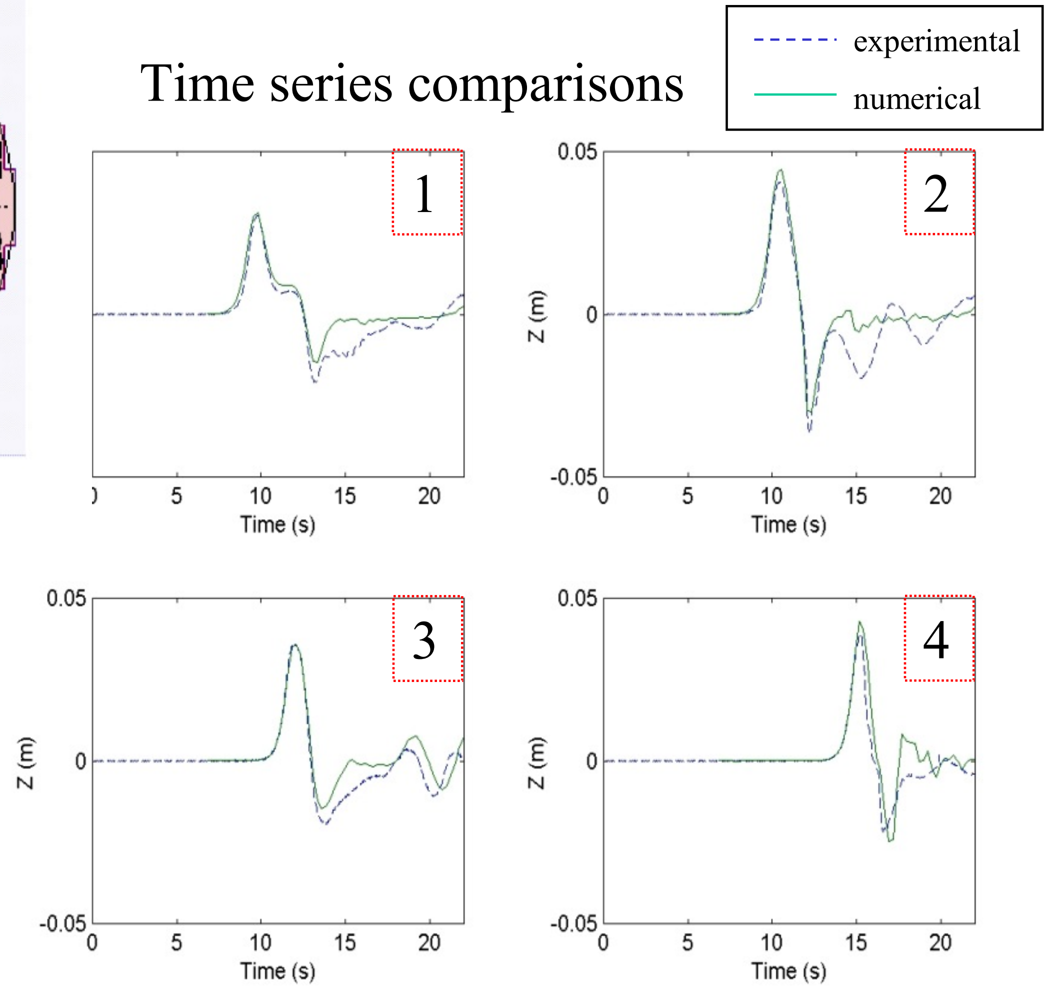
3D animation of runup of solitary wave around a circular island

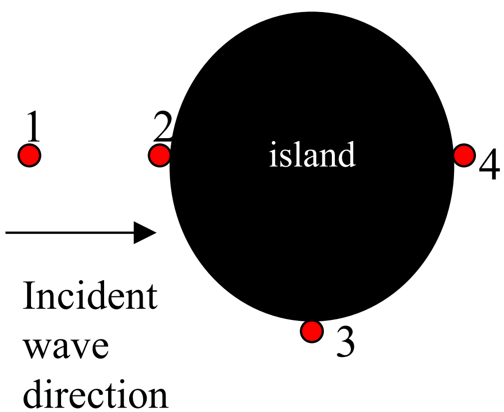


Validation of Runup Algorithm



Time series comparisons





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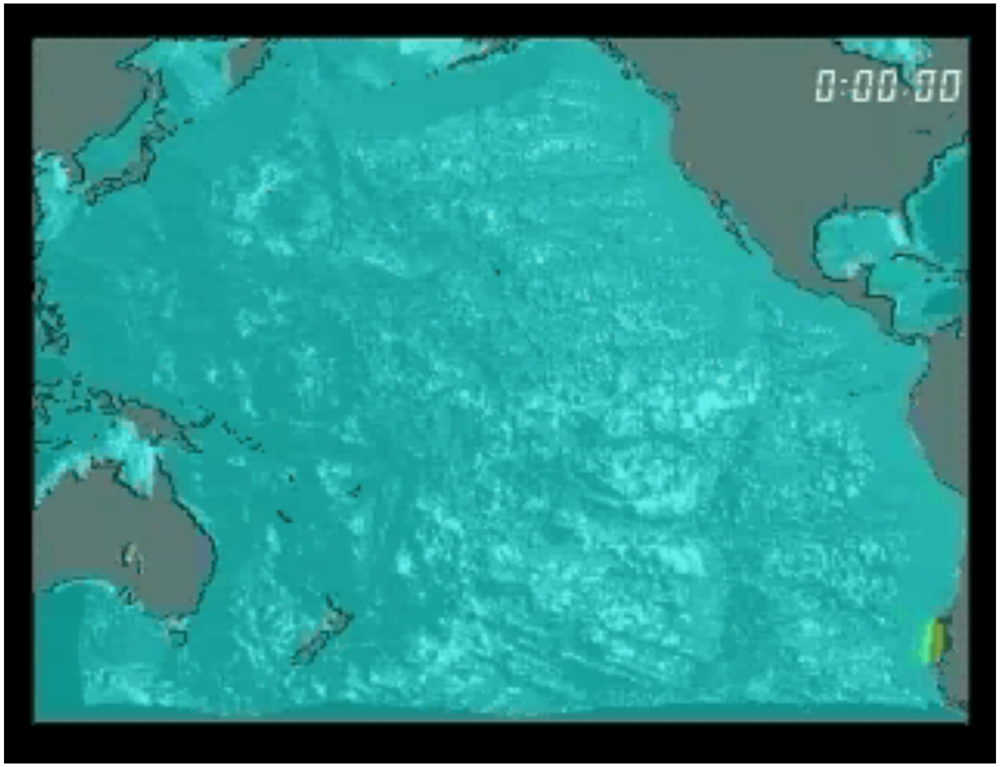
1960 Chilean Tsunami



The epicenter of the 1960 Chilean earthquake was located about 100 km offshore of the Chilean coast. The fault zone was roughly 800 km long and 200 km wide, and the displacement of the fault was 24 m. The orientation of the fault was N10 E. The focal depth of the slip was estimated at 53 km with a 90 degree slip angle and a 10 degree dip angle. Using these estimated fault parameters, we can calculate the initial free-surface displacement (Mansinha and Smylie, 1971). The wavelength of the initial tsunami form was roughly 1,000 km and the wave height was roughly 10 m.

1960 Chilean Tsunami Inundation in Hilo Bay

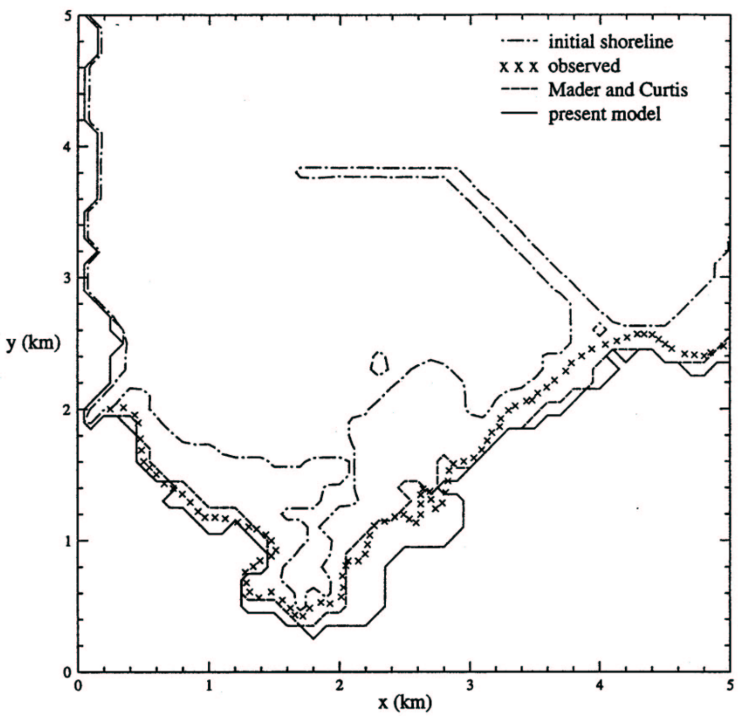


Figure 4-10: The comparison of maximum inundation area.

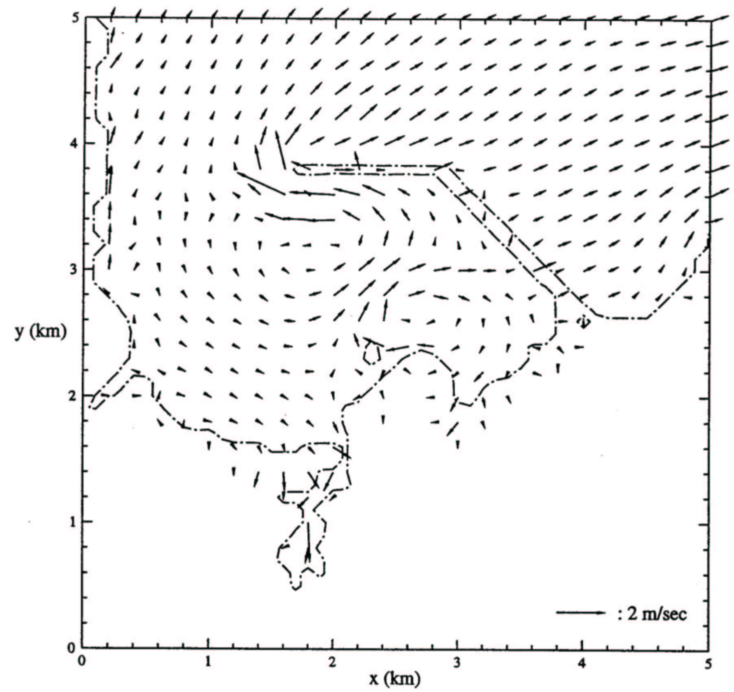


Figure 4-12 (c). The snapshot of velocity distribution (time = 15 hr 20 min).

Some common difficulties in using depth-integrated wave equations

- Most of numerical algorithms are dissipative, especially the moving shoreline algorithms;
- Most of models do not include wave breaking;
- Most of models specify bottom friction coefficients and wave breaking parameters empirically with limited validation;
- Depth-integrated wave equations can not adequately address the wave-structure interaction issues.

Other open issues

- Coupling the hydrodynamic models with sediment transport models
- Coupling the hydrodynamic models with debris flow models
- Coupling the hydrodynamic models with soil (foundation) dynamic models

3D/2D Numerical Modeling of Tsunamis in Nearshore Environment and Their Interaction with Structures

$$\mathbf{u} = 0$$

$$\frac{\partial(\ \mathbf{u})}{\partial t} + \ (\ \mathbf{uu}) = \ p + \ \%\!\!\theta + \ \mathbf{g} \qquad \%\!\!\theta = \ (\ \mathbf{u} + \ ^T\mathbf{u})$$

Turbulence Models