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A Model for a Motor Unit Train Recorded during Constant Force Isometric Contractions

Carlo J. De Luca

Department of Orthopaedic Surgery, Harvard Medical School, Children's Hospital Medical Center, Boston, Mass., USA and Department of Mechanical Engineering, Massachusetts Institute of Technology

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Abstract

A model for the motor-unit action-potential train is developed, based on previously obtained empirical information. The auto and cross-correlation functions are calculated. The autocorrelation function is used to derive the mean rectified value, the variance and the root-mean-squared value of a motor-unit action-potential train. These parameters are solved by using two approximations for the motor-unit action-potential; a piece-wise approximation and a Dirac Delta function approximation. The Dirac Delta function approximation sufficiently simplifies the mathematics so that the model can be extended to myoelectric signals. The cross-correlation function contains information about the synchronization of motor-unit action-potential trains that may be useful as an objective indicator of muscle fatigue.

Introduction

Several investigations have attempted to formulate mathematical expressions for the motor-unit action-potential train (MUAPT) [2, 14, 10, 9, 26, 5]. Of the above investigators, only Libkind [14] employed an empirically derived model for the inter-pulse of the MUAPT's. In a previous paper De Luca and Forrest [11] described some properties of MUAPT's recorded during constant force isometric contractions. Those properties will be used to derive a set of equations for some parameters of the MUAPT recorded during a constant force isometric contraction.

The mathematical development will focus on the derivation of the time dependent auto and cross-correlation functions of the MUAPT's. The correlation function may be used to obtain expressions for measurable parameters such as: a) the mean rectified value, b) the mean integrated rectified value, c) the root-mean-squared value, and d) the power density spectrum. These parameters may not be useful when referring to a MUAPT; however, they assume a practical importance when they are assigned to the myoelectric signal. The analysis for the myoelectric signal, based on the derivations in this paper, is presently in progress and will be reported soon.

Background

In a previous paper, De Luca and Forrest [11] reported some properties of the MUAPT's recorded from the middle fibers of the deltoid muscle during constant force isometric abduction. Two of these properties are listed here because they will be required for the ensuing discussion.

1. The Weibull probability distribution function with time and force dependent parameters provides an acceptably good fit for the distribution of the inter-pulse intervals of a MUAPT.

2. The inter-pulse intervals between two adjacent motor-unit action-potentials (MUAP's) in the same MUAPT have a tendency to be statistically independent, but the independence of adjacent pulses is not as strong as that between every other pulse in the same train.

The MUAPT's may be represented by the pulse random process illustrated in Fig. 1. Consider the Dirac Delta impulse random process

$$\delta_i(t) = \sum_{k=1}^{n} \delta(t - t_k).$$

The impulses are located at time t_k, where t is a real continuous random variable. If the impulse process is passed through a system with an impulse response $h(t)$, a pulse random process $u(t)$ is formed. If $h(t)$ is the equation of a MUAP, the random process $u(t)$ represents MUAPT's which can be expressed as

$$u_i(t) = \sum_{k=1}^{n} h(t - t_k) \quad (1)$$

where $t_k = \sum_{l=1}^{k} x_l$ for $k, l = 1, 2, 3, \cdots, n$.

The real continuous random variable x represents the inter-pulse interval and n is the total number of inter-pulse intervals in a MUAPT. The distribution of x in a MUAPT is described by the Weibull probability distribution function

$$f_X(x) = \frac{\kappa(t, f)}{\beta(t, f)} \left(\frac{x - \alpha}{\beta(t, f)} \right)^{\kappa(t, f) - 1} \exp\left(- \left(\frac{x - \alpha}{\beta(t, f)} \right)^{\kappa(t, f)} \right) \quad (2)$$

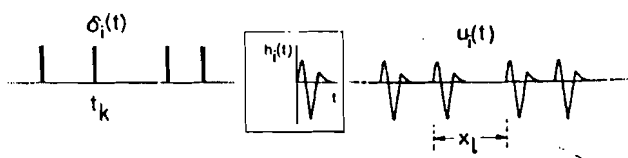

Fig. 1. Schematic model for the motor-unit action-potential train

where t represents the contraction time and f the constant force of the isometric contraction. The skewness parameter κ and the scale parameter β are dependent on t and f; whereas, the location parameter, α, is independent of t and f.

Correlation Functions of the Motor-Unit Action-Potential Trains

Because the MUAPT's, $u_i(t)$ and $u_j(t)$, may be synthesized by convolution of $h_i(t)$ and $h_j(t)$ with the time-dependent Dirac Delta impulse trains, the time-dependent correlation function of the MUAPT's may be expressed as

$$R_{u_i u_j}(t_a, t_b) = R_{\delta_i \delta_j}(t_a, t_b) * h_i(t_a) * h_j(t_b) \tag{3}$$

where $R_{\delta_i \delta_j}(t_a, t_b)$ is the correlation function of the time-dependent Dirac Delta impulse trains. To facilitate the expansion of the latter function consider the two Dirac Delta impulse trains in Fig. 2. The height of each impulse is represented by ξ. The correlation function may be calculated as an ensemble average

$$R_{\delta_i \delta_j}(t_a, t_b) = \sum_{k=0}^{1} \sum_{l=0}^{1} \delta_{ik}\, \delta_{jl}\, P_{\delta_i \delta_j}(\delta_{ik}, \delta_{jl}; t_a, t_b). \tag{4}$$

Now, δ_{ik} and δ_{jl} can assume values of either 0 or ξ; yielding four possible values of $\delta_{ik}\delta_{jl}$:

$$0 \times \xi, \quad \xi \times 0, \quad 0 \times 0, \quad \xi \times \xi.$$

In the case where $i = j$, $R_{\delta_i \delta_i}(t_a, t_b)$ represents the autocorrelation function

$$R_{\delta_i \delta_i}(t_a, t_b) = \xi^2 P_{\delta_i \delta_i}(\delta_{i1}, \delta_{i1}; t_a, t_b) \tag{5}$$

for $t_a = t_b = t$

$$\begin{aligned} R_{\delta_i \delta_i}(t) &= \xi^2 P_{\delta_i \delta_i}(\delta_{i1}, \delta_{i1}; t) \\ &= \xi^2 P_{\delta_i}(\delta_{i1}, t) \end{aligned} \tag{6}$$

and for $t_a \neq t_b$

$$R_{\delta_i \delta_i}(t_a, t_b) = \xi^2 P_{\delta_i}(\delta_{i1}; t_a)\, P_{\delta_i}(\delta_{i1}; t_b) \tag{7}$$

if the occurence of the Dirac Delta impulses is independent of any previous occurences in the same train. Clamann [8] and De Luca and Forrest [11] have reported that for the purpose of signal analysis the inter-pulse intervals of adjacent MUAP's may be considered to be statistically independent. According to the MUAPT properties listed in the Background Section, Eq. (7) is particularly valid if the interval $t_b - t_a$ is greater than an inter-pulse interval. A diagrammatic representation of the correlation of two Dirac Delta impulse trains can be seen in Fig. 2.

The probability of having an impulse in the interval dt is

$$P_{\delta_i}(\delta_{i1} = \xi, t, f) = \lambda_i(t, f)\, dt \tag{8}$$

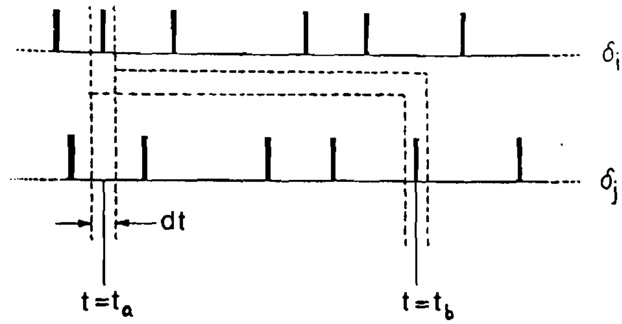

Fig. 2. Dirac Delta function impulse trains graphically arranged to demonstrate autocorrelation and cross-correlation

where $\lambda_i(t, f)$ is the firing rate of a MUAPT. The firing rate may be obtained by taking the inverse of the mean value of the Weibull probability distribution function, hence,

$$\lambda_i(t, f) = \frac{1}{\mu_i(t, f)} = \frac{1000}{\beta_i(t, f)\, \Gamma\left(1 + \dfrac{1}{\kappa_i(t, f)}\right) + \alpha_i} \tag{9}$$

pulses per sec.

The equations for $\beta_i(t, f)$, $\kappa_i(t, f)$ and α_i were determined in a previous paper [11]. Note that the constant-force variable has been introduced because the Weibull probability distribution function describing the inter-pulse intervals was found to be dependent on the constant-force level of the isometric contraction.

By direct substitution, Eqs. (6) and (7) become

$$\begin{aligned} R_{\delta_i \delta_i}(t, f) &= \xi^2 dt\, \lambda_i(t, f) & t_a &= t_b \\ R_{\delta_i \delta_i}(t_a, t_b, f) &= \xi dt\, \lambda_i(t_a, f)\, \xi dt\, \lambda_i(t_b, f) & t_a &\neq t_b \end{aligned} \tag{10}$$

But for a Dirac Delta pulse as $\xi \to \infty$ and $dt \to 0$, $\xi dt = 1$, therefore,

$$R_{\delta_i \delta_i}(t_a, t_b) = \lambda_i(t_a, f)\, \delta_i(t_a) + \lambda_i(t_a)\, \lambda_i(t_b). \tag{11}$$

This is the autocorrelation function of the time dependent Dirac Delta impulse train.

In the case where $i \neq j$, $R_{\delta_i \delta_j}(t_a, t_b)$ represents the cross-correlation function of two impulse trains. In this situation, the probability that an impulse is simultaneously present at $t = t_a$ in δ_i and $t_a = t_b$ in δ_j is the same as the probability that an impulse is present at $t = t_a$ in δ_i and $t \neq t_b$ in δ_j. Furthermore, according to Shiavi [24], two MUAPT's recorded during a constant-force isometric contraction have no significant interaction, hence,

$$\begin{aligned} P_{\delta_i \delta_j}(\delta_{i1}, \delta_{j1}; t_a, t_b) &= P_{\delta_i}(\delta_{i1}, t_a)\, P_{\delta_j}(\delta_{j1}, t_b) \\ &= \lambda_i(t_a, f)\, dt\, \lambda_j(t_b, f)\, dt. \end{aligned} \tag{12}$$

Employing the same argument used to derive the autocorrelation function, the following expression of the cross-correlation function can be obtained

$$R_{\delta_i\delta_j}(t_a, t_b, f) = \lambda_i(t_a, f)\,\lambda_j(t_b, f)\,. \tag{13}$$

It may be possible that two motor units will fire in unison (i.e., they are phase-locked) with $\lambda_i(t, f) = \lambda_j(t, f)$. In such a circumstance the cross correlation function will be identical to the autocorrelation function.

Now, by introducing Eq. (11) into Eq. (3), the equation for the time dependent autocorrelation function of a MUAPT is expressed as

$$\begin{aligned} R_{u_iu_i}(t_a, t_b, f) = & \int_0^\infty \lambda_i(\hat{t}, f)\,h_i(t_a - \hat{t})\,d\hat{t} \\ & \cdot \int_0^\infty \lambda_i(\hat{t}, f)\,h_i(t_b - \hat{t})\,d\hat{t} \\ & + \int_0^\infty \lambda_i(\hat{t}, f)\,h_i(t_a - \hat{t})\,h_i(t_b - \hat{t})\,d\hat{t} \end{aligned} \tag{14}$$

where $\hat{t}$ is a dummy variable. The lower limit of the integration may be set to zero because the MUAPT is only present for positive time. The time dependent cross-correlation function of two MUAPT's with firing rates $\lambda_i(t, f) \neq \lambda_j(t, f)$ can be obtained by placing Eq. (13) into Eq. (3)

$$\begin{aligned} R_{\substack{u_iu_j \\ i \neq j}}(t_a, t_b, f) = & \int_0^\infty \lambda_i(\hat{t}, f)\,h_i(t_a - \hat{t})\,d\hat{t} \\ & \cdot \int_0^\infty \lambda_j(\hat{t}, f)\,h_j(t_b - \hat{t})\,d\hat{t}\,. \end{aligned} \tag{15}$$

Two motor units firing in unison with identical firing rates $\lambda_i(t, f) = \lambda_j(t, f)$ will have a time dependent cross-correlation function identical to Eq. (15) if the MUAPT's have a relative displacement (T) greater than or equal to the time duration of $h_i(t)$ or $h_j(t)$. When T is less than the time duration of $h_i(t)$ or $h_j(t)$ and $\lambda_i(t, f) = \lambda_j(t, f)$ the time dependent cross-correlation function will be

$$\begin{aligned} R_{\substack{u_iu_j \\ i \neq j}}(t_a, t_b, f) = & \int_0^\infty \lambda_i(\hat{t}, f)\,h_i(t_a - \hat{t})\,d\hat{t} \\ & \cdot \int_0^\infty \lambda_i(\hat{t}, f)\,h_j(t_b - \hat{t})\,d\hat{t} \\ & + \int_0^\infty \lambda_i(\hat{t}, f)\,h\,(t_a - \hat{t} \pm T)\,h_j(t_b - \hat{t})\,d\hat{t}\,. \end{aligned} \tag{16}$$

In such a case, the MUAPT's will be considered to be *synchronized*.

Piecewise Linear Approximation for the Motor-Unit Action-Potential

Up to this point, the properties of the MUAP, $h(t)$, have not been mentioned. Some knowledge of $h(t)$ is required to continue the solution of the correlation functions. It is well known that the shape of $h(t)$ can assume several different forms depending on: a) the particular muscle from which it is recorded [22, 20, 1], b) the type of electrode that is used to record the MUAP [6, 7], and c) the relative position of the recording electrode and the active muscle fibers [6, 7]. In addition to the above variables the amplitude and duration of $h(t)$ also depend on the elapsed time of a muscle contraction (De Luca and Forrest, 1973), the force level of the muscle contraction [20, 12, 19] and the depth of insertion in a particular muscle [6].

For the purpose of this study a tri-phasic MUAP (plotted in Fig. 3) with a duration of 7.5 msec was adopted to carry out the ensuing analysis. The arbitrary shape and duration are representative of the MUAP's recorded from muscles in the arm and forearm. Average time duration values ranging from 5.40 msec [1] to 9.15 msec [6] have been recorded from upper limb muscles.

Previous attempts at representing $h(t)$ with mathematical functions can be found in the literature. Bernshtein [3] approximated $h(t)$ by a series of exponentials, but ran into difficulties in relating his expression to empirical MUAP's. Person and Mishin [21] idealized $h(t)$ by a symmetrical positive and negative triangular pulse, and with this function proceeded to calculate some parameters of the myoelectric signal.

In this paper, a piecewise linear approximation of $h(t)$ will be used. Such an approximation preserves the basic useful information of $h(t)$, such as the number of phases, the amplitude, the zero crossing and the time duration. Consider the empirically recorded MUAP in Fig. 3. The MUAP can be expressed as the sum of its constituent phases, a_k,

$$h(t) = \sum_{k=1}^{p} a_k(t - t_{2k-1}) \tag{17}$$

where $k = 1, 2, \cdots, p$, and p represents the total number of phases in $h(t)$. Each phase may be approximated by a triangle, which in turn may be expressed by three ramps, for example:

$$\begin{aligned} a_1(t) \simeq & \; m_1(t - t_0)\,w(t) + (m_2 - m_1)(t - t_1)\,w(t - t_1) \\ & - m_2(t - t_2)\,w(t - t_2) \end{aligned} \tag{18}$$

where t_0, t_1, t_2 are the time locations of successive zero crossings or peak points and m_1 and m_2 represents the slopes of the ramps. In the case of the MUAP in

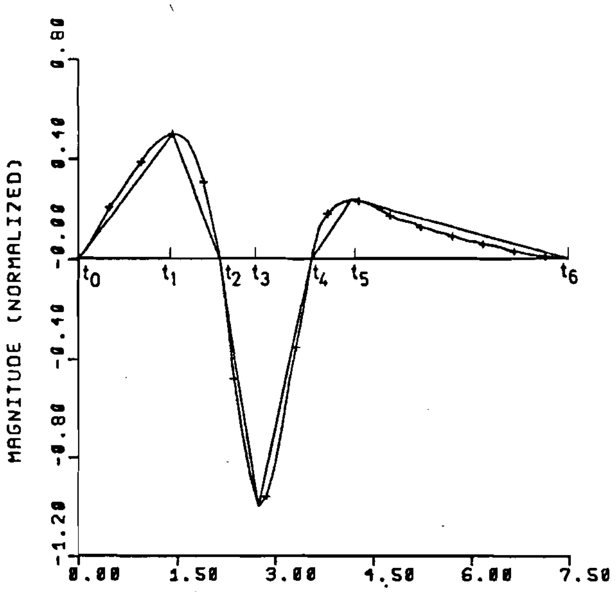

Fig. 3. An empirically recorded motor-unit action-potential and the piecewise linear approximation. The amplitude is arbitrarily normalized with respect to the peak value of the negative phase. The time markings indicate the location of the peaks and zero crossings

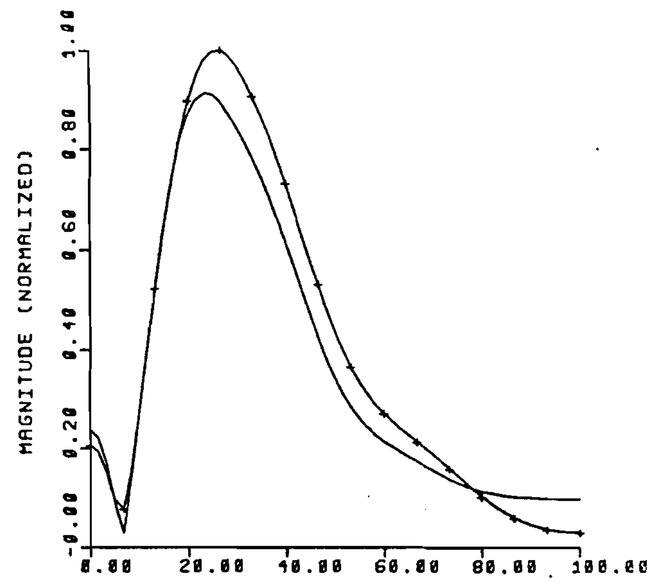

Fig. 4. Frequency spectrum of the motor-unit action-potential in Fig. 3; + + + + continuous pulse, ——— piecewise linearly approximated pulse

Fig. 3, m_1 is a positive quantity and m_2 is a negative quantity. The function $w(t)$ is the unit step function. If a negative phase were considered, the signs of m_1 and m_2 would be reversed. The phases of a MUAP always alternate in sign, therefore, it follows that

$$h(t) \simeq \sum_{k=0}^{2p} (m_{k+1} - m_k)(t - t_k)\, w(t - t_k) \qquad (19)$$

where $m_0 = m_{2p+1} = 0$.

The frequency spectrum of the piecewise linear approximation of $h(t)$ can be obtained by taking the Fourier transform of Eq. (19). The magnitude of the frequency spectrum can be expressed as

$$|H(f)|^2 \simeq \frac{1}{f^4}\left\{\left[\sum_{k=0}^{2p} (m_k - m_{k+1})\cos(f t_k)\right]^2 + \left[\sum_{k=0}^{2p} (m_k - m_{k+1})\sin(f t_k)\right]^2\right\}. \qquad (20)$$

The above equation has a second order pole at $f = 0$. The value of the frequency spectrum at $f = 0$ can be determined by applying L'Hôpital's rule. It can be shown that

$$|H(0)|^2 \simeq \frac{1}{4}\left[\sum_{k=0}^{2p} t_k^2 (m_{k+1} - m_k)\right]^2. \qquad (21)$$

As in Eq. (19), Eqs. (20) and (21) also have the restriction that

$$m_0 = m_{2p+1} = 0\,.$$

A detailed discussion of the above derivation is presented by De Luca [10].

It is possible to obtain a further indication of the utility of the piecewise linear approximation by comparing the frequency spectrum of the MUAP in Fig. 3 with that of the piecewise approximation described by Eqs. (20) and (21). The comparison of the two spectra presented in Fig. 4 shows that for the particular MUAP considered the approximated MUAP has a lower amplitude spectrum that is slightly shifted towards the low frequency end. The difference is not critical considering the great simplification in the representation of the MUAP.

Concept of a Generalized Firing Rate

Now that a mathematical representation has been made for $h(t)$, an expression for the MUAPT firing rate is required to solve the correlation functions. If a MUAPT can be recorded such that it is distinguishable, then the individual firing rate $\lambda_i(t, f)$ can be measured and expressed mathematically.

In myoelectric signals composed of several MUAPT's, the component MUAPT's are not continuously distinguishable throughout the signal. Hence, their individual firing rates are unattainable. However, it is still possible to obtain a meaningful solution to Eqs. (14), (15) and (16) by replacing the individual firing rates with the *generalized firing rate*, $\lambda(\tau, \phi)$. The *generalized firing rate* is defined as the firing rate of a typical MUAPT and has the form of Eq. (9);

$$\lambda(\tau, \phi) = \frac{1000}{\beta(\tau, \phi)\,\Gamma\left(1 + \dfrac{1}{\kappa(\tau, \phi)}\right) + \alpha} \text{ pulses per sec} \quad (22)$$

with

$$\kappa(\tau, \phi) = 1.16 - 0.19\tau + 0.18\phi \quad \text{for } 0 < \tau < 1 \quad 0 < \phi < 1$$

$$\beta(\tau, \phi) = \exp(4.60 + 0.67\tau - 1.16\phi) \text{ msec}$$

$$\alpha = 3.89 \text{ msec}$$

where τ represents the normalized contraction-time and ϕ the constant force normalized with respect to the force of maximum voluntary contraction [11]. The values of $\kappa(\tau, \phi)$ and $\beta(\tau, \phi)$ were obtained from MUAPT's recorded from four subjects performing constant force isometric contractions of various levels. It was necessary to normalize the contraction time and constant force level to obtain the relationships.

In its present form, Eq. (22) is not suitable for solving the integral formulas of the auto and cross-correlation functions. The function can be expressed as a polynomial by taking a Taylor expansion with respect to τ and performing a polynomial least-squared regression on the coefficients as a function of ϕ. The simplified expression is

$$\lambda(\tau, \phi) \simeq \sum_{l=0}^{3} \eta_l(\phi)\, \tau^l \quad (23)$$

where

$$\begin{aligned}
\eta_0 &= 1.028258 \times 10^{-2} + 1.024522 \times 10^{-2}\phi + 1.010051 \times 10^{-2}\phi^2 \\
\eta_1 &= -7.022323 \times 10^{-3} - 7.141921 \times 10^{-3}\phi - 4.659232 \times 10^{-3}\phi^2 \\
\eta_2 &= 2.170056 \times 10^{-3} + 1.996963 \times 10^{-3}\phi + 1.139468 \times 10^{-3}\phi^2 - 5.330590 \times 10^{-4}\phi^3 \\
\eta_3 &= -4.006277 \times 10^{-4} - 2.650333 \times 10^{-4}\phi - 1.027011 \times 10^{-4}\phi^2 + 2.780395 \times 10^{-4}\phi^3 .
\end{aligned}$$

The polynomial is truncated at the third order with a residual error of less than 0.75% in the worst possible case. This is an acceptable error well within the experimental bounds.

Solution of Parameters

Equation (14) represents the autocorrelation function of a particular MUAPT for all values of t_a and t_b. Consider the case where $t_a = t_b$, then the expression reduces to

$$\begin{aligned}
R_{u_i u_i}(t, f) &= \left[\int_0^{\infty} \lambda_i(\hat{t}, f)\, h_i(t - \hat{t})\, d\hat{t}\right]^2 + \\
&\quad + \int_0^{\infty} \lambda_i(\hat{t}, f)\, h_i^2(t - \hat{t})\, d\hat{t} \\
&= [E\{u_i(t)\}]^2 + \sigma_{u_i}^2(t, f) \\
&= (\text{mean})^2 + \text{variance} \\
&= (\text{rms})^2 .
\end{aligned} \quad (24)$$

The above equation involves convolutions of a MUAP with its firing rate. By using the piecewise linear approximation for $h(t)$ it is only necessary to know the time and amplitude values of the peaks and zero crossings of $h(t)$. If the actual firing rate of the MUAPT is not known, a meaningful solution to Eq. (24) can be obtained by using the approximation for the *generalized firing rate* of Eq. (23).

The solution of the convolutions in Eq. (24) will be greatly simplified if the linearly approximated $h(t)$ is differentiated twice, thus reducing to a sequence of Dirac Delta pulses. The convolution of $\lambda(t, f)$ with the Dirac pulses is reduced to a multiplication $\lambda(t, f)$ with the Dirac Delta pulse. The result is then integrated twice to complete the original convolution. The details are presented in the Appendix.

Some parameters of the MUAPT that have practical application to the signal processing problem of myoelectric signals are the mean rectified value, the variance and the root-mean-squared (rms) value. These parameters can all be obtained from Eq. (24). Their solution appears in Eq. (A7) in the Appendix. The following expressions are general and, therefore, will be written as functions of normalized contraction-time (τ) and force (ϕ).

1. Mean Rectified Value

The MUAP is rectified by inverting the sign of the slopes in the negative phase. This procedure must be performed when the values of the slopes are placed in the calculations. The resulting expression is

$$E\{|u_i(\tau)|\} \simeq \sum_{k=0}^{2p} \sum_{l=0}^{3} (m_{k+1} - m_k)\, \frac{\eta_l(\phi)}{(l+1)(l+2)}\, (\tau - \tau_k)^{l+2} \quad (25)$$

where $m_0 = m_{2p+1} = 0$ and the individual terms are zero for $\tau < \tau_k$.

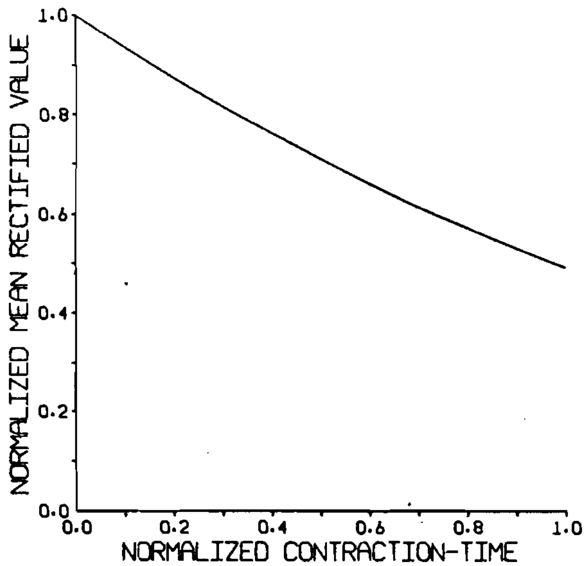

Fig. 5. The normalized mean rectified value of the motor-unit action-potential train as a function of the normalized contraction-time

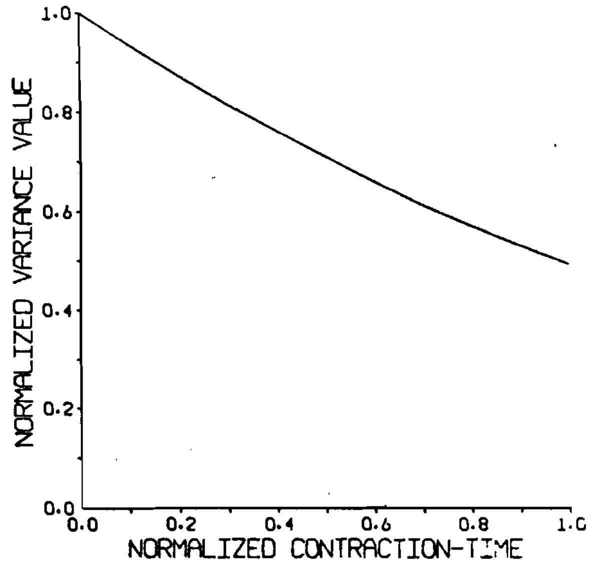

Fig. 6. The normalized variance of the motor-unit action-potential train as a function of the normalized contraction-time

2. *Variance*

The expression involves a convolution of $\lambda(\tau, \phi)$ and $h^2(\tau)$; a somewhat more involved manipulation than that of the mean rectified value. The resulting expression is

$$\begin{aligned} \mathrm{VAR}\{u_i(\tau)\} \simeq & \sum_{k=0}^{2p} \sum_{l=0}^{3} (2m_{k+1}^2 - 2m_k^2) \\ & \cdot \frac{\eta_l(\phi)}{(l+1)(l+2)(l+3)} (\tau - \tau_k)^{l+3} \\ & + \sum_{k=1}^{p} \sum_{l=0}^{3} [2m_{2k-1}^2 (t_{2k-2} - t_{2k-1}) \\ & + 2m_{2k}^2 (t_{2k-1} - t_{2k})] \\ & \cdot \frac{\eta_l(\phi)}{(l+1)(l+2)} (\tau - \tau_{2k-1})^{l+2} \end{aligned} \tag{26}$$

where $m_0 = m_{2p+1} = 0$, the values of the first term are zero for $\tau < \tau_k$ and the values of the second term are zero for $\tau < \tau_{2k-1}$.

3. *Root-Mean-Squared-Value*

The expression for the rms value can be obtained by arranging Eqs. (25) and (26) so that

$$\mathrm{RMS} = [\mathrm{MEAN}^2 + \mathrm{VARIANCE}]^{\frac{1}{2}}. \tag{27}$$

The preceding equations are easily implemented on a digital computer. The data of the approximated MUAP of Fig. 3 and that of the generalized firing rate was placed in Eqs. (25), (26), and (27). The calculated values were normalized with respect to the value of

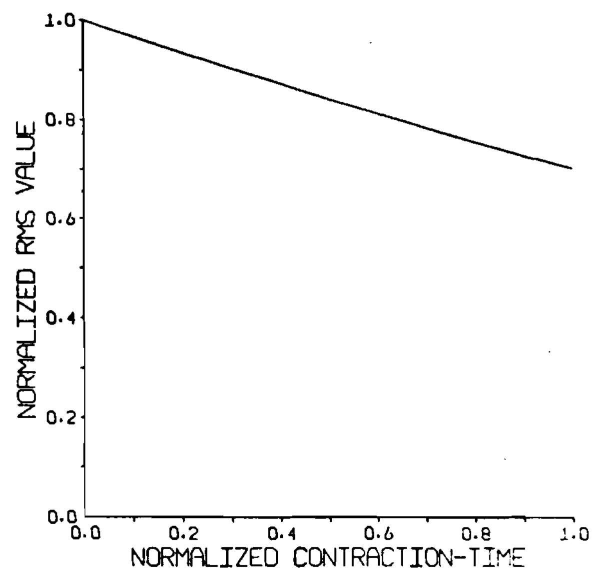

Fig. 7. The normalized root-mean-squared value of the motor-unit action-potential trains as a function of the normalized contraction-time

$\tau = 0$, which in the case of the mean rectified value is

$$\lambda(0) \int_0^{\tau} |h(\tau)|\, d\tau$$

and for the variance is

$$\lambda(0) \int_0^{\tau} h^2(\tau)\, d\tau .$$

It is assumed that these expressions, i.e., the area of the MUAP, remain constant throughout a contraction.

The normalized results are plotted in Figs. 5, 6, and 7. In these plots the time duration of the sustained contraction was considered to be 45 sec and the normalized force level was 0.85. The duration of the MUAP was normalized accordingly. Similar calculations were made for a weaker contraction with a time duration of 132 sec and normalized force level of 0.25. The largest difference between the values of the normalized parameters for the two contractions was calculated for the mean rectified value. The discrepancy was measured to be 0.3212%; an insignificant value considering the much larger experimental errors associated with MUAPT signal recordings and the limited accuracy of the approximated *generalized firing rate*. Hence, the curves in Figs. 5, 6, and 7 are valid for weak and strong contractions.

Dirac Delta Function Approximation of the Motor-Unit Action-Potential

It would be convenient from a mathematical point of view to replace the function $h(t)$ by Dirac Delta impulses, $\delta(t)$. The convolutions of Eq. (24) become trivial because one of the most important properties of the Dirac Delta function is

$$\int_0^{\infty} \lambda(\hat{t}, f)\, \delta(t - \hat{t})\, d\hat{t} = \lambda(t, f). \tag{28}$$

Hence, the autocorrelation function of the MUAPT could be obtained directly from Eq. (11).

There are two direct ways of pursuing the impulse approximation. Each phase of the MUAP can be expressed by an impulse whose amplitude is equal to the area of the phase and is located at the peak of each phase. The other possibility is to represent the complete MUAP by one impulse whose amplitude is equal to the total area of the MUAP and is located in the middle of the MUAP. It follows from Eqs. (24) and (28) that the simplified expressions for the second case can be written as

$$E\{|u_i(\tau)|\} \simeq \underline{|h_i(\tau)|}\, \lambda(\tau, \phi) \tag{29}$$

$$VAR\{u_i(\tau)\} \simeq \underline{h_i^2(\tau)}\, \lambda(\tau, \phi) \tag{30}$$

$$RMS\{u_i(\tau)\} \simeq \{[\underline{h_i(\tau)}\, \lambda(\tau, \phi)]^2 + \underline{h_i^2(\tau)}\, \lambda(\tau, \phi)\}^{\frac{1}{2}} \tag{31}$$

where

$$\underline{h_i(\tau)} = \int_0^{\infty} h_i(\tau)\, d\tau$$

$$\underline{|h_i(\tau)|} = \int_0^{\infty} |h_i(\tau)|\, d\tau \tag{32}$$

$$\underline{h_i^2(\tau)} = \int_0^{\infty} h_i^2(\tau)\, d\tau.$$

The deviation between the values of the parameters calculated by the piecewise linear approximation and the Dirac Delta impulse approximation is 0.0086% in the worst case. This error is considerably smaller than the experimental error.

Discussion

The model developed in this paper supplies precise expressions for parameters of the motor-unit action-potential train (MUAPT) as a function of contraction time at a constant muscle-force during isometric contractions. The resulting equations depend on the firing rate and the area of the motor-unit action-potential (MUAP). The latter requirement is gratifying because it eliminates intimate knowledge and description of the MUAP shape, which varies considerably between recorded MUAPT's.

The solutions to the mean rectified value, the variance and the rms value that are plotted in Figs. 5, 6, and 7 were obtained by assuming that the area of a MUAP remains constant throughout a sustained contraction. Reports of both increase and decrease of the amplitude of the MUAP can be found in the literature. Knowlton *et al.* [13] and Stålberg [25] reported an increase in the amplitude with increasing contraction time. On the other hand, several investigators [17, 23, 4, 15, 18, 11] have reported that the amplitude decreases with increasing contraction time. De Luca and Forrest [11] also stated that the time duration of the MUAP has a tendency to decrease with contraction time. Lindström *et al.* [16] postulated that variations in the shape of a MUAP could be attributed to a decreasing conduction velocity of the muscle fibers during a sustained contraction. If, indeed, the amplitude decreases and the time duration increases, then the area of $h_i(t)$ may not vary substantially (if at all) with contraction time.

The normalized values of the mean rectified value and the variance plotted in Figs. 5 and 6 appear to be identical. This result is not obvious when comparing the equations that generated the curves, i.e., Eqs. (25) and (26). The similarity can be explained by referring to Eqs. (29) and (30), in which $h(t)$ has been replaced by the Dirac Delta function, $\delta(t)$. In the previous section, this replacement was shown to be valid.

In general, the cross-correlation function of the MUAPT's does not contain the variance term that is present in the autocorrelation function. However, if the MUAPT's are synchronized (i.e., have similar firing rate and fire in unison) the cross-correlation function [Eq. (16)] contains a co-variance term. The co-variance term increases the amplitude of the cross-correlation function. This calculable change in the cross-correlation function may provide a precise and objective measurement of *synchronization*. This is a particularly interesting outcome because synchronization is considered to be a phenomenon associated with muscle fatigue. The prospect of yielding an objective measure of a phenomenon associated with muscle fatigue enhances the usefulness of the MUAPT model.

The MUAPT model also provides the groundwork for mathematical analyses of the myoelectric signal formed by the superposition of several MUAPT's. In the myoelectric signal, the individual MUAPT's are generally not separable or distinguishable. Therefore, the individual firing rates of the MUAPT's cannot usually be measured. In this application, the concept of the *generalized firing rate* will be useful. The feasibility of accurately representing the MUAP by a Dirac Delta impulse will greatly facilitate the mathematical derivations.

It is expected that time dependent parameters of the myoelectric signals such as: a) the mean rectified value, b) the mean integrated rectified value, c) the rms value, and d) the power density spectrum will be described by equations. These parameters of the myoelectric signal have basic practical applications in clinical diagnostics and myoelectric prostheses design.

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Appendix

By using the piecewise approximation of $h(t)$, the convolutions in Eq. (24) can be performed according to the following method

$$R_{u_i u_i}(t,f) = [\lambda_i(t,f) * h_i(t)]^2 + \lambda_i(t,f) * h_i^2(t)$$
$$= \left[h(t)'' \int_0^t\!\!\int \lambda_i(t,f)\,dt\,dt \right]^2 + h_i^2(t)''' \int_0^t\!\!\int\!\!\int \lambda_i(t,f)\,dt\,dt\,dt \qquad \text{(A1)}$$

where $h(t)''$ = the second derivative of $h(t)$

and $h^2(t)'''$ = the third derivative of $h^2(t)$.

The piecewise approximation of $h(t)$ consists of a series of ramp functions, hence $h(t)''$ *and* $h^2(t)'''$ become a series of Dirac Delta functions. The convolution of $\lambda(t,f)$ with $h_i(t)''$ reduces to a multiplication. The second derivative, $h(t)''$, can be obtained by two successive differentiations of Eq. (19)

$$h(t)'' \simeq \sum_{k=0}^{2p} (m_{k+1} - m_k)\,\delta(t - t_k) \qquad \text{(A2)}$$

where $m_0 = m_{2p+1} = 0$ and p = the number of phases in a MUAP.

The third derivative of $h^2(t)$ is not as straightforward to calculate, because it contains more complicated discontinuities. The problem is simplified by rearranging Eq. (19) as follows

$$h(t) \simeq \sum_{k=1}^{p} \{ m_{2k-1}(t - t_{2k-2})[u(t - t_{2k-2}) - u(t - t_{2k-1})]$$
$$+ m_{2k}(t - t_{2k})[u(t - t_{2k-1}) - u(t - t_{2k})] \} \qquad \text{(A3)}$$

where $u(t)$ is the unit step function. This function can now be squared

$$h^2(t) \simeq \sum_{k=1}^{p} \{ m_{2k-1}^2(t - t_{2k-2})^2 [u(t - t_{2k-2}) - u(t - t_{2k-1})]$$
$$+ m_{2k}^2(t - t_{2k})^2 [u(t - t_{2k-1}) - u(t - t_{2k})] \}. \qquad \text{(A4)}$$

Two consecutive differentiations of this equation yield

$$h^2(t)'' \simeq \sum_{k=0}^{2p} (2m_{k-1}^2 - 2m_k^2)\,u(t - t_k)$$
$$+ \sum_{k=1}^{p} [2m_{2k-1}^2(t_{2k-2} - t_{2k-1}) \qquad \text{(A5)}$$
$$+ 2m_{2k}^2(t_{2k-1} - t_{2k})]\,\delta(t - t_{2k-1}).$$

The second part of the equation is reduced to a Dirac Delta function expression, but the first part requires one additional differentiation. The approximated solution to the autocorrelation function can be obtained by collecting Eqs. (A2) and (A5) and multiplying them with the appropriate integrals of $\lambda(t,f)$. It follows that the autocorrelation function of a three phase MUAPT may be expressed as:

$$R_{u_i u_i}(t,f) \simeq \left[\sum_{k=0}^{2p} (m_{k-1} - m_k) \int_{t_k}^{t}\!\!\int \lambda_i(t - t_k)\,dt\,dt \right]^2$$
$$+ \sum_{k=0}^{2p} (2m_{k-1}^2 - 2m_k^2) \int_{t_k}^{t}\!\!\int\!\!\int \lambda_i(t - t_k)\,dt\,dt\,dt$$
$$+ \sum_{k=1}^{p} [2m_{2k-1}^2(t_{2k-2} - t_{2k-1}) \qquad \text{(A6)}$$
$$+ 2m_{2k}^2(t_{2k-2} - t_{2k})] \int_{t_k}^{t}\!\!\int \lambda_i(t - t_k)\,dt\,dt.$$

By substituting the *generalized firing rate* approximation of Eq. (23) in the above equation, the following solution can be obtained:

$$R_{u_i u_i}(\tau,f) \simeq \left[\sum_{k=0}^{2p} \sum_{l=0}^{3} (m_{k-1} - m_k) \frac{\eta_l(\phi)}{(l-1)(l+2)} (\tau - \tau_k)^{l-2} \right]^2$$
$$+ \sum_{k=0}^{2p} \sum_{l=0}^{3} (2m_{k-1}^2 - 2m_k^2) \frac{\eta_l(\phi)}{(l-1)(l-2)(l-3)} (\tau - \tau_k)^{l-3}$$
$$+ \sum_{k=1}^{p} \sum_{l=0}^{3} [2m_{2k-1}^2(\tau_{2k-2} - \tau_{2k-1}) \qquad \text{(A7)}$$
$$+ 2m_{2k}^2(\tau_{2k-1} - \tau_{2k})] \frac{\eta_l(\phi)}{(l+1)(l+2)} (\tau - \tau_{2k-1})^{l-2}$$

where $m_0 = m_{2p+1} = 0$ and values of the first and second terms are zero for $\tau < \tau_k$; the third term is zero for $\tau < \tau_{2k-1}$.

Equation (A7) is a solution to the autocorrelation function in Eq. (24). Hence, the squared term is a solution for the mean value and the other two terms for the variance.

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Dr. Carlo J. De Luca
Rehabilitation Engineering Center
Bader 6 Rm 3636
Children's Hosp. Med. Ctr.
300 Longwood Ave.
Boston, Mass. 02115, USA